Computational Complexity of Evolutionary Computation in Combinatorial Optimisation

Frank Neumann[1] Carsten Witt[2]

[1]Algorithms and Complexity Group
Max-Planck-Institute for Informatics
Saarbrücken, Germany

[2]Fakultät für Informatik, LS 2
Technische Universität Dortmund
Germany

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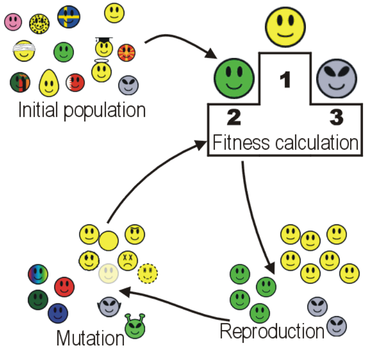
Most famous search heuristic: Evolutionary Algorithms (EAs)

- a bio-inspired heuristic
- paradigm: evolution in nature, "survival of the fittest"



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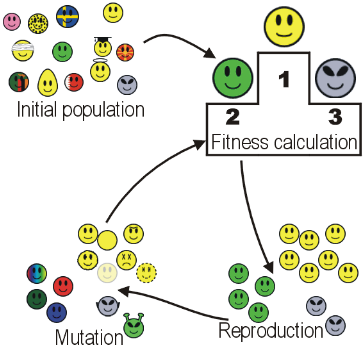
- actually it's only an algorithm, a randomised search heuristic (RSH)



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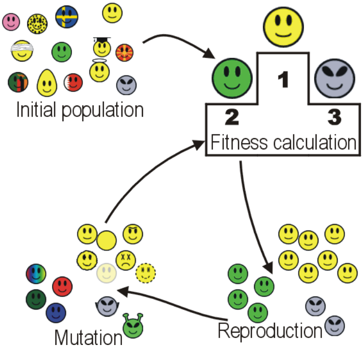
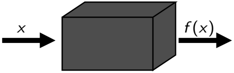


- Goal: optimisation
- Here: discrete search spaces, combinatorial optimisation, in particular pseudo-boolean functions

Optimise $f\colon \{0,1\}^n \rightarrow \mathbb{R}$

Why Do We Consider Randomised Search Heuristics?

- Not enough resources (time, money, knowledge) for a tailored algorithm

- Black Box Scenario 

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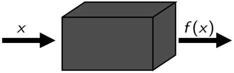
- Black Box Scenario 

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Point of view

Do not only consider RSHs empirically. We need a solid theory to understand how (and when) they work.

What RSHs Do We Consider?

Theoretically considered RSHs

- (1+1) EA
- (1+λ) EA (offspring population)
- (μ+1) EA (parent population)
- (μ+1) GA (parent population and crossover)
- GIGA (crossover)
- SEMO, DEMO, FEMO, ... (multi-objective)
- Randomised Local Search (RLS)
- Metropolis Algorithm/Simulated Annealing (MA/SA)
- Ant Colony Optimisation (ACO)
- Particle Swarm Optimisation (PSO)
- ...

First of all: define the simple ones

The Most Basic RSHs

(1+1) EA, RLS, MA and SA for maximisation problems

(1+1) EA

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   1. Create y by flipping each bit of x_t indep. with probab. $1/n$.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$ else $x_{t+1} := x_t$.

The Most Basic RSHs

(1+1) EA, RLS, MA and SA for maximisation problems

RLS

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(1+1) EA, RLS, MA and SA for maximisation problems

MA

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   1. Create y by flipping one bit of x_t uniformly.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$
      else $x_{t+1} := y$ with probability $e^{(f(x_t)-f(y))/T}$ anyway
      and $x_{t+1} := x_t$ otherwise.

T is fixed over all iterations.

(1+1) EA, RLS, MA and SA for maximisation problems

SA

1. Choose $x_0 \in \{0,1\}^n$ uniformly at random.
2. For $t := 0, \ldots, \infty$
   1. Create y by flipping one bit of x_t uniformly.
   2. If $f(y) \geq f(x_t)$ set $x_{t+1} := y$
      else $x_{t+1} := y$ with probability $e^{(f(x_t)-f(y))/T_t}$ anyway
      and $x_{t+1} := x_t$ otherwise.

T_t is dependent on t, typically decreasing

What Kind of Theory Are We Interested in?

- Not interesting here: convergence (often trivial), local progress, models of EAs (e. g., infinite populations), ...
- Treat RSHs as randomised algorithm!
- Analyse their "runtime" (computational complexity) on selected problems

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- Analyse their "runtime" (computational complexity) on selected problems

Definition

Let RSH A optimise f. Each f-evaluation is counted as a time step. The *runtime* $T_{A,f}$ of A is the random first point of time such that A has sampled an optimal search point.

- Often considered: expected runtime, distribution of $T_{A,f}$
- Asymptotical results w. r. t. n

How Do We Obtain Results?

We use (rarely in their pure form):

- Coupon Collector's Theorem
- Principle of Deferred Decisions
- Concentration inequalities:
  Markov, Chebyshev, Chernoff, Hoeffding, ... bounds
- Markov chain theory: waiting times, first hitting times
- Rapidly Mixing Markov Chains
- Random Walks: Gambler's Ruin, drift analysis (Wald's equation), martingale theory, electrical networks
- Random graphs (esp. random trees)
- Identifying typical events and failure events
- Potential functions and amortised analysis
- ...

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Adapt tools from the analysis of randomised algorithms; understanding the stochastic process is often the hardest task.

Early Results

Analysis of RSHs already in the 1980s:

- Sasaki/Hajek (1988): SA and Maximum Matchings
- Sorkin (1991): SA vs. MA
- Jerrum (1992): SA and Cliques
- Jerrum/Sorkin (1993, 1998): SA/MA for Graph Bisection
- ...

These were high-quality results, however, limited to SA/MA (nothing about EAs) and hard to generalise.

