

Computational Intelligence

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Computational Intelligence

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- Fuzzy relations
- Fuzzy logic
 - Linguistic variables and terms
 - Inference from fuzzy statements

relations with conventional sets $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!

$\Rightarrow$ all set operations remain valid!

crisp membership function (of x to relation R)

$$R(x_1, x_2, \dots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \dots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition

Fuzzy relation = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$ ■

→ each tuple $(x_1, \dots, x_n)$ has a degree of membership to relation

→ degree of membership expresses
strength of relationship between elements of tuple

appropriate representation: n-dimensional membership matrix

example: Let $X = \{ \text{Bejing, New York, Dortmund} \}$ and $Y = \{ \text{New York, Paris} \}$.

relation R = “very far away”

membership matrix →

relation R	New York	Paris
Bejing	1.0	0.9
New York	0.0	0.7
Dortmund	0.6	0.3

Definition

Let $R(X, Y)$ be a fuzzy relation with membership matrix R . The **inverse fuzzy relation** to $R(X, Y)$, denoted $R^{-1}(Y, X)$, is a relation on $Y \times X$ with membership matrix R' . ■

Remark: R' is the transpose of membership matrix R .

Evidently: $(R^{-1})^{-1} = R$ since $(R')' = R$

Definition

Let $P(X, Y)$ and $Q(Y, Z)$ be fuzzy relations. The operation $\circ$ on two relations, denoted $P(X, Y) \circ Q(Y, Z)$, is termed **max-min-composition** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$
 ■

Theorem

- a) max-min composition on relations is associative.
- b) max-min composition on relations is not commutative.
- c) $(P(X,Y) \circ Q(Y,Z))^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y, X)$.

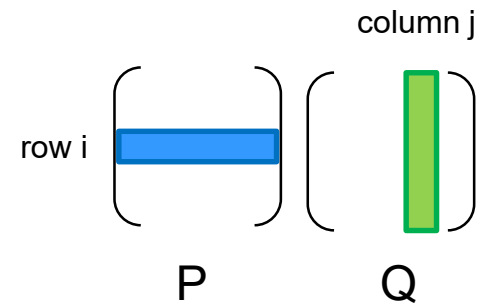
membership matrix of max-min composition
determinable via “fuzzy matrix multiplication”: $R = P \circ Q$

fuzzy matrix multiplication

$$r_{ij} = \max_k \min\{p_{ik}, q_{kj}\}$$

crisp matrix multiplication

$$r_{ij} = \sum_k p_{ik} \cdot q_{kj}$$



further methods for realizing compositions of relations:

max-prod composition

$$(P \odot Q)(x, z) = \max_{y \in \mathcal{Y}} \{P(x, y) \cdot Q(y, z)\}$$

generalization: sup-t composition

$$(P \circ Q)(x, z) = \sup_{y \in \mathcal{Y}} \{t(P(x, y), Q(y, z))\}, \quad \text{where } t(\dots) \text{ is a t-norm}$$

e.g.: $t(a, b) = \min\{a, b\} \Rightarrow \text{max-min-composition}$

$t(a, b) = a \cdot b \Rightarrow \text{max-prod-composition}$

Binary fuzzy relations on $X \times X$: properties

• **reflexive** $\Leftrightarrow \forall x \in X : R(x,x) = 1$

• **irreflexive** $\Leftrightarrow \exists x \in X : R(x,x) < 1$

• **antireflexive** $\Leftrightarrow \forall x \in X : R(x,x) < 1$

• **symmetric** $\Leftrightarrow \forall (x,y) \in X \times X : R(x,y) = R(y,x)$

• **asymmetric** $\Leftrightarrow \exists (x,y) \in X \times X : R(x,y) \neq R(y,x)$

• **antisymmetric** $\Leftrightarrow \forall (x,y) \in X \times X : R(x,y) \neq R(y,x)$

• **transitive** $\Leftrightarrow \forall (x,z) \in X \times X : R(x,z) \geq \max_{y \in X} \min \{ R(x,y), R(y,z) \}$

• **intransitive** $\Leftrightarrow \exists (x,z) \in X \times X : R(x,z) < \max_{y \in X} \min \{ R(x,y), R(y,z) \}$

