

Computational Intelligence

Winter Term 2025/26

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Computational Intelligence

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- Approximate Reasoning
- Fuzzy Control

So far:

- p: IF X is A THEN Y is B

$$\rightarrow R(x, y) = \text{Imp}(A(x), B(y))$$

rule as relation; fuzzy implication

- rule: IF X is A THEN Y is B
fact: X is A'
conclusion: Y is B'

$$\rightarrow B'(y) = \sup_{x \in X} t(A'(x), R(x, y))$$

composition rule of inference

Thus:

- $B'(y) = \sup_{x \in X} t(A'(x), \text{Imp}(A(x), B(y)))$

given : fuzzy rule

input : fuzzy set A'

output : fuzzy set B'

special case:

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases} \quad \boxed{\text{crisp input!}}$$

$$B'(y) = \sup_{x \in X} t(A'(x), \text{Imp}(A(x), B(y)))$$

$$= \begin{cases} \sup_{x \neq x_0} t(0, \text{Imp}(A(x), B(y))) & \text{for } x \neq x_0 \\ t(1, \text{Imp}(A(x_0), B(y))) & \text{for } x = x_0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x \neq x_0 & \text{since } t(0, a) = 0 \\ \text{Imp}(A(x_0), B(y)) & \text{for } x = x_0 & \text{since } t(a, 1) = a \quad [A1] \end{cases}$$

Lemma:

- a) $t(a, 1) = a$
- b) $t(a, b) \leq \min \{ a, b \}$
- c) $t(0, a) = 0$

Proof:

ad a) Identical to axiom 1 of t-norms.

ad b) From monotonicity (axiom 2) follows for $b \leq 1$, that $t(a, b) \leq t(a, 1) = a$.

Commutativity (axiom 3) and monotonicity lead in case of $a \leq 1$ to $t(a, b) = t(b, a) \leq t(b, 1) = b$. Thus, $t(a, b)$ is less than or equal to a as well as b , which in turn implies $t(a, b) \leq \min\{ a, b \}$.

ad c) From b) follows $0 \leq t(0, a) \leq \min \{ 0, a \} = 0$ and therefore $t(0, a) = 0$. ■

by a)



Multiple rules:

IF X is A_1 , THEN Y is B_1

IF X is A_2 , THEN Y is B_2

IF X is A_3 , THEN Y is B_3

...

IF X is A_n , THEN Y is B_n

X is A'

Y is B'

$$\rightarrow R_1(x, y) = \text{Imp}_1(A_1(x), B_1(y))$$

$$\rightarrow R_2(x, y) = \text{Imp}_2(A_2(x), B_2(y))$$

$$\rightarrow R_3(x, y) = \text{Imp}_3(A_3(x), B_3(y))$$

...

$$\rightarrow R_n(x, y) = \text{Imp}_n(A_n(x), B_n(y))$$

Multiple rules for fuzzy input: $A'(x)$ is given

$$B_1'(y) = \sup_{x \in X} t(A'(x), R_1(x, y))$$

...

$$B_n'(y) = \sup_{x \in X} t(A'(x), R_n(x, y))$$

} aggregation of rules or
local inferences necessary!

aggregate! $\Rightarrow B'(y) = \text{aggr}\{B_1'(y), \dots, B_n'(y)\}$, where $\text{aggr} = \begin{cases} \min \\ \max \end{cases}$

Why?

Axioms of Aggregation

[cf. Fung/Fu 1975; quoted from W. Cholewa: Fuzzy Sets & Systems 17:249-258, 1985]

Let $A, A_1, A_2, \dots$ be fuzzy sets over X . The aggregate is denoted by $A_1 \oplus A_2$.

(A1) $\exists$ function $\circ: [0,1] \times [0,1] \rightarrow [0,1]$ with $(A_1 \oplus A_2)(x) = A_1(x) \circ A_2(x) \quad \forall x \in X$

(A2) $\forall A: A \oplus A = A$

(A3) $\forall i, j: A_i \oplus A_j = A_j \oplus A_i$

(A4) For $m \geq 3: A_1 \oplus \dots \oplus A_m = (A_1 \oplus \dots \oplus A_{m-1}) \oplus A_m$

(A5) $\forall i, j, k: A_i \oplus (A_j \oplus A_k) = (A_i \oplus A_j) \oplus A_k$

(A6) Let $A_1 = A \oplus A_3$ and $A_2 = A \oplus A_4$. If $A_3(x) > A_4(x)$ then $A_1(x) > A_2(x) \quad \forall x \in X$

Theorem

If Axioms (A1) – (A6) hold, then only three types of aggregation are possible:

1. $a \circ b = \min(a, b)$
2. $a \circ b = \max(a, b)$
3. $a \circ b = \min(a, b)$ for $a, b \geq \theta$; $= \max(a, b)$ for $a, b \leq \theta$; $= \theta$ otherwise $(0 < \theta < 1)$

FITA: “First inference, then aggregate!”

1. Each rule of the form **IF X is A_k THEN Y is B_k** must be transformed by an appropriate fuzzy implication $\text{Imp}_k(\cdot, \cdot)$ to a relation R_k :
$$R_k(x, y) = \text{Imp}_k(A_k(x), B_k(y)).$$
2. Determine $B'_k(y) = R_k(x, y) \circ A'(x)$ for all $k = 1, \dots, n$ (local inference).
3. Aggregate to $B'(y) = \beta(B'_1(y), \dots, B'_n(y))$.

FATI: “First aggregate, then inference!”

1. Each rule of the form **IF X is A_k THEN Y is B_k** must be transformed by an appropriate fuzzy implication $\text{Imp}_k(\cdot, \cdot)$ to a relation R_k :
$$R_k(x, y) = \text{Imp}_k(A_k(x), B_k(y)).$$
2. Aggregate $R_1, \dots, R_n$ to a **superrelation** with aggregating function $\alpha(\cdot)$:
$$R(x, y) = \alpha(R_1(x, y), \dots, R_n(x, y)).$$
3. Determine $B'(y) = R(x, y) \circ A'(x)$ w.r.t. superrelation (inference).