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Since the early 1990s

Systematic approach for the analysis of RSHs,
building up a completely new research area

This Tutorial

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA
2. Combinatorial optimisation problems
   - (1+1) EA and Eulerian cycles
   - (1+1) EA and minimum spanning trees
   - (1+1) EA and maximum matchings
   - (1+1) EA and the partition problem
   - Multi-objective optimisation and the set cover problem
   - SA beats MA in combinatorial optimisation
   - ACO and minimum spanning trees
3. End
4. References

How the Systematic Research Began — Toy Problems

Simple example functions (test functions)

- $\text{OneMax}(x_1, \ldots, x_n) = x_1 + \cdots + x_n$
- $\text{LeadingOnes}(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$
- $\text{BinVal}(x_1, \ldots, x_n) = \sum_{i=1}^{n} 2^{n-i} x_i$
- polynomials of fixed degree

Goal: derive first runtime bounds and methods

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Goal: derive first runtime bounds and methods

Artificially designed functions

- with sometimes really horrible definitions
- but for the first time these allow rigorous statements

Goal: prove benefits and harm of RSH components,
e. g., crossover, mutation strength, population size ...

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Example: OneMax

Theorem (e. g., Droste/Jansen/Wegener, 1998)

The expected runtime of the RLS, (1+1) EA, (μ+1) EA, (1+λ) EA on OneMax *is* $\Omega(n \log n)$.

Proof by modifications of Coupon Collector's Theorem.

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The expected runtime of the RLS, (1+1) EA, (μ+1) EA, (1+λ) EA on $\textsc{OneMax}$ *is* $\Omega(n \log n)$.

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Theorem (e. g., Mühlenbein, 1992)

The expected runtime of RLS and the (1+1) EA on $\textsc{OneMax}$ *is* $O(n \log n)$.

Holds also for population-based (μ+1) EA and
for (1+λ) EA with small populations.

Proof of the $O(n \log n)$ bound

- *Fitness levels:* $L_i := \{x \in \{0,1\}^n \mid \textsc{OneMax}(x) = i\}$

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Proof of the $O(n \log n)$ bound

- *Fitness levels:* $L_i := \{x \in \{0,1\}^n \mid \textsc{OneMax}(x) = i\}$
- (1+1) EA never decreases its current fitness level.
- From i to some higher-level set with prob. at least

$$\underbrace{\binom{n-i}{1}}_{\text{choose a 0-bit}} \cdot \underbrace{\left(\frac{1}{n}\right)}_{\text{flip this bit}} \cdot \underbrace{\left(1-\frac{1}{n}\right)^{n-1}}_{\text{keep the other bits}} \geq \frac{n-i}{en}$$

- Expected time to reach a higher-level set is at most $\frac{en}{n-i}$.
- Expected runtime is at most

$$\sum_{i=0}^{n-1} \frac{en}{n-i} = O(n \log n). \qquad \square$$

Later Results Using Toy Problems

- Find the theoretically optimal mutation strength ($1/n$ for OneMax!).
- optimal population size (often 1!)
- crossover vs. no crossover $\rightarrow$ Real Royal Road Functions
- multistarts vs. populations
- frequent restarts vs. long runs
- dynamic schedules
- ...

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Further reading: Droste/Jansen/Wegener (2002), He/Yao (2002, 2003), Jansen (2002), Jansen/De Jong/Wegener (2005), Jansen/Wegener (2001, 2005), Storch/Wegener (2004), Witt (2006)

RSHs for Combinatorial Optimisation

- Analysis of runtime and approximation quality on well-known combinatorial optimisation problems, e. g.,
  - sorting problems (is this an optimisation problem?),
  - shortest path problems,
  - subsequence problems,
  - vertex cover,
  - Eulerian cycles,
  - minimum spanning trees,
  - maximum matchings,
  - partition problem,
  - set cover problem,
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- What we do not hope: to be better than the best problem-specific algorithms
- In the following no fine-tuning of the results

Eulerian Cycle Problem

Given: Undirected connected Eulerian (degree of each vertex is even) graph $G = (V, E)$ with n vertices and m edges

Find: A Cycle (permutation of the edges) such that each edge is used exactly once.

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Eulerian Cycle (Hierholzer)

1. Find a cycle C in G
2. Delete the edges of C from G
3. If G is not empty go to step 1.
4. Construct the Eulerian cycle from the cycles produced in Step 1.

Fitness Function

Representation: permutation of edges

Fitness function

Consider the edges of the permutation after another and build up a path p of length l.

$$path(\pi) := \text{ length of the path } p \text{ implied by } \pi$$

Example: $\pi = (\{2,3\}, \{1,2\}, \{1,5\}, \{3,4\}, \{4,5\}) \Longrightarrow |p| = 3$

The Algorithms

(1+1) EA

1. Choose $\pi \in S_m$ uniform at random.
2. Choose s according to a Poisson distribution with parameter $\lambda = 1$. Perform sequentially $s+1$ jump operations to produce π' from π.

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   ({2,3},{1,2},{3,4},{1,5},{4,5}) produces
   ({2,3},{3,4},{1,5},{1,2},{4,5})
3. Replace π by π' if $path(\pi') \geq path(\pi)$.
4. Repeat Steps 2 and 3 forever.

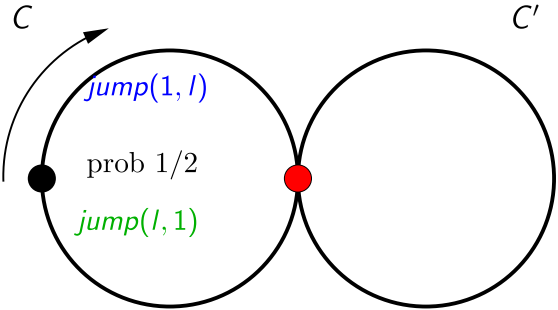
Upper Bound, (1+1) EA

Theorem (Neumann, 2007)

The expected time until (1+1) EA working on the fitness function path constructs an Eulerian cycle is bounded by $O(m^5)$.