• **antitransitive** $\Leftrightarrow \forall (x,z) \in X \times X : R(x,z) < \max_{y \in X} \min \{ R(x,y), R(y,z) \}$

actually, here: max-min-transitivity ($\rightarrow$ in general: sup-t-transitivity)

binary fuzzy relation on $X \times X$: example

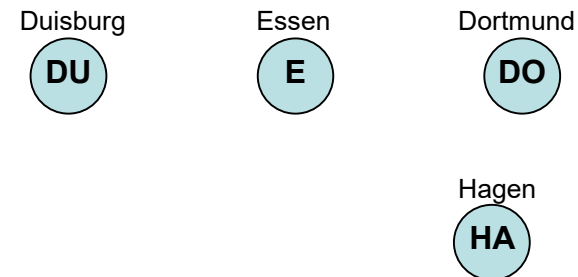
Let X be a subset of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to“.

- $R(x,x) = 1$, since every city is certainly very close to itself.
 $\Rightarrow$ **reflexive**
- $R(x,y) = R(y,x)$: if city x is very close to city y , then also vice versa.
 $\Rightarrow$ **symmetric**

R	DU	E	DO	HA
DU	1	0.7	0.5	0.4
E	0.7	1	0.8	0.8
DO	0.5	0.8	1	0.9
HA	0.4	0.8	0.9	1

$\Rightarrow$ **intransitive**



$$R(DO, DU) = 0.5 < \max_y \min\{R(DO, y), R(y, DU)\} = 0.7$$

$$R(E, DO) = 0.8 \geq \max_y \min\{R(E, y), R(y, DO)\} = 0.8$$

crisp:

relation R is equivalence relation , R reflexive, symmetric, transitive

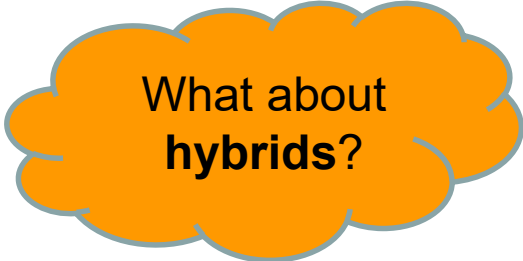
fuzzy:

relation R is similarity relation , R reflexive, symmetric, (max-min-) transitive

examples:

- *equivalence relation*: farm animals
cattle, pigs, chicken, ...
 $R(\text{cow}, \text{ox}) = 1$ but $R(\text{cow}, \text{hen}) = 0$

- *similarity relation*: farm animals
cattle, pigs, chicken, horse, donkey, mule, ...
 $R(\text{mule}, (\text{male}) \text{ donkey}) = 0.5$ and $R(\text{mule}, (\text{female}) \text{ horse}) = 0.5$



What about
hybrids?

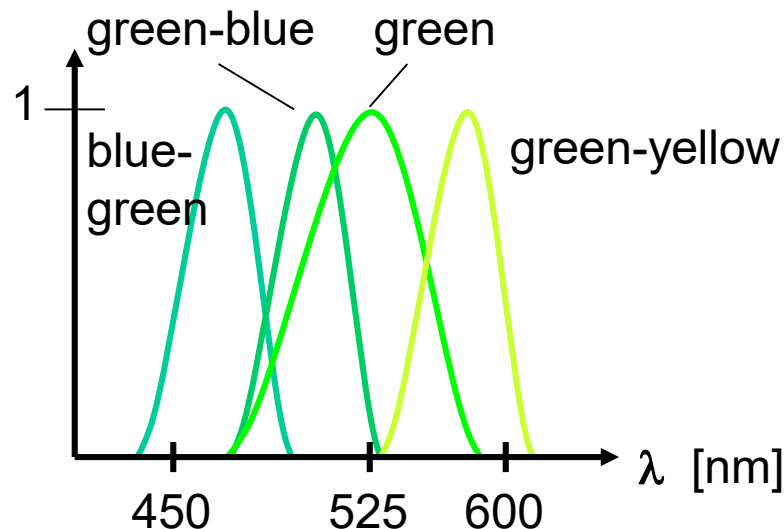
linguistic variable:

variable that can attain several values of linguistic / verbal nature

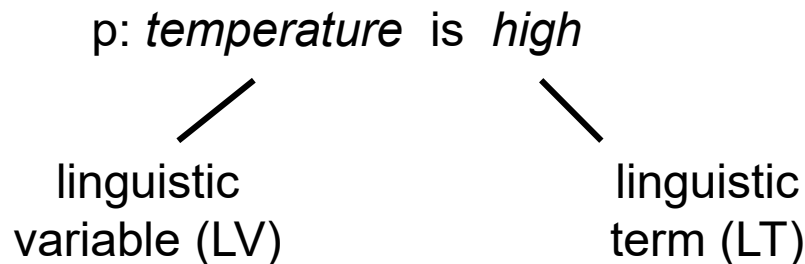
e.g.: **color** can attain values **red**, **green**, **blue**, **yellow**, ...