1. Which principle is better? FITA or FATI?

2. Equivalence of FITA and FATI ?

FITA:

$$\begin{aligned} B'(y) &= \beta(B_1'(y), \dots, B_n'(y)) \\ &= \beta(R_1(x, y) \circ A'(x), \dots, R_n(x, y) \circ A'(x)) \end{aligned}$$

FATI:

$$\begin{aligned} B'(y) &= R(x, y) \circ A'(x) \\ &= \alpha(R_1(x, y), \dots, R_n(x, y)) \circ A'(x) \end{aligned}$$

→ general case: no further analysis without simplifying assumptions ...

special case:

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

On the equivalence of FITA and FATI:

FITA:

$$\begin{aligned} B'(y) &= \beta(B_1'(y), \dots, B_n'(y)) \\ &= \beta(\text{Imp}_1(A_1(x_0), B_1(y)), \dots, \text{Imp}_n(A_n(x_0), B_n(y))) \end{aligned}$$

FATI:

$$\begin{aligned} B'(y) &= R(x, y) \circ A'(x) \\ &= \sup_{x \in X} t(A'(x), R(x, y)) && \text{(from now: special case)} \\ &= R(x_0, y) \\ &= \alpha(\text{Imp}_1(A_1(x_0), B_1(y)), \dots, \text{Imp}_n(A_n(x_0), B_n(y))) \end{aligned}$$

FATI = FITA if sup-t-composition with same t-norm, $\alpha(\cdot) = \beta(\cdot)$, same $\text{Imp}_i()$, and ...

• AND-connected premises

IF $X_1 = A_{11}$ AND $X_2 = A_{12}$ AND ... AND $X_m = A_{1m}$ THEN $Y = B_1$

...

IF $X_n = A_{n1}$ AND $X_2 = A_{n2}$ AND ... AND $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$$A_k(x_1, \dots, x_m) = \min \{ A_{k1}(x_1), A_{k2}(x_2), \dots, A_{km}(x_m) \} \quad \text{or in general: t-norm}$$

• OR-connected premises

IF $X_1 = A_{11}$ OR $X_2 = A_{12}$ OR ... OR $X_m = A_{1m}$ THEN $Y = B_1$

...

IF $X_n = A_{n1}$ OR $X_2 = A_{n2}$ OR ... OR $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

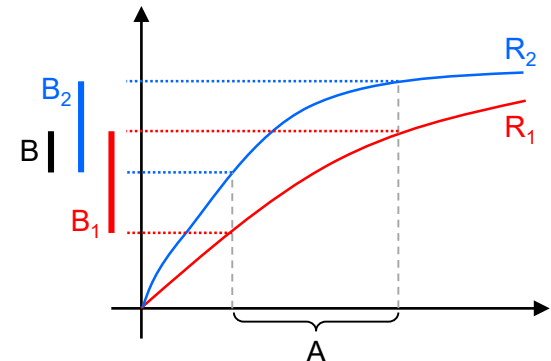
$$A_k(x_1, \dots, x_m) = \max \{ A_{k1}(x_1), A_{k2}(x_2), \dots, A_{km}(x_m) \} \quad \text{or in general: s-norm}$$

important:

- if rules of the form **IF X is A THEN Y is B** interpreted as logical implication
 $\Rightarrow R(x, y) = \text{Imp}(A(x), B(y))$ makes sense
- we obtain: $B'(y) = \sup_{x \in X} t(A'(x), R(x, y))$

interpretation of output set $B'(y)$:

- $B'_k(y)$ is the set of values that are possible under the particular rule k
 - each rule leads to a different restriction of the values that are possible
 - must determine set of values that are possible for **all** rules
- $\Rightarrow$ resulting fuzzy sets $B'_k(y)$ obtained from single rules must be mutually intersected!
- $\Rightarrow$ aggregation via $B'(y) = \min \{ B'_1(y), \dots, B'_n(y) \}$



important:

- if rules of the form **IF X is A THEN Y is B** are not interpreted as logical implications, then the function $\text{Fct}(\bullet)$ in

$$R(x, y) = \text{Fct}(A(x), B(y))$$

can be chosen as required for desired interpretation.

- frequent choice (especially in fuzzy control):

$$- R(x, y) = \min \{ A(x), B(y) \} \quad \text{Mamdani – “implication”}$$

$$- R(x, y) = A(x) \cdot B(y) \quad \text{Larsen – “implication”}$$

⇒ of course, they are no implications but specific t-norms!

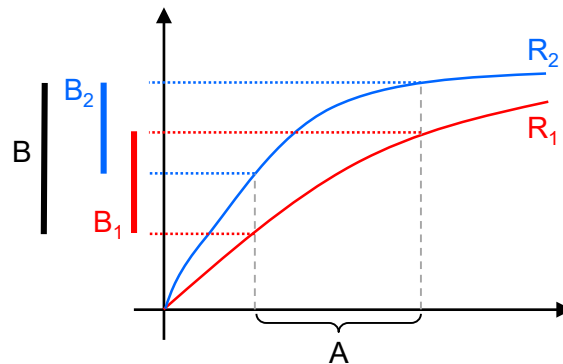
⇒ thus, if relation $R(x, y)$ is given,
then the *composition rule of inference*

$$B'(y) = A'(x) \circ R(x, y) = \sup_{x \in X} \min \{ A'(x), R(x, y) \}$$

still can lead to a conclusion via fuzzy logic.

interpretation of output set $B'(y)$:

- $B'_k(y)$ is the set of values that are possible under the particular rule k
 - technical system must work for all values that are possible
 - each rule may extend the set of the values that are possible
- $\Rightarrow$ resulting fuzzy sets $B'_k(y)$ obtained from single rules must be mutually united!
- $\Rightarrow$ aggregation via $B'(y) = \mathbf{max} \{ B'_1(y), \dots, B'_n(y) \}$



example: [JM96, S. 244ff.]