Proof outline:

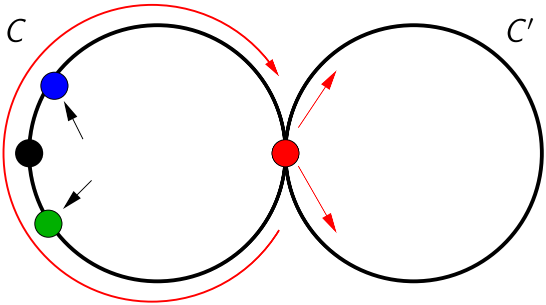
- p is not a cycle:
  1 improving jump $\Longrightarrow$ expected time for an improvement is $O(m^2)$
- p is a cycle:
  Show: Expected time for an improvement is bounded by $O(m^4)$
- $O(m)$ improvements $\Longrightarrow$ theorem



Typical run:

- k-step (accepted mutation with k-jumps that change p)
- Only 1-steps: $O(m^4)$ steps for an improvement
- No k-step, $k \geq 4$, in $O(m^4)$ steps with prob. $1 - o(1)$
- $O(1)$ 2- or 3-steps in $O(m^4)$ steps with prob. $1 - o(1)$

1-Steps



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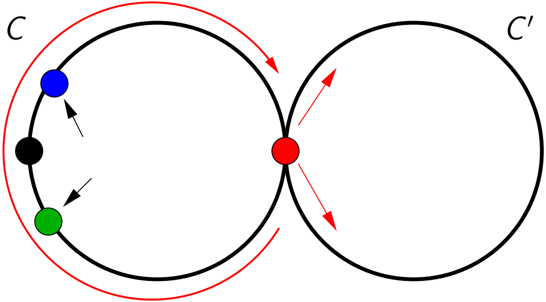

- time $O(m^2)$ to move black vertex
- black performs random walk
- length of cycle is at most m.
- fair random walk $\Longrightarrow O(m^2)$ movements are enough to reach red vertex
- expected time for an improvement $O(m^4)$

Further Results

- lower bound $\Omega(m^4)$

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- restricted jumps (always jump to position 1)
  - no random walk, but directed walk
  - upper bound $O(m^3)$ (Doerr/Hebbinghaus/Neumann, 2007)

Further Results

- lower bound $\Omega(m^4)$
- restricted jumps (always jump to position 1)
  - no random walk, but directed walk
  - upper bound $O(m^3)$ (Doerr/Hebbinghaus/Neumann, 2007)
- use of more sophisticated representations and mutation operators:
  - $O(m^2 \log m)$ (Doerr/Klein/Storch, 2007)
  - $O(m \log m)$ (Doerr/Johannsen, 2007)

This Tutorial

Minimum Spanning Trees

Problem

Given: Undirected connected graph $G = (V, E)$ with n vertices and m edges with positive integer weights.
Find: Edge set $E' \subseteq E$ with minimal weight connecting all vertices.

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Fitness function

Decrease number of connected components, find minimum spanning tree:

$$f(s) := (c(s), w(s)).$$

Minimization of f with respect to the lexicographic order.

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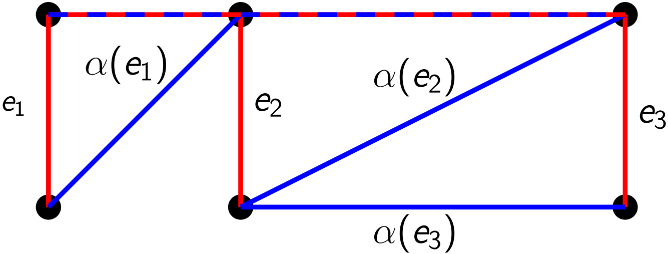
Connected graph

- Connected graph in expected time $O(m \log n)$ (fitness level arguments)

Bijection (Mayr/Plaxton, 1992)



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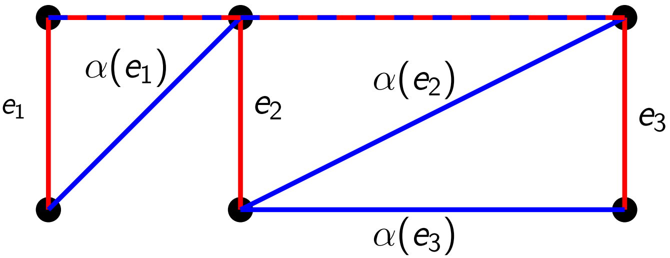

- $k := |E(T^*) \setminus E(T)|$
- Bijection $\alpha : E(T^*) \setminus E(T) \to E(T) \setminus E(T^*)$
- $\alpha(e_i)$ on the cycle of $E(T) \cup \{e_i\}$
- $w(e_i) \leq w(\alpha(e_i))$

$\Longrightarrow$ k accepted 2-bit flips that turn T into T^*

Upper Bound

Theorem (Neumann/Wegener, 2007)

The expected time until (1+1) EA constructs a minimum spanning tree is bounded by $O(m^2(\log n + \log w_{\max}))$.

Sketch of proof:

- $w(s)$ weight current solution s
- w_{opt} weight minimum spanning tree T^*

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Sketch of proof:

- $w(s)$ weight current solution s
- w_{opt} weight minimum spanning tree T^*
- set of $m+1$ operations to reach T^*
  - $m' = m - (n-1)$ 1-bit flips concerning non-T^* edges $\Longrightarrow$ spanning tree T
  - k 2-bit flips defined by bijection
  - $n-k$ non accepted 2-bit flips
- $\Longrightarrow$ average weight decrease $(w(s) - w_{opt})/(m+1)$

Upper Bound

- 1-step (larger total weight decrease of 1-bit flips)
- 2-step (larger total weight decrease of 2-bit flips)

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- 2-step (larger total weight decrease of 2-bit flips)

Consider 2-steps:

- Expected weight decrease by a factor $1 - (1/(2n))$
- Probability $\Theta(n/m^2)$ for a good 2-bit flip
- Expected time until r 2-steps $O(rm^2/n)$

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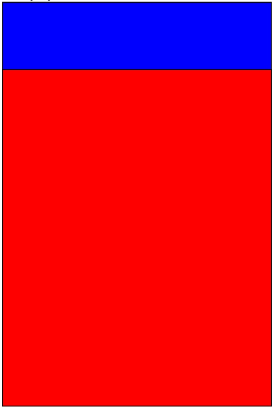
1-steps faster $\Longrightarrow$ show bound for 2-steps.