values (red, green, ...) of linguistic variable are called **linguistic terms**

linguistic terms are associated with fuzzy sets

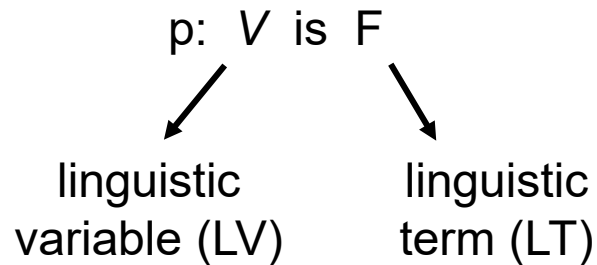


fuzzy proposition



- LV may be associated with several LT : *high, medium, low, ...*
- *high, medium, low* temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given **concrete crisp** temperature value v is interpreted as equal to the degree of membership $high(v)$ of the fuzzy set *high*

fuzzy proposition



actually:

$p: V \text{ is } F(v)$

and

$T(p) = F(v)$ for a concrete crisp value v

↙
trueness(p)

establishes
connection between
degree of membership
of a fuzzy set and the
degree of trueness
of a fuzzy proposition

fuzzy proposition

p: IF *heating* is *hot*, THEN *energy consumption* is *high*

|
LV

|
LT

|
LV

|
LT

expresses relation between

- a) temperature of heating and
- b) quantity of energy consumption

p: (*heating*, *energy consumption*) $\in R$ relation

fuzzy proposition

p: IF X is A, THEN Y is B

LV	LT	LV	LT

How can we determine / express degree of trueness $T(p)$?

- For crisp, given values x, y we know $A(x)$ and $B(y)$
- $A(x)$ and $B(y)$ must be processed to single value via relation R
- $R(x, y) = \text{function}(A(x), B(y))$ is fuzzy set over $X \times Y$
- as before: interpret $T(p)$ as degree of membership $R(x, y)$

fuzzy proposition

p: IF X is A , THEN Y is B

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over $X \times Y$

$\forall (x,y) \in X \times Y: R(x, y) = \text{Imp}(A(x), B(y))$

What is $\text{Imp}(\cdot, \cdot)$?

$\Rightarrow$ „appropriate“ fuzzy implication $[0,1] \times [0,1] \rightarrow [0,1]$

assumption: we know an „appropriate“ $\text{Imp}(a,b)$.

How can we determine the *degree of trueness* $T(p)$?

example: (discrete case)

let $\text{Imp}(a, b) = \min\{ 1, 1 - a + b \}$ and consider fuzzy sets

A:

x_1	x_2
0.5	0.9

B:

y_1	y_2	y_3
0.1	0.8	0.4

$\Rightarrow$

R	y_1	y_2	y_3
x_1	0.6	1.0	0.9
x_2	0.2	0.9	0.5

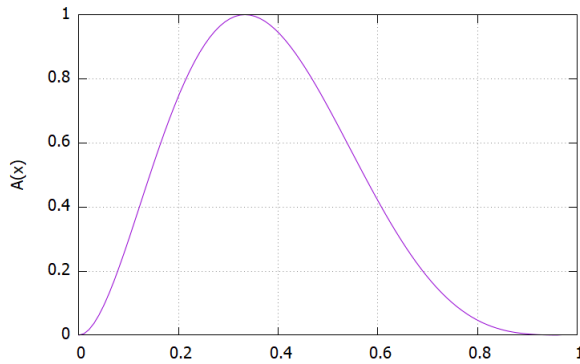
z.B.

$$R(x_2, y_1) = \text{Imp}(A(x_2), B(y_1)) = \text{Imp}(0.9, 0.1) = \min\{1.0, 1.0 - 0.9 + 0.1\} = 0.2$$