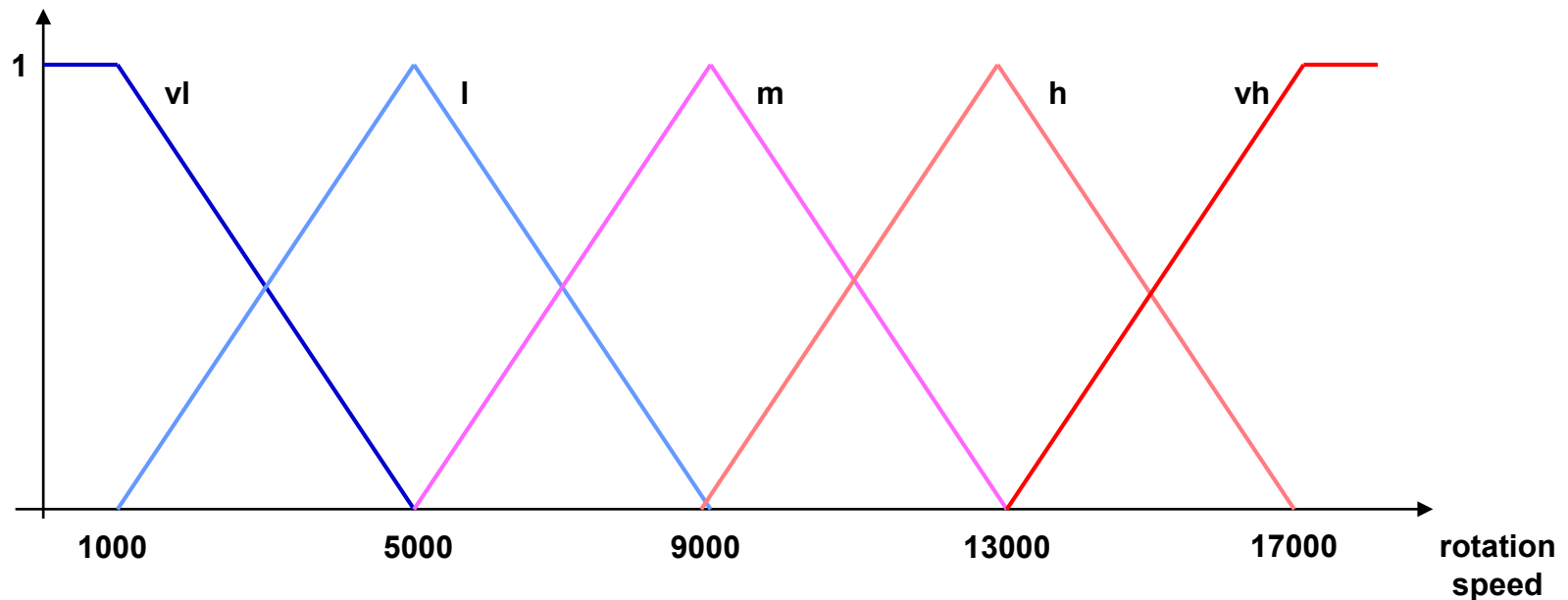
industrial drill machine → control of cooling supply

modelling

linguistic variable : **rotation speed**

linguistic terms : ***very low, low, medium, high, very high***

ground set : X with $0 \leq x \leq 18000$ [rpm]



example: (continued)

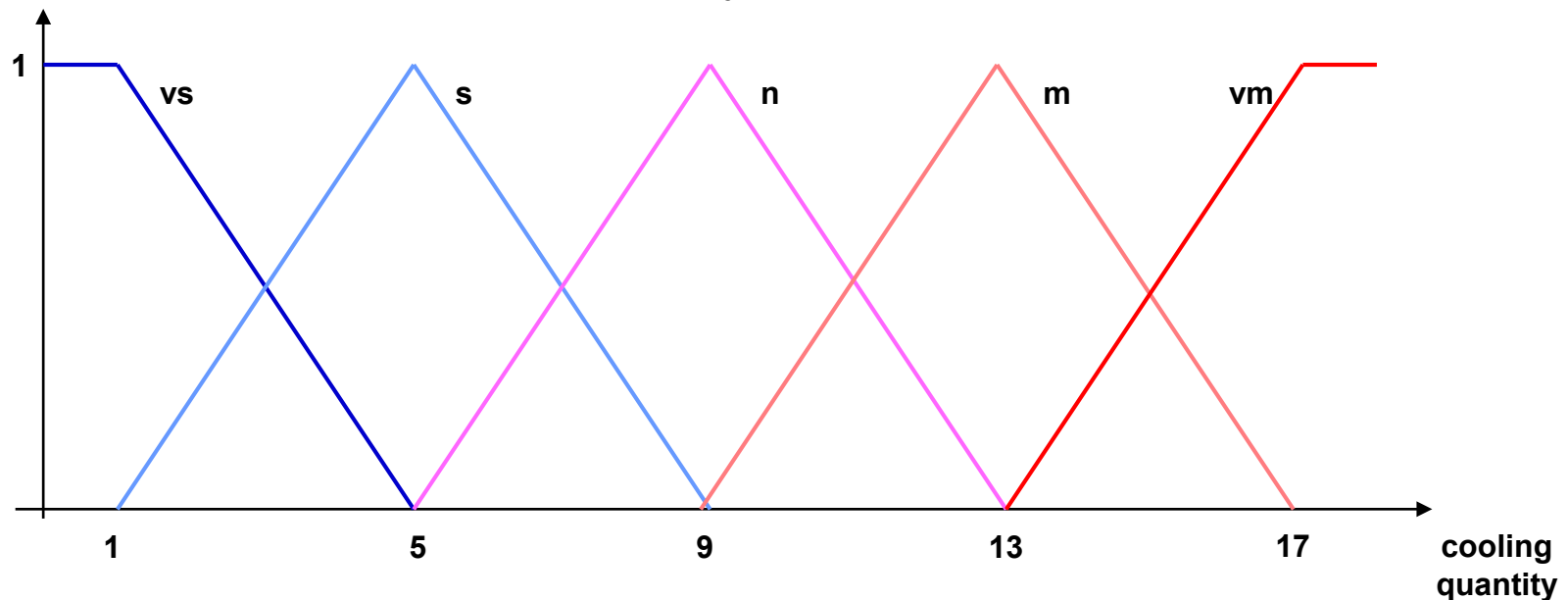
industrial drill machine → control of cooling supply

modelling

linguistic variable : **cooling quantity**

linguistic terms : ***very small, small, normal, much, very much***

ground set : Y with $0 \leq y \leq 18$ [liter / time unit]



example: (continued)

industrial drill machine → control of cooling supply

rule base

IF rotation speed IS *very low* THEN cooling quantity IS *very small*

low

small

medium

normal

high

much

very high

very much



sets $S_{vl}, S_l, S_m, S_h, S_{vh}$

“rotation speed”



sets $C_{vs}, C_s, C_n, C_m, C_{vm}$

“cooling quantity”

example: (continued)

industrial drill machine $\rightarrow$ control of cooling supply

1. **input:** crisp value $x_0 = 10\,000\text{ min}^{-1}$ (not a fuzzy set!)