Expected Number of 2-Steps

$w(s) \leq D := m \cdot w_{max}$



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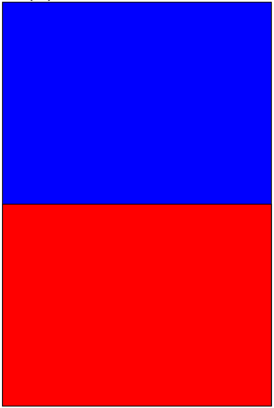

$(1 - \frac{1}{2n})(w(s) - w_{opt})$

$(1 - \frac{1}{2n})^N(w(s) - w_{opt})$

w_{opt}

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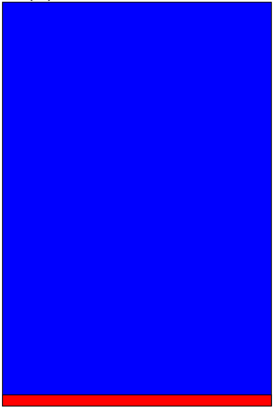


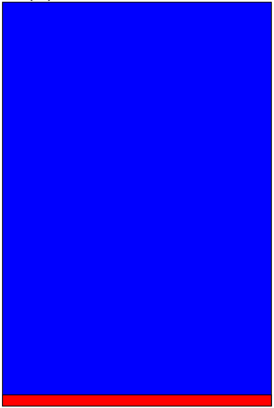
$(1 - \frac{1}{2n})(w(s) - w_{opt})$

$(1 - \frac{1}{2n})^N(w(s) - w_{opt})$

$N := \lceil 2 \cdot (\ln 2) \cdot n \cdot (\log D + 1) \rceil$

$\Longrightarrow (1 - \frac{1}{2n})^N(w(s) - w_{opt}) \leq 1/2$

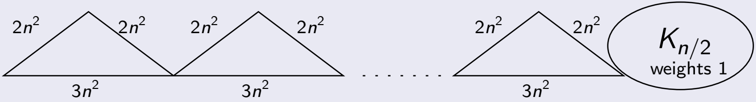
w_{opt}



- Expected number of 2-steps $2N = O(n(\log n + \log w_{max}))$ (Markov)
- Expected time $O(Nm^2/n) = O(m^2(\log n + \log w_{\max}))$.

Further Results

Lower Bound $\Omega(n^4 \log n)$



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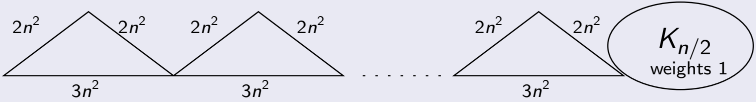

Related Results

- Experimental investigations (Briest et al., 2004)
- Biased mutation operators (Raidl/Koller/Julstrom, 2006)
- $O(mn^2)$ for a multi-objective approach (Neumann/Wegener, 2006)
- Approximations for multi-objective minimum spanning trees (Neumann, 2007)
- SA/MA/ACO and minimum spanning trees (Later!)

This Tutorial

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA
2. Combinatorial optimisation problems
   - (1+1) EA and Eulerian cycles
   - (1+1) EA and minimum spanning trees
   - (1+1) EA and maximum matchings
   - (1+1) EA and the partition problem
   - Multi-objective optimisation and the set cover problem
   - SA beats MA in combinatorial optimisation
   - ACO and minimum spanning trees
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$n+1$ nodes, n edges: bit string from $\{0,1\}^n$ selects edges

Fitness function: size of matching/negative for non-matchings

$n + 1$ nodes, n edges: bit string from $\{0, 1\}^n$ selects edges

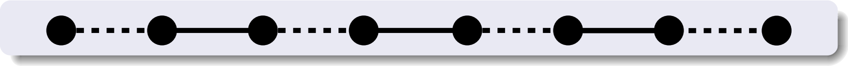
Fitness function: size of matching/negative for non-matchings

Theorem (Giel/Wegener, 2003)

The expected time until the (1+1) EA finds a maximum matching on a path of n edges is $O(n^4)$.

Proof idea:

- Consider a second-best matching.
- Is there a free edge? Flip one bit! $\rightarrow$ probability $\Theta(1/n)$.
- Else 2-bit flips $\rightarrow$ probability $\Theta(1/n^2)$.



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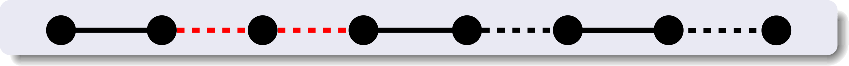

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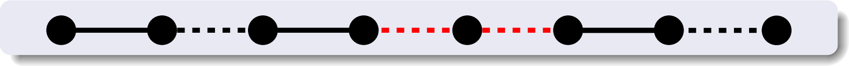

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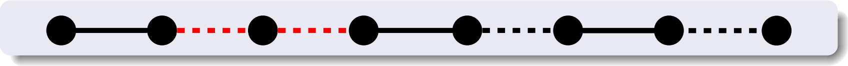

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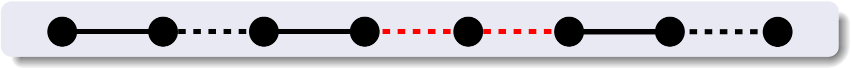

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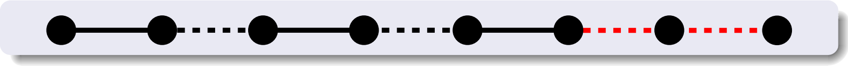
- Then flip the free edge!