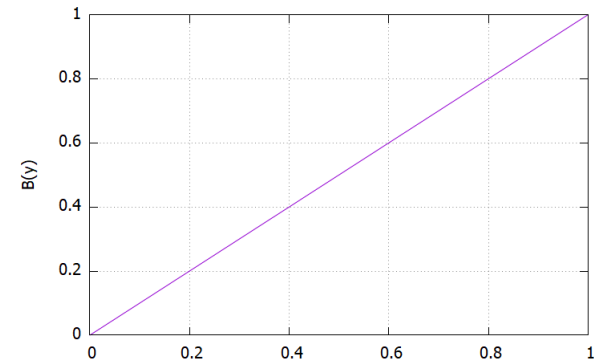
and $T(p)$ for (x_2, y_1) is $R(x_2, y_1) = 0.2$ ■

example: (continuous case)

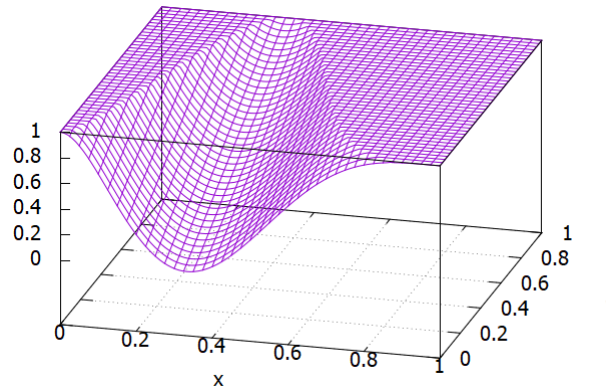
let $\text{Imp}(a, b) = \min\{1, 1 - a + b\}$ and consider fuzzy sets



$$A(x) = \frac{729}{16} x^2 (1 - x)^4 \text{ for } x \in [0, 1]$$



$$B(y) = y \text{ for } y \in [0, 1]$$

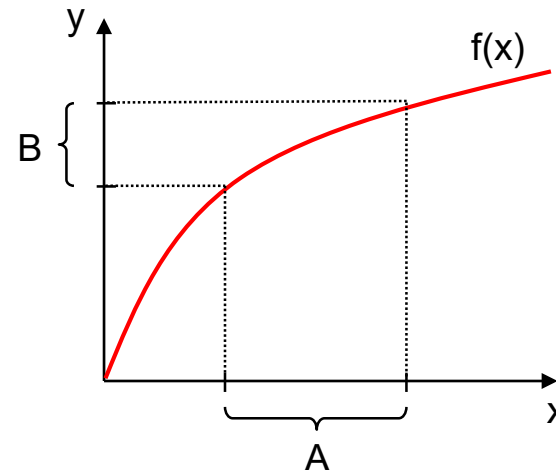
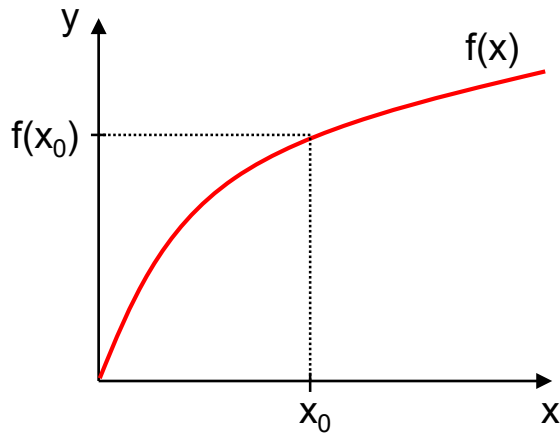


$$\Rightarrow R(x, y) = \min\{1, 1 - A(x) + B(y)\} = \min\{1, 1 - \frac{729}{16} x^2 (1 - x)^4 + y\}$$

toward inference from fuzzy statements:

- let $R = \{ (x, y) : y = f(x) \}$ for a function $f: \mathbb{R} \rightarrow \mathbb{R}$
- IF $X = \{ x_0 \}$ THEN $Y = \{ f(x_0) \}$
- IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y} : y = f(x), x \in A \}$