$\rightarrow$ **fuzzification** = determine membership for each fuzzy set over X

$\rightarrow$ yields $S' = (0, 0, \frac{3}{4}, \frac{1}{4}, 0)$ via $x \mapsto (S_{vl}(x_0), S_l(x_0), S_m(x_0), S_h(x_0), S_{vh}(x_0))$

2. FITA: local **inference**

$\Rightarrow$ note: $\text{Imp}(0, a) = 1$ (axiom 3)

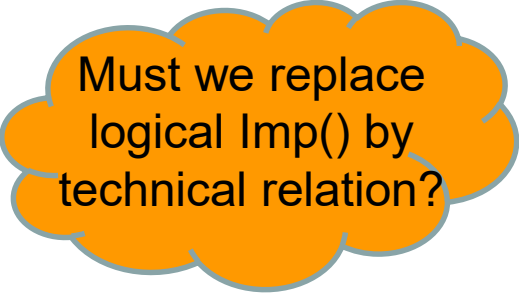
$$S_{vl}: C'_{vs}(y) = \text{Imp}(0, C_{vs}(y))$$

$$S_l: C'_s(y) = \text{Imp}(0, C_s(y))$$

$$S_m: C'_n(y) = \text{Imp}(\frac{3}{4}, C_n(y))$$

$$S_h: C'_m(y) = \text{Imp}(\frac{1}{4}, C_m(y))$$

$$S_{vh}: C'_{vm}(y) = \text{Imp}(0, C_{vm}(y))$$



Must we replace
logical $\text{Imp}()$ by
technical relation?

example: (continued)

industrial drill machine → control of cooling supply

in case of control task typically **no logic-based interpretation:**

→ max-aggregation and

→ relation $R(x,y)$ not interpreted as implication.

often: $R(x,y) = \min(A(x), B(y))$ „**Mamdani controller**“

2. FITA: local inference

$$S_{vl}: C'_{vs}(y) = \min(0, C_{vs}(y)) = 0$$

$$S_l: C'_s(y) = \min(0, C_s(y)) = 0$$

$$S_m: C'_n(y) = \min(\frac{3}{4}, C_n(y)) \geq 0$$

$$S_h: C'_m(y) = \min(\frac{1}{4}, C_m(y)) \geq 0$$

$$S_{vh}: C'_{vm}(y) = \min(0, C_{vm}(y)) = 0$$

⇒ since $\min(0,a) = 0$ and max-aggr.
we only need to consider C_n and C_m

example: (continued)

industrial drill machine → control of cooling supply

3. aggregation:

$$C'(y) = \mathbf{aggr} \{ C'_n(y), C'_m(y) \} = \max \{ \min(\frac{3}{4}, C_n(y)), \min(\frac{1}{4}, C_m(y)) \}$$

Remark:

This approach can be applied with every t-norm and max-aggregation

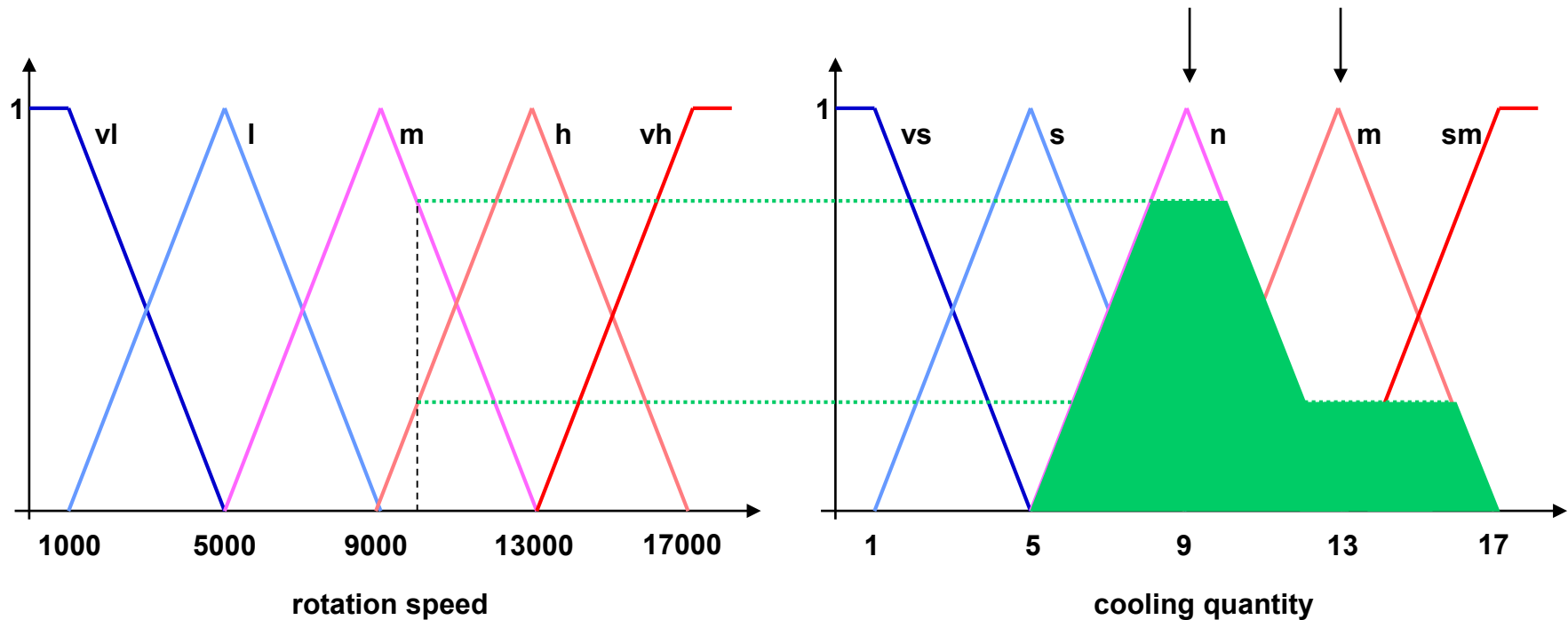
$$\Rightarrow C'(y) = \max \{ t(\frac{3}{4}, C_n(y)), t(\frac{1}{4}, C_m(y)) \}$$