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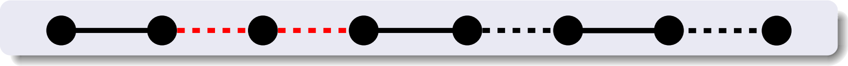

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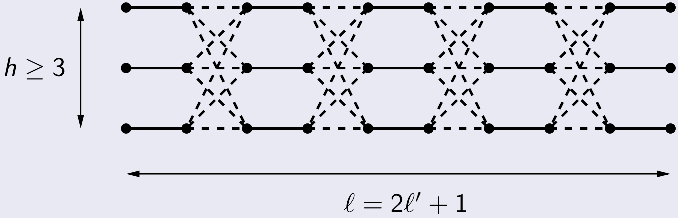
- Length changes according to a fair random walk (Gambler's Ruin Problem)
  $\rightarrow$ Expected runtime $O(n^2) \cdot O(n^2) = O(n^4)$.

Worst-case graph (Sasaki/Hajek, 1988)



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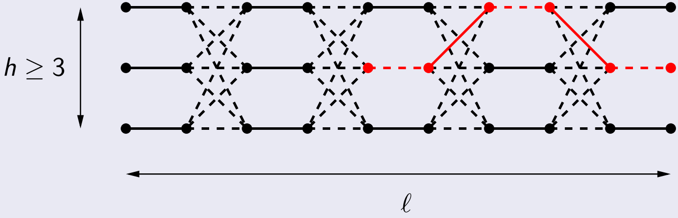
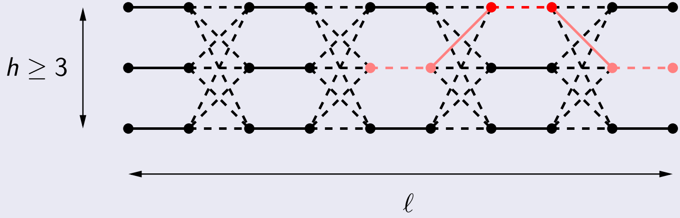


Augmenting path

Worst-case graph (Sasaki/Hajek, 1988)



Augmenting path can get shorter

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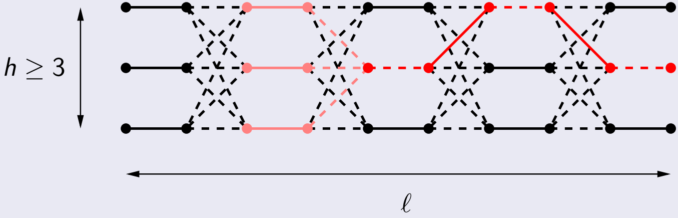

Augmenting path can get shorter but is more likely to get longer.

Theorem

For $h \geq 3$, the (1+1) EA has exponential expected runtime $2^{\Omega(\ell)}$ on $G_{h,\ell}$.

Proof by drift analysis

Insight: do not hope for exact solutions but for approximations

Theorem (Giel/Wegener, 2003)

For $\varepsilon > 0$, the (1+1) EA finds a $(1+\varepsilon)$-approximation of a maximum matching in expected time $O(m^{2\lceil 1/\varepsilon \rceil})$ and is a polynomial-time randomised approximation scheme (PRAS).

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Proof idea:

- Look into the analysis of the Hopcroft/Karp algorithm.
- Current solution worse than $(1+\varepsilon)$-approximate $\rightarrow$ many augmenting paths, in partic. a short one of length $\leq 2\lceil\varepsilon^{-1}\rceil$
- Wait for the (1+1) EA to optimise this short path.

A More General View

Minimum spanning trees and bipartite matching are special cases of matroid optimisation problems.

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Let E be a finite set and $\mathcal{F} \subseteq 2^E$. $M = (E, \mathcal{F})$ is a *matroid* if

(i) $\emptyset \in \mathcal{F}$,
(ii) $\forall X \subseteq Y \in \mathcal{F}\colon X \in \mathcal{F}$, and
(iii) $\forall X, Y \in \mathcal{F}, |X| > |Y|\colon \exists x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.

Adding a function $w\colon E \to \mathbb{N}$ yields a weighted matroid.

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Exemplary Results (Reichel and Skutella, 2007)

The (1+1) EA and RLS solve the matroid optimisation problems

- min. weight basis exactly in time $O(|E|^2(\log|E| + \log w_{\max}))$.
- unweighted intersection up to $1 - \varepsilon$ in time $O(|E|^{2\lceil 1/\varepsilon \rceil})$.

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Very abstract/general, a step towards a characterisation of polynomially solvable problems on which EAs are efficient

This Tutorial

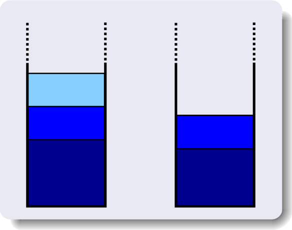
(1+1) EA and the Partition Problem

What about NP-hard problems? $\rightarrow$ Study approximation quality

What about NP-hard problems? → Study approximation quality

For $w_1, \ldots, w_n$, find $I \subseteq \{1, \ldots, n\}$ minimising

$$\max\left\{\sum_{i \in I} w_i, \sum_{i \notin I} w_i\right\}.$$



What about NP-hard problems? → Study approximation quality

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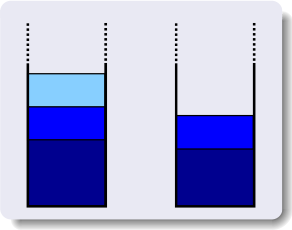

This is an "easy" NP-hard problem:

- not strongly NP-hard,
- FPTAS exist,
- ...

