

Computational Intelligence

Winter Term 2025/26

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- Fuzzy relations
- Fuzzy logic
 - Linguistic variables and terms
 - Inference from fuzzy statements

relations with conventional sets $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!

⇒ all set operations remain valid!

crisp membership function (of x to relation R)

$$R(x_1, x_2, \dots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \dots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition

Fuzzy relation = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$ ■

→ each tuple $(x_1, \dots, x_n)$ has a degree of membership to relation

→ degree of membership expresses
strength of relationship between elements of tuple

appropriate representation: n-dimensional membership matrix

example: Let $X = \{ \text{Beijing, New York, Dortmund} \}$ and $Y = \{ \text{New York, Paris} \}$.

relation $R = \text{"very far away"}$

membership matrix →

relation R	New York	Paris
Beijing	1.0	0.9
New York	0.0	0.7
Dortmund	0.6	0.3

Definition

Let $R(X, Y)$ be a fuzzy relation with membership matrix R . The **inverse fuzzy relation** to $R(X, Y)$, denoted $R^{-1}(Y, X)$, is a relation on $Y \times X$ with membership matrix R' . ■

Remark: R' is the transpose of membership matrix R .

Evidently: $(R^{-1})^{-1} = R$ since $(R')' = R$

Definition

Let $P(X, Y)$ and $Q(Y, Z)$ be fuzzy relations. The operation $\circ$ on two relations, denoted $P(X, Y) \circ Q(Y, Z)$, is termed **max-min-composition** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}. \quad \blacksquare$$

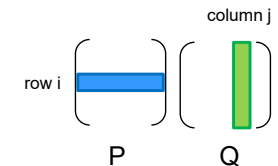
Theorem

- max-min composition on relations is associative.
- max-min composition on relations is not commutative.
- $(P(X, Y) \circ Q(Y, Z))^{-1} = Q^{-1}(Z, Y) \circ P^{-1}(Y, X)$.

membership matrix of max-min composition
determinable via “fuzzy matrix multiplication”: $R = P \circ Q$

fuzzy matrix multiplication $r_{ij} = \max_k \min \{ p_{ik}, q_{kj} \}$

crisp matrix multiplication $r_{ij} = \sum_k p_{ik} \cdot q_{kj}$



further methods for realizing compositions of relations:

max-prod composition

$$(P \odot Q)(x, z) = \max_{y \in \mathcal{Y}} \{ P(x, y) \cdot Q(y, z) \}$$

generalization: sup-t composition

$$(P \circ Q)(x, z) = \sup_{y \in \mathcal{Y}} \{ t(P(x, y), Q(y, z)) \}, \quad \text{where } t(\dots) \text{ is a t-norm}$$

e.g.: $t(a, b) = \min\{a, b\} \Rightarrow \text{max-min-composition}$
 $t(a, b) = a \cdot b \Rightarrow \text{max-prod-composition}$

Binary fuzzy relations on $X \times X$: properties

- reflexive** $\Leftrightarrow \forall x \in X : R(x, x) = 1$
- irreflexive** $\Leftrightarrow \exists x \in X : R(x, x) < 1$
- antireflexive** $\Leftrightarrow \forall x \in X : R(x, x) < 1$
- symmetric** $\Leftrightarrow \forall (x, y) \in X \times X : R(x, y) = R(y, x)$
- asymmetric** $\Leftrightarrow \exists (x, y) \in X \times X : R(x, y) \neq R(y, x)$
- antisymmetric** $\Leftrightarrow \forall (x, y) \in X \times X : R(x, y) \neq R(y, x)$
- transitive** $\Leftrightarrow \forall (x, z) \in X \times X : R(x, z) \geq \max_{y \in X} \min \{ R(x, y), R(y, z) \}$
- intransitive** $\Leftrightarrow \exists (x, z) \in X \times X : R(x, z) < \max_{y \in X} \min \{ R(x, y), R(y, z) \}$
- antitransitive** $\Leftrightarrow \forall (x, z) \in X \times X : R(x, z) < \max_{y \in X} \min \{ R(x, y), R(y, z) \}$

actually, here: max-min-transitivity ($\rightarrow$ in general: sup-t-transitivity)

binary fuzzy relation on $X \times X$: example

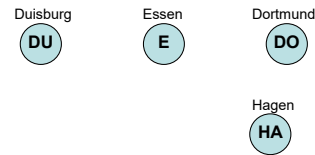
Let X be a subset of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to“.