→ graphical illustration

example: (continued)

industrial drill machine → control of cooling supply

$$C'(y) = \max \{ \min \{ \frac{3}{4}, C_n(y) \}, \min \{ \frac{1}{4}, C_m(y) \} \}, x_0 = 10\,000 \text{ [rpm]}$$



open and closed loop control:

affect the dynamical behavior of a system
in a desired manner

- **open loop control**

control is aware of reference values and has a model of the system

⇒ control values can be adjusted,

such that process value of system is equal to reference value

problem: noise! ⇒ deviation from reference value not detected

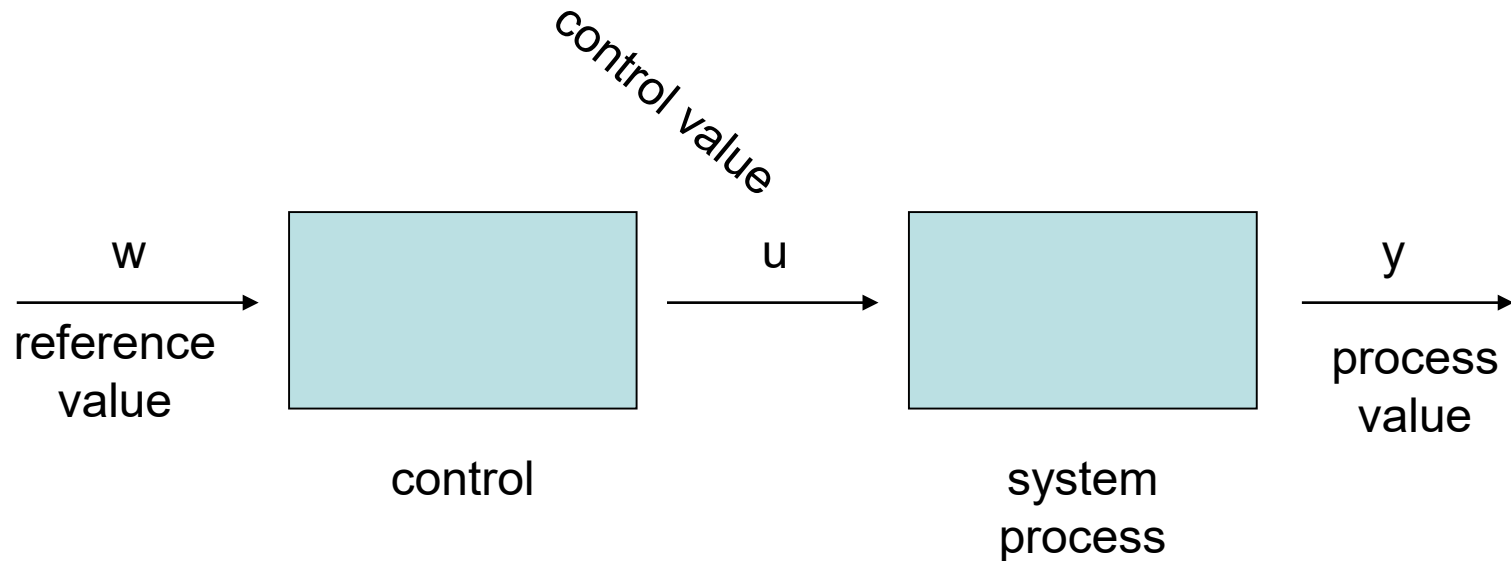
- **closed loop control**

now: detection of deviations from reference value possible

(by means of measurements / sensors)