Coding: bit string $\{0,1\}^n$ characteristic vector of I

Fitness function: weight of fuller bin

Theorem (Witt, 2005)

On any instance for the partition problem, the (1+1) EA reaches a solution with approximation ratio $4/3$ *in expected time* $O(n^2)$.

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On any instance for the partition problem, the (1+1) EA reaches a solution with approximation ratio $4/3$ in expected time $O(n^2)$.

Theorem (Witt, 2005)

There is an instance such that the (1+1) EA needs with prob. $\Omega(1)$ at least $n^{\Omega(n)}$ steps to find a solution with a better ratio than $4/3 - \varepsilon$.

Proof ideas: study effect of local steps and local optima

Theorem (Witt, 2005)

On any instance, the (1+1) EA with prob. $\geq 2^{-c\lceil 1/\varepsilon\rceil \ln(1/\varepsilon)}$ finds a $(1+\varepsilon)$-approximation within $O(n \ln(1/\varepsilon))$ steps.

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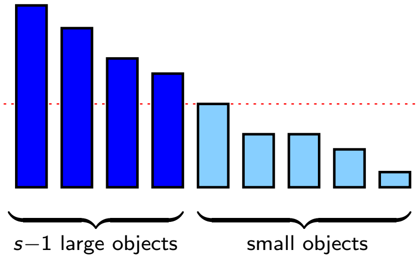
- $2^{O(\lceil 1/\varepsilon\rceil \ln(1/\varepsilon))}$ parallel runs find a $(1+\varepsilon)$-approximation with prob. $\geq 3/4$ in $O(n\ln(1/\varepsilon))$ parallel steps.
- Parallel runs form a PRAS!

Set $s := \left\lceil \frac{2}{\varepsilon} \right\rceil$ and $w := \sum_{i=1}^{n} w_i$.

Assuming $w_1 \geq \cdots \geq w_n$, we have $w_i \leq \varepsilon \frac{w}{2}$ for $i \geq s$.



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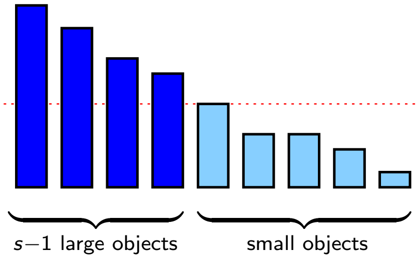

Analyse probability of distributing

- large objects in an optimal way,
- small objects greedily $\Rightarrow$ additive error $\leq \varepsilon w/2$,

This is the algorithmic idea by Graham (1969).

Models: each weight drawn independently at random, namely

1. uniformly from the interval $[0, 1]$,
2. exponentially distributed with parameter 1 (i. e., $\text{Prob}(X \geq t) = e^{-t}$ for $t \geq 0$).

Approximation ratio no longer meaningful, we investigate:
discrepancy = absolute difference between weights of bins.

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How close to discrepancy 0 do we come?

Deterministic, problem-specific heuristic LPT

Sort weights decreasingly,
put every object into currently emptier bin.

Analysis in both random models:

After LPT has been run, additive error is $O((\log n)/n)$ (Frenk/Rinnooy Kan, 1986).

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Can RLS or the (1+1) EA reach a discrepancy of $o(1)$?

Theorem (Witt, 2005)

In both models, the (1+1) EA reaches discrepancy $O((\log n)/n)$ after $O(n^{c+4}\log^2 n)$ steps with probability $1 - O(1/n^c)$.

Almost the same result as for LPT!

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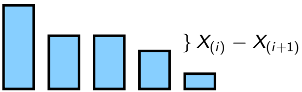
Proof exploits order statistics:

W. h. p.
$X_{(i)} - X_{(i+1)} = O((\log n)/n)$
for $i = \Omega(n)$.



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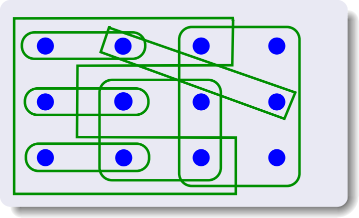
The Set Cover Problem

Another NP-hard problem



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- ground set S,
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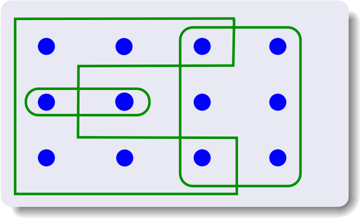

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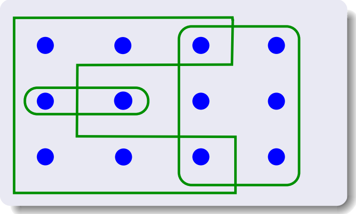

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Traditional single-objective approach

Fitness = cost of selection of subsets, penalty for non-covers

Theorem

There is a Set Cover instance parameterised by $c > 0$ such that RLS and the (1+1) EA for any c need an infinite resp. exponential expected time to obtain a c-approximation.

Multi-objective Optimisation

Fitness $f\colon \{0,1\}^n \to \mathbb{R} \times \mathbb{R}$ has two objectives:

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Simple Evolutionary Multi-objective Optimiser (SEMO)

1. Choose $x \in \{0,1\}^n$ uniformly at random.
2. Determine $f(x)$.
3. $P \leftarrow \{x\}$.
4. Repeat
   - Choose $x \in P$ uniformly at random.
   - Create x' by flipping one randomly chosen bit of x.
   - Determine $f(x')$.
   - If x' is not dominated by any other search point in P, include x' into P and delete all other solutions $z \in P$ with $f(x') \preccurlyeq f(z)$ from P.

Achieving Almost Best-possible Approximations

Theorem (Friedrich, He, Hebbinghaus, Neumann, Witt, 2007)

For any instance of the Set Cover problem, SEMO finds an $(\ln|S| + 1)$*-approximate solution in expected time* $O(n|S|^2 + n|S|(\log n + \log c_{\max}))$*.*

Proof idea:

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It probably cannot be done better in polynomial time.