- $R(x,x) = 1$, since every city is certainly very close to itself.
⇒ **reflexive**
- $R(x,y) = R(y,x)$: if city x is very close to city y , then also vice versa.
⇒ **symmetric**

R	DU	E	DO	HA
DU	1	0.7	0.5	0.4
E	0.7	1	0.8	0.8
DO	0.5	0.8	1	0.9
HA	0.4	0.8	0.9	1

⇒ **intransitive**



$$R(DO, DU) = 0.5 < \max_y \min\{R(DO, y), R(y, DU)\} = 0.7$$

$$R(E, DO) = 0.8 \geq \max_y \min\{R(E, y), R(y, DO)\} = 0.8$$

crisp:

relation R is equivalence relation, R reflexive, symmetric, transitive

fuzzy:

relation R is similarity relation, R reflexive, symmetric, (max-min-) transitive

examples:

- *equivalence relation*: farm animals
cattle, pigs, chicken, ...
 $R(\text{cow}, \text{ox}) = 1$ but $R(\text{cow}, \text{hen}) = 0$

- *similarity relation*: farm animals
cattle, pigs, chicken, horse, donkey, mule, ...
 $R(\text{mule}, (\text{male}) \text{ donkey}) = 0.5$ and $R(\text{mule}, (\text{female}) \text{ horse}) = 0.5$

What about hybrids?

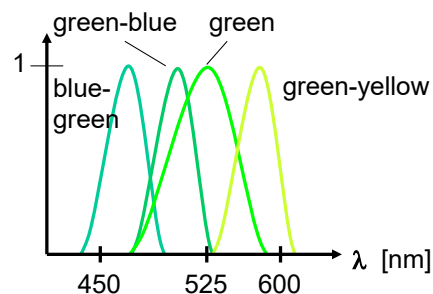
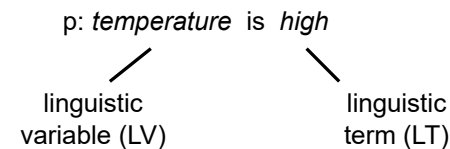
linguistic variable:

variable that can attain several values of linguistic / verbal nature

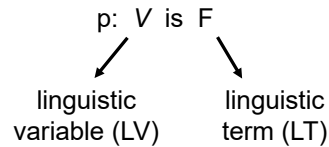
e.g.: **color** can attain values **red, green, blue, yellow, ...**

values (red, green, ...) of linguistic variable are called **linguistic terms**

linguistic terms are associated with fuzzy sets

**fuzzy proposition**

- LV may be associated with several LT : *high, medium, low, ...*
- *high, medium, low* temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given **concrete crisp** temperature value v is interpreted as equal to the degree of membership $high(v)$ of the fuzzy set *high*

fuzzy proposition

actually:

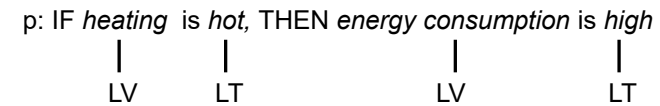
$p: V \text{ is } F(v)$

and

$T(p) = F(v)$ for a concrete crisp value v

↘
trueness(p)

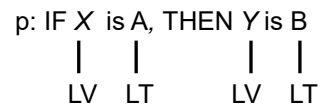
establishes
connection between
degree of membership
of a fuzzy set and the
degree of trueness
of a fuzzy proposition

fuzzy proposition

expresses relation between

- a) temperature of heating and
- b) quantity of energy consumption

$p: (\text{heating, energy consumption}) \in R$
↖ relation

fuzzy proposition

How can we determine / express degree of trueness $T(p)$?