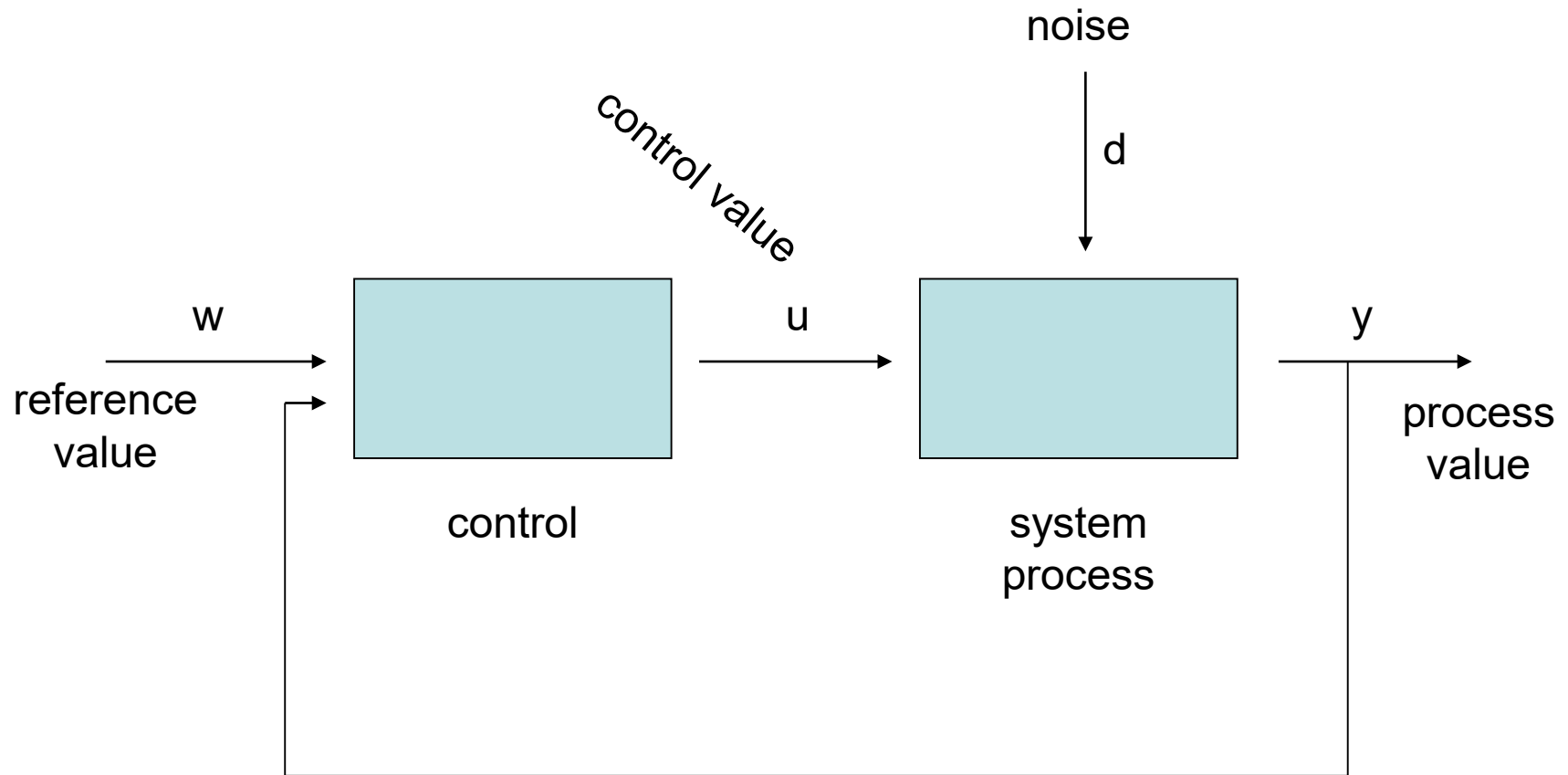
and new control values can take into account the amount of deviation

open loop control



assumption: undisturbed operation $\Rightarrow$ process value = reference value

closed loop control



$$\text{control deviation} = \text{reference value} - \text{process value}$$

required:

model of system / process

→ as differential equations or difference equations (DEs)

→ well developed theory available

so, why fuzzy control?

- if there exists no process model in form of DEs etc.
(operator/human being has realized control manually)
- if process with high-dimensional nonlinearities → no classic methods available
- if control goals are vaguely formulated (e.g. „soft“ changing gears in cars)

fuzzy description of control behavior

IF X is A_1 , THEN Y is B_1

IF X is A_2 , THEN Y is B_2

IF X is A_3 , THEN Y is B_3

...

IF X is A_n , THEN Y is B_n

X is A'

Y is B'

} similar to approximative reasoning

note: fact A' is not a fuzzy set but a crisp input

→ actually, it is the current process value

fuzzy controller executes inference step

→ yields fuzzy output set $B'(y)$

but crisp control value required for the process / system

→ defuzzification (= “condense” fuzzy set to crisp value)

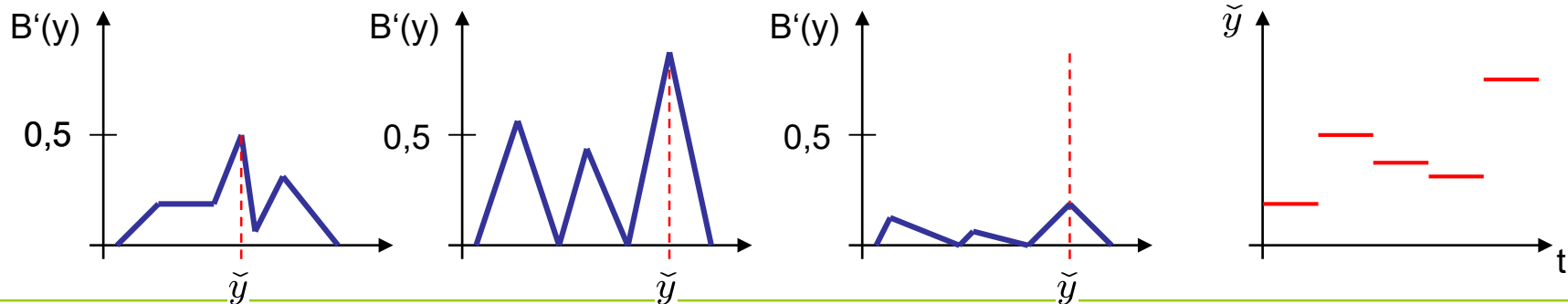
defuzzification

Def: rule k active $\Leftrightarrow A_k(x_0) > 0$

- maximum method

- only active rule with largest activation level is taken into account
 - suitable for pattern recognition / classification
 - decision for a single alternative among finitely many alternatives
- selection independent from activation level of rule (0.05 vs. 0.95)
- if used for control: discontinuous curve of output values (leaps)

$$\tilde{y} = \operatorname{argmax} B'(y)$$

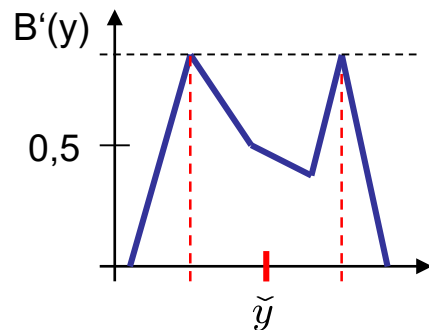


defuzzification

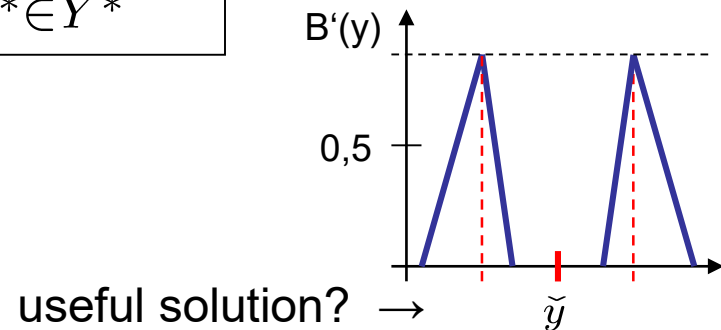
$$Y^* = \{ y \in Y: B'(y) = \text{hgt}(B') \}$$