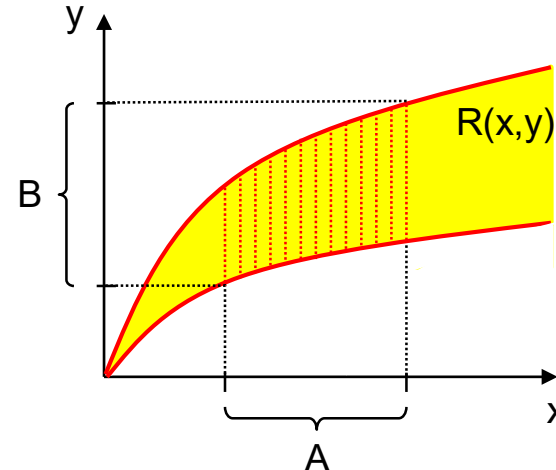
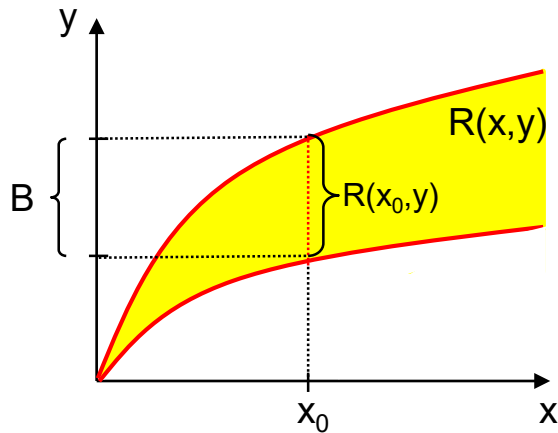
crisp case:
functional
relationship



toward inference from fuzzy statements:

- let relationship between x and y be a relation R on $\mathcal{X} \times \mathcal{Y}$
- IF $X = x_0$ THEN $Y \in B = \{y \in \mathcal{Y} : (x_0, y) \in R\}$
- IF $X \in A$ THEN $Y \in B = \{y \in \mathcal{Y} : (x, y) \in R, x \in A\}$

crisp case:
relational
relationship



toward inference from fuzzy statements:

IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y} : (x, y) \in R, x \in A \}$

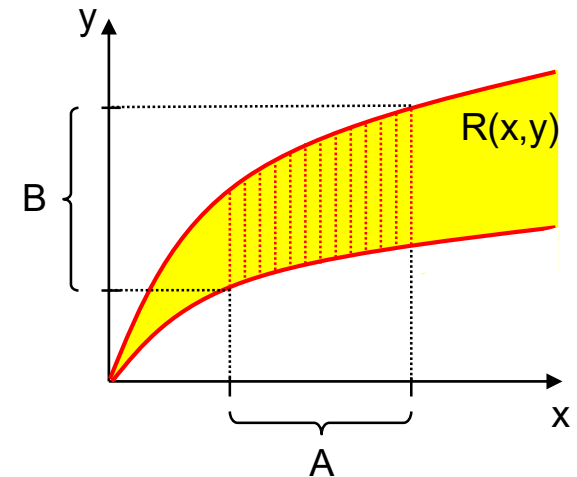
also expressible via characteristic functions of sets A, B, R:

$$B(y) = 1 \text{ iff } \exists x: A(x) = 1 \text{ and } R(x, y) = 1$$

$$\Leftrightarrow \exists x: \min\{ A(x), R(x, y) \} = 1$$

$$\Leftrightarrow \max_{x \in \mathcal{X}} \min\{ A(x), R(x, y) \} = 1$$

$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min\{ A(x), R(x, y) \}$$



inference from fuzzy statements

Now: A' , B' fuzzy sets over $\mathcal{X}$ resp. $\mathcal{Y}$

Assume: $R(x,y)$ and $A'(x)$ are given.

Idea: Generalize characteristic function of $B(y)$ to membership function $B'(y)$

$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$$

characteristic functions

$$\forall y \in \mathcal{Y}: B'(y) = \sup_{x \in \mathcal{X}} \min \{ A'(x), R(x, y) \}$$

membership functions

composition rule of inference (in matrix form): $B^T = A \circ R$

Note:

$A'(x)$ is **not** the derivative of $A(x)$!
It is the membership function
of fuzzy set A' .

inference from fuzzy statements

- conventional:
modus ponens

$$\begin{array}{l} a \Rightarrow b \\ a \\ \hline b \end{array}$$

- fuzzy:
generalized modus ponens (GMP)