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Jerrum/Sinclair (1996)

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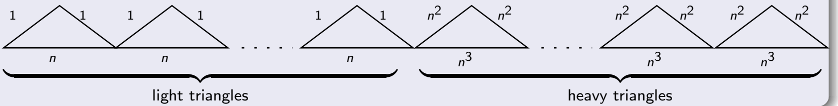
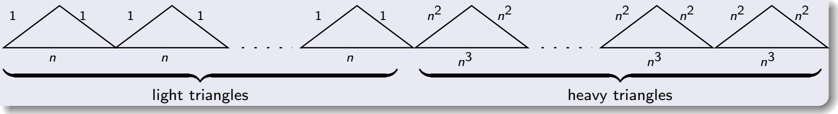
Solution (Wegener, 2005): MSTs are such an example.

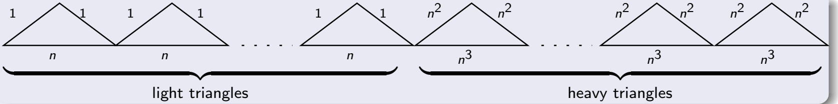
A bad instance for MA





Theorem (Wegener, 2005)

The MA with arbitrary temperature computes the MST for this instance only with probability $e^{-\Omega(n)}$ in polynomial time. SA with temperature $T_t := n^3(1 - \Theta(1/n))^t$ computes the MST in $O(n \log n)$ steps with probability $1 - O(1/poly(n))$.



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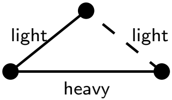
Proof idea: need different temperatures to optimise all triangles.

Concentrate on wrong triangles:
one heavy, one light edge chosen



Simulated Annealing Beats Metropolis in Combinatorial Optimisation

Proof Idea

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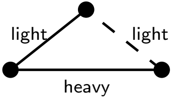

- Soon after initialization $\Omega(n)$ wrong triangles, both in heavy and light part of the graph
- To correct such triangle, light edge must be flipped in.

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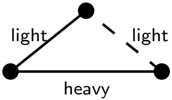

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  $\rightarrow$ need high temperature T^* to correct wrong heavy triangles.

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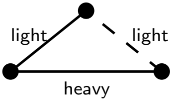

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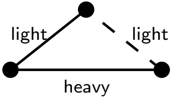

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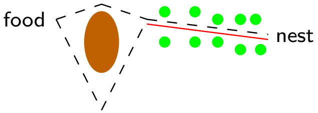
- Light edges of heavy triangles still much heavier than heavy edges of light triangles $\rightarrow$ at temperature T^* almost random search on light triangles $\rightarrow$ many light triangles remain wrong.
- SA first corrects heavy triangles at temperature T^*.
- After temperature has dropped, SA corrects light triangles, without destroying heavy ones.

This Tutorial

1. The origins: example functions and toy problems
   - A simple toy problem: OneMax for (1+1) EA
2. Combinatorial optimisation problems
   - (1+1) EA and Eulerian cycles
   - (1+1) EA and minimum spanning trees
   - (1+1) EA and maximum matchings
   - (1+1) EA and the partition problem
   - Multi-objective optimisation and the set cover problem
   - SA beats MA in combinatorial optimisation
   - ACO and minimum spanning trees
3. End
4. References



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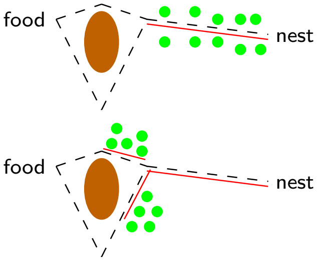

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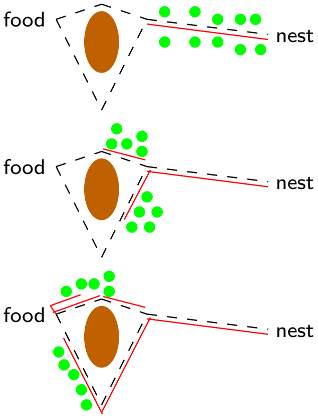

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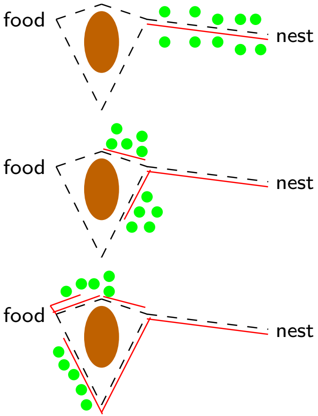

Ant Colony Optimisation (ACO) is yet another biologically inspired search heuristic.

Applications: combinatorial optimisation problems, e. g., TSP

Broder's Algorithm

Problem: Minimum Spanning Trees

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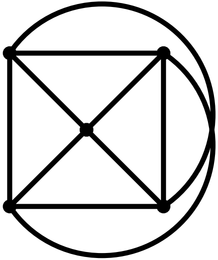
Spanning tree can be chosen uniformly at random using random walk algorithms (e. g. Broder, 1989).