- maximum mean value method

- all active rules with largest activation level are taken into account
 - interpolations possible, but need not be useful
 - obviously, only useful for neighboring rules with max. activation
- selection independent from activation level of rule (0.05 vs. 0.95)
- if used in control: incontinuous curve of output values (leaps)



$$\tilde{y} = \frac{1}{|Y^*|} \sum_{y^* \in Y^*} y^*$$



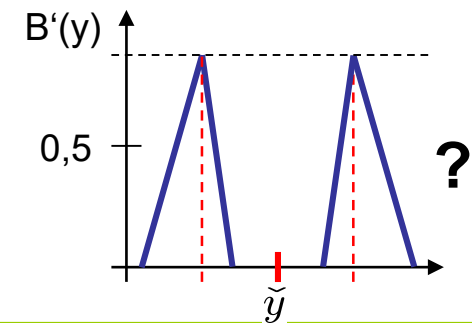
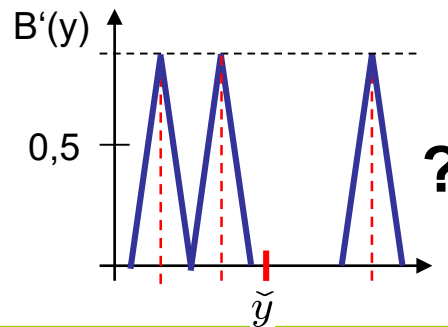
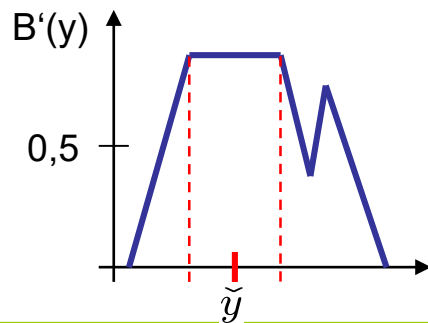
defuzzification

$$Y^* = \{ y \in Y: B'(y) = \text{hgt}(B') \}$$

- center-of-maxima method (COM)

- only **extreme** active rules with largest activation level are taken into account
 - interpolations possible, but need not be useful
 - obviously, only useful for neighboring rules with max. activation level
- selection independent from activation level of rule (0.05 vs. 0.95)
- in case of control: incontinuous curve of output values (leaps)

$$\tilde{y} = \frac{\inf Y^* + \sup Y^*}{2}$$



defuzzification

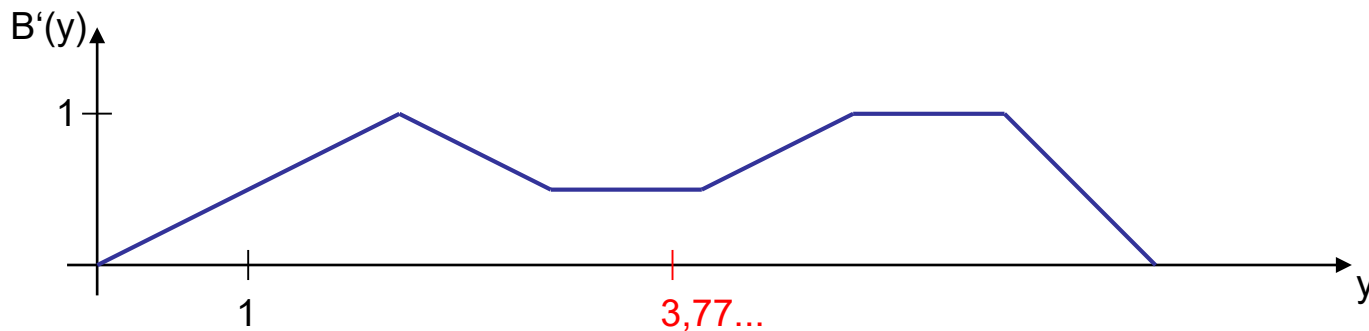
- Center of Gravity (COG)

- all active rules are taken into account
 - but numerically expensiveonly valid for HW solution, today!
 - borders cannot appear in output ($\exists$ work-around)
- if only single active rule: independent from activation level
- continuous curve for output values