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ X \text{ is } A' \\ \hline Y \text{ is } B' \end{array}$$

e.g.: IF *heating* is hot, THEN *energy consumption* is high
 heating is warm

 energy consumption is normal

example: GMP

consider

A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

$\Rightarrow$

R	y_1	y_2
x_1	1.0	0.9
x_2	1.0	0.4
x_3	1.0	0.8

with the rule: IF X is A THEN Y is B

using

$$\text{Imp}(a,b) = \min\{1, 1-a+b\}$$

given fact

A':

x_1	x_2	x_3
0.6	0.9	0.7

thus: $A' \circ R = B'$

with max-min-composition

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

■

inference from fuzzy statements

- conventional:
modus tollens

$$\begin{array}{r} a \Rightarrow b \\ \overline{b} \\ \hline \overline{a} \end{array}$$

- fuzzy:
generalized modus tollens (GMT)

$$\begin{array}{r} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ Y \text{ is } B' \\ \hline X \text{ is } A' \end{array}$$

e.g.: IF *heating* is hot, THEN *energy consumption* is high
 energy consumption is normal
 heating is warm

example: GMT

consider

A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

with the rule: IF X is A THEN Y is B

using $\text{Imp}(a,b) = \min\{1, 1-a+b\}$



given fact

B':

y_1	y_2
0.9	0.7

R'	x_1	x_2	x_3
y_1	1.0	1.0	1.0
y_2	0.9	0.4	0.8

$$\text{thus: } B' \circ R^{-1} = A' \quad \left(\begin{array}{cc} 0.9 & 0.7 \end{array} \right) \circ \left(\begin{array}{ccc} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{array} \right) = \left(\begin{array}{ccc} 0.9 & 0.9 & 0.9 \end{array} \right)$$

with max-min-composition

inference from fuzzy statements

- conventional:
hypothetic syllogism

$$\begin{array}{l} a \Rightarrow b \\ b \Rightarrow c \\ \hline a \Rightarrow c \end{array}$$

- fuzzy:
generalized HS

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ \text{IF } Y \text{ is } B, \text{ THEN } Z \text{ is } C \\ \hline \text{IF } X \text{ is } A, \text{ THEN } Z \text{ is } C \end{array}$$

e.g.: IF *heating* is hot, THEN *energy consumption* is high
 IF *energy consumption* is high, THEN *living* is expensive

 IF *heating* is hot, THEN *living* is expensive

example: GHS

let fuzzy sets $A(x)$, $B(y)$, $C(z)$ be given

$\Rightarrow$ determine the three relations according to Imp function

$$R_1(x,y) = \text{Imp}(A(x),B(y))$$

$$R_2(y,z) = \text{Imp}(B(y),C(z))$$

$$R_3(x,z) = \text{Imp}(A(x),C(z))$$

and express them as matrices R_1 , R_2 , R_3

We say:

GHS is valid if $R_1 \circ R_2 = R_3$

So, ... what makes sense for $\text{Imp}(\cdot, \cdot)$?

$\text{Imp}(a, b)$ ought to express fuzzy version of implication ($a \Rightarrow b$)

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b$

But how can we calculate with fuzzy “boolean” expressions?

request: must be compatible to crisp version (and more) for $a, b \in \{0, 1\}$

a	b	$a \wedge b$	$t(a, b)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

a	b	$a \vee b$	$s(a, b)$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

a	$\bar{a}$	$c(a)$
0	1	1
1	0	0

So, ... what makes sense for $\text{Imp}(\cdot, \cdot)$?

1st approach: S implications

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b$

fuzzy: $\text{Imp}(a, b) = s(c(a), b)$

2nd approach: R implications

conventional: $a \Rightarrow b$ identical to $\max\{x \in \{0, 1\} : a \wedge x \leq b\}$

fuzzy: $\text{Imp}(a, b) = \max\{x \in [0, 1] : t(a, x) \leq b\}$

3rd approach: QL implications

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b \equiv \bar{a} \vee (a \wedge b)$ law of absorption

fuzzy: $\text{Imp}(a, b) = s(c(a), t(a, b))$ (dual tripel ?)

example: S implication

$$\text{Imp}(a, b) = s(c_s(a), b) \quad (c_s : \text{std. complement})$$

1. Kleene-Dienes implication

$$s(a, b) = \max\{a, b\} \quad (\text{standard}) \quad \text{Imp}(a, b) = \max\{1-a, b\}$$

2. Reichenbach implication

$$s(a, b) = a + b - ab \quad (\text{algebraic sum}) \quad \text{Imp}(a, b) = 1 - a + ab$$