Reward chosen edges ⇒ next solution will be similar to constructed one
But: local improvements are possible

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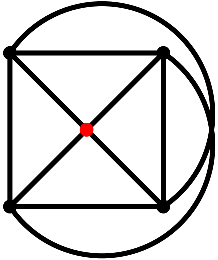

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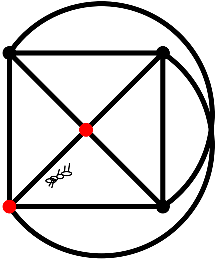

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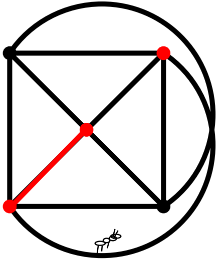

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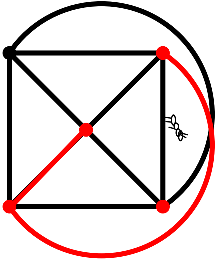

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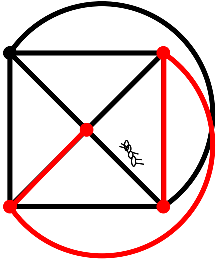

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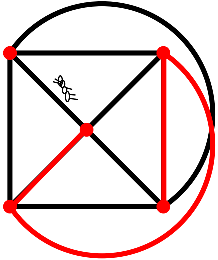

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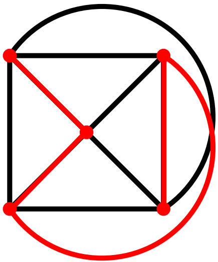

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Component-based Construction Graph

- Vertices correspond to edges of the input graph
- Construction graph $C(G) = (N, A)$ satisfies $N = \{0, \ldots, m\}$ (start vertex 0) and $A = \{(i, j) \mid 0 \leq i \leq m, 1 \leq j \leq m, i \neq j\}$.

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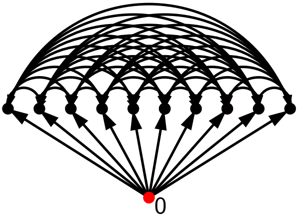

For a given path $v_1, \ldots, v_k$ select the next edge from its neighborhood $N(v_1, \ldots, v_k) :=$ $(E \setminus \{v_1, \ldots, v_k\}) \setminus \{e \in E \mid (V, \{v_1, \ldots, v_k, e\}) \text{ contains a cycle}\}$

(problem-specific aspect of ACO).

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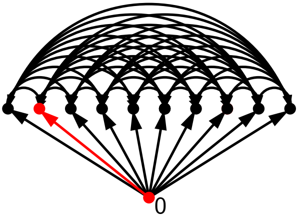

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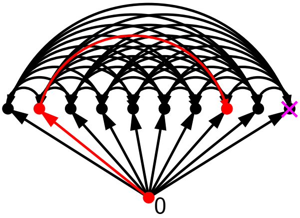

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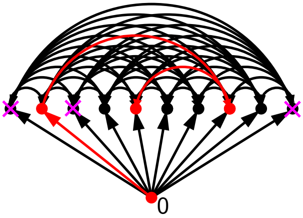

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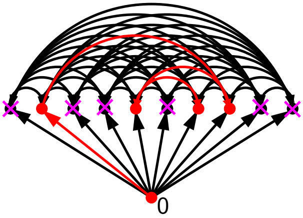

Reward: all edges, that point to visited vertices
(neglect order of chosen edges)

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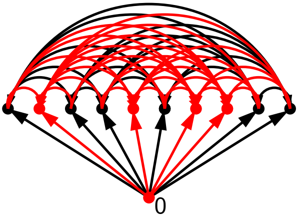

1-ANT:

- two pheromone values
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- Let v_k the current vertex and N_{v_k} be its neighborhood.
- Prob(to choose neighbor y of v_k) $= \frac{[\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}{\sum_{y \in N(v_k)} [\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}$ with $\alpha, \beta \geq 0$.
- Consider special cases where either $\beta = 0$ or $\alpha = 0$.

Results for Pheromone Updates

Case $\alpha = 1$, $\beta = 0$: proportional influence of pheromone values

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Theorem (Broder-based construction graph)

Choosing $h/\ell = n^3$, the expected time until the 1-ANT with the Broder-based construction graph has found an MST is $O(n^6(\log n + \log w_{max}))$.

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Theorem (Component-based construction graph)

Choosing $h/\ell = (m - n + 1)\log n$*, the expected time until the 1-ANT with the component-based construction graph has found an MST is* $O(mn(\log n + \log w_{max}))$.

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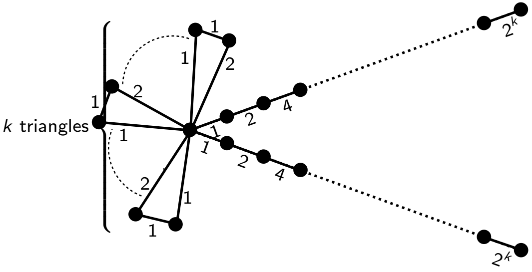
Better than (1+1) EA!

Example graph G^* with $n = 4k + 1$ vertices.

- k triangles of weight profile $(1, 1, 2)$
- two paths of length k with exponentially increasing weights.



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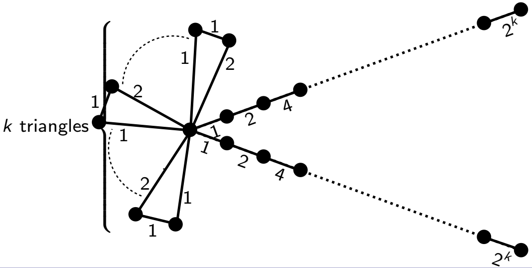

Theorem (Broder-based construction graph)

Let $\alpha = 0$ and β be arbitrary, then the probability that the 1-ANT using the Broder construction procedure does not find an MST in polynomial time with probability $1 - 2^{-\Omega(n)}$.

Component-based Construction Graph/Heuristic Information

Theorem (Component-based construction graph)

Choosing $\alpha = 0$ and $\beta \geq 6w_{\mathrm{max}} \log n$, the expected time of the 1-ANT with the component-based construction graph to find an MST is constant.

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Proof Idea

- Choose edges as Kruskal's algorithm.
- Calculation shows: probability of choosing a lightest edge is at least $(1 - 1/n)$.
- $n - 1$ steps $\Longrightarrow$ probability for an MST is $\Omega(1)$.

Summary and Conclusions

- Analysis of RSHs in combinatorial optimisation
- Starting from toy problems to real problems
- Surprising results
- Interesting techniques
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$\longrightarrow$ The analysis of RSHs is an exciting research direction.

Thank you!

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