$$\tilde{y} = \frac{\int y \cdot B'(y) dy}{\int B'(y) dy}$$

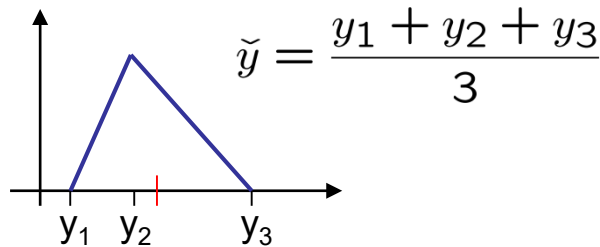
Excursion: COG

$$\tilde{y} = \frac{\int y \cdot B'(y) dy}{\int B'(y) dy}$$

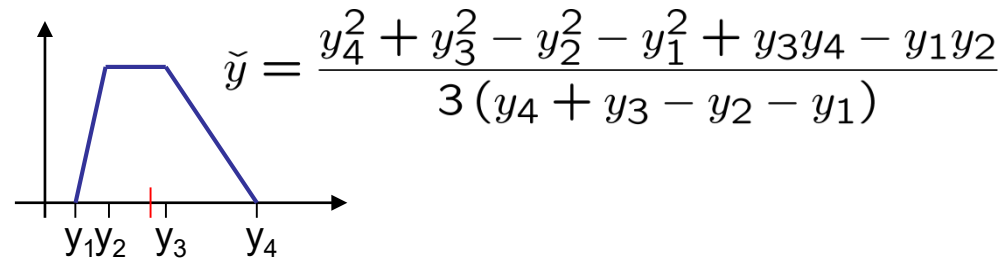


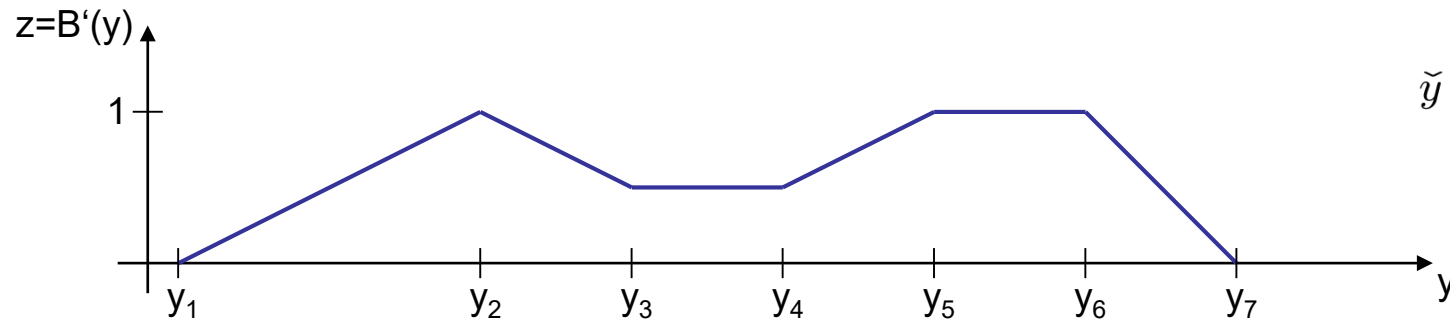
pendant in
probability theory:
expectation value

triangle:



trapezoid:





$$\tilde{y} = \frac{\int y \cdot B'(y) dy}{\int B'(y) dy}$$

assumption: fuzzy membership functions piecewise linear

output set $B'(y)$ represented by sequence of points $(y_1, z_1), (y_2, z_2), \dots, (y_n, z_n)$

$\Rightarrow$ area under $B'(y)$ and weighted area can be determined additively piece by piece

$\Rightarrow$ linear equation $z = m y + b \rightarrow$ insert (y_i, z_i) and (y_{i+1}, z_{i+1})

$\Rightarrow$ yields m and b for each of the $n-1$ linear sections

$$\Rightarrow F_i = \int_{y_i}^{y_{i+1}} (m y + b) dy = \frac{m}{2} (y_{i+1}^2 - y_i^2) + b(y_{i+1} - y_i)$$

$$\Rightarrow G_i = \int_{y_i}^{y_{i+1}} y (m y + b) dy = \frac{m}{3} (y_{i+1}^3 - y_i^3) + \frac{b}{2} (y_{i+1}^2 - y_i^2)$$

$$\tilde{y} = \frac{\sum_i G_i}{\sum_i F_i}$$

Defuzzification

- Center of Area (COA)
 - developed as an approximation of COG
 - let $\hat{y}_k$ be the COGs of the output sets $B'_k(y)$:

$$\tilde{y} = \frac{\sum_k A_k(x_0) \cdot \hat{y}_k}{\sum_k A_k(x_0)}$$

how to:

assume that fuzzy sets $A_k(x)$ and $B_k(x)$ are triangles or trapezoids

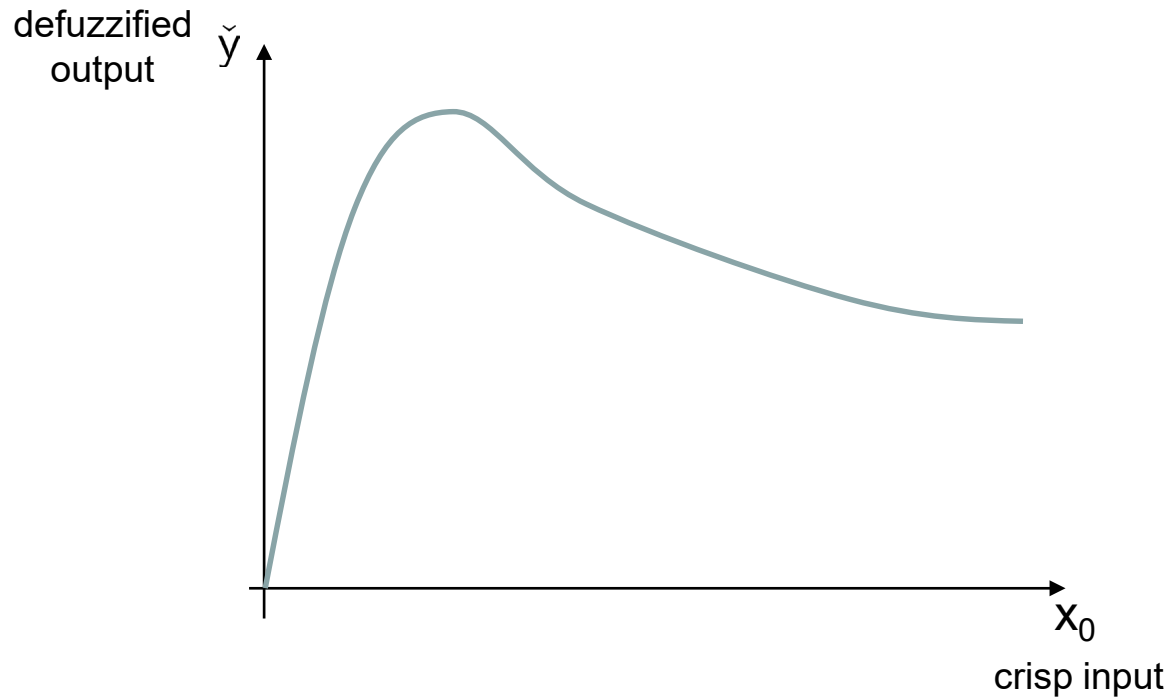
let x_0 be the crisp input value

for each fuzzy rule “IF A_k is X THEN B_k is Y”

determine $B'_k(y) = R(A_k(x_0), B_k(y))$, where $R(.,.)$ is the relation

find $\hat{y}_k$ as center of gravity of $B'_k(y)$

Putting all together:



→ map controller (in german: *Kennfeldregler*)