3. Łukasiewicz implication

$$s(a, b) = \min\{1, a + b\} \quad (\text{bounded sum}) \quad \text{Imp}(a, b) = \min\{1, 1 - a + b\}$$

example: R implicationen

$$\text{Imp}(a, b) = \max\{ x \in [0, 1] : t(a, x) \leq b \}$$

1. Gödel implication

$$t(a, b) = \min\{ a, b \} \quad (\text{std.})$$

$$\text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ b & , \text{ else} \end{cases}$$

2. Goguen implication

$$t(a, b) = ab \quad (\text{algeb. product})$$

$$\text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ \frac{b}{a} & , \text{ else} \end{cases}$$

3. Łukasiewicz implication

$$t(a, b) = \max\{ 0, a + b - 1 \} \quad (\text{bounded diff.})$$

$$\text{Imp}(a, b) = \min\{ 1, 1 - a + b \}$$

example: QL implication

$$\text{Imp}(a, b) = s(c(a), t(a, b))$$

1. Zadeh implication

$$t(a, b) = \min \{ a, b \} \quad (\text{std.})$$

$$s(a, b) = \max \{ a, b \} \quad (\text{std.})$$

$$\text{Imp}(a, b) = \max \{ 1 - a, \min \{ a, b \} \}$$

2. „NN“ implication ☺ (Klir/Yuan 1994)

$$t(a, b) = ab \quad (\text{algebr. prd.})$$

$$s(a, b) = a + b - ab \quad (\text{algebr. sum})$$

$$\text{Imp}(a, b) = 1 - a + a^2b$$

3. Kleene-Dienes implication

$$t(a, b) = \max \{ 0, a + b - 1 \} \quad (\text{bounded diff.})$$

$$s(a, b) = \min \{ 1, a + b \} \quad (\text{bounded sum})$$

$$\text{Imp}(a, b) = \max \{ 1 - a, b \}$$

axioms for fuzzy implications

- | | |
|--|--------------------------|
| 1. $a \leq b$ implies $\text{Imp}(a, x) \geq \text{Imp}(b, x)$ | monotone in 1st argument |
| 2. $a \leq b$ implies $\text{Imp}(x, a) \leq \text{Imp}(x, b)$ | monotone in 2nd argument |
| 3. $\text{Imp}(0, a) = 1$ | dominance of falseness |
| 4. $\text{Imp}(1, b) = b$ | neutrality of trueness |
| 5. $\text{Imp}(a, a) = 1$ | identity |
| 6. $\text{Imp}(a, \text{Imp}(b, x)) = \text{Imp}(b, \text{Imp}(a, x))$ | exchange property |
| 7. $\text{Imp}(a, b) = 1$ iff $a \leq b$ | boundary condition |
| 8. $\text{Imp}(a, b) = \text{Imp}(c(b), c(a))$ | contraposition |
| 9. $\text{Imp}(\cdot, \cdot)$ is continuous | continuity |

Caution!

Not all S-, R-, QL- implications obey all axioms for fuzzy implications!

Implication	Valid Axioms									
Kleene-Dienes	1	2	3	4	–	6	–	8	9	
Reichenbach	1	2	3	4	–	6	–	8	9	
Łukasiewicz	1	2	3	4	5	6	7	8	9	←
Gödel	1	2	3	4	5	6	7	–	–	
Goguen	1	2	3	4	5	6	7	–	9	
Zadeh	1	2	3	4	–	–	–	–	9	
Klir-Yuan	–	2	3	4	–	–	–	–	9	

characterization of fuzzy implication

Theorem:

$\text{Imp}: [0,1] \times [0,1] \rightarrow [0,1]$ satisfies axioms 1 - 9 for fuzzy implications for a certain fuzzy complement $c(\cdot)$ $\Leftrightarrow$

$\exists$ strictly monotone increasing, continuous function $f: [0,1] \rightarrow [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0,1]: \text{Imp}(a, b) = f^{-1}(\min\{ f(1) - f(a) + f(b), f(1) \})$
- $\forall a \in [0,1]: c(a) = f^{-1}(f(1) - f(a))$

Proof: Smets & Magrez (1987), p. 337f. ■

examples: (in tutorial)

choosing an „appropriate“ fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem.“

guideline:

GMP, GMT, GHS should be compatible with MP, MT, HS

for fuzzy implication in calculations with relations:

$$B(y) = \sup \{ t(A(x), \text{Imp}(A(x), B(y))) : x \in X \}$$

example:

Gödel implication for t-norm = bounded difference