- For crisp, given values x, y we know $A(x)$ and $B(y)$
- $A(x)$ and $B(y)$ must be processed to single value via relation R
- $R(x, y) = \text{function}(A(x), B(y))$ is fuzzy set over $X \times Y$
- as before: interpret $T(p)$ as degree of membership $R(x, y)$

fuzzy proposition

$p: \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B$

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over $X \times Y$

$\forall (x, y) \in X \times Y: R(x, y) = \text{Imp}(A(x), B(y))$

What is $\text{Imp}(\cdot, \cdot)$?

$\Rightarrow$ „appropriate“ fuzzy implication $[0, 1] \times [0, 1] \rightarrow [0, 1]$

assumption: we know an „appropriate“ $\text{Imp}(a,b)$.

How can we determine the *degree of trueness* $T(p)$?

example: (discrete case)

let $\text{Imp}(a, b) = \min\{1, 1 - a + b\}$ and consider fuzzy sets

A:

x_1	x_2
0.5	0.9

B:

y_1	y_2	y_3
0.1	0.8	0.4

$\Rightarrow$

R	y_1	y_2	y_3
x_1	0.6	1.0	0.9
x_2	0.2	0.9	0.5

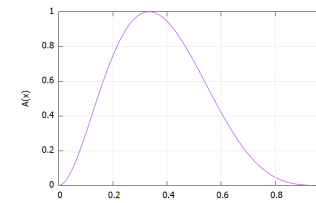
z.B.

$$R(x_2, y_1) = \text{Imp}(A(x_2), B(y_1)) = \text{Imp}(0.9, 0.1) = \min\{1.0, 1.0 - 0.9 + 0.1\} = 0.2$$

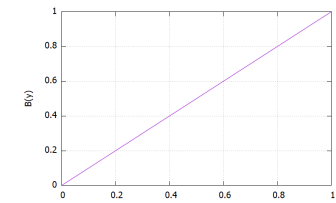
and $T(p)$ for (x_2, y_1) is $R(x_2, y_1) = 0.2$ ■

example: (continuous case)

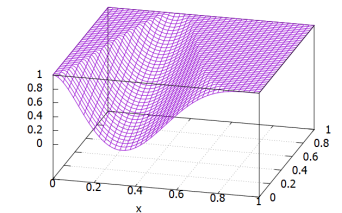
let $\text{Imp}(a, b) = \min\{1, 1 - a + b\}$ and consider fuzzy sets



$$A(x) = \frac{729}{16} x^2 (1 - x)^4 \text{ for } x \in [0, 1]$$



$$B(y) = y \text{ for } y \in [0, 1]$$



$$\Rightarrow R(x, y) = \min\{1, 1 - A(x) + B(y)\} = \min\{1, 1 - \frac{729}{16} x^2 (1 - x)^4 + y\}$$

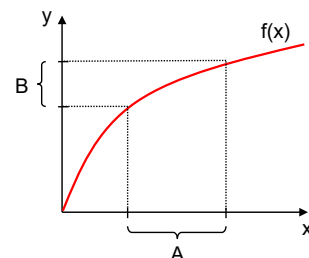
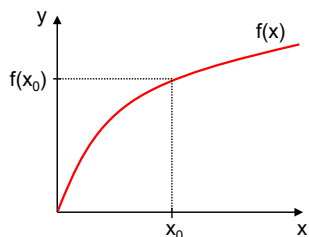
toward inference from fuzzy statements:

- let $R = \{(x, y) : y = f(x)\}$ for a function $f: \mathbb{R} \rightarrow \mathbb{R}$

$$\text{IF } X = \{x_0\} \text{ THEN } Y = \{f(x_0)\}$$

- IF $X \in A$ THEN $Y \in B = \{y \in \mathcal{Y} : y = f(x), x \in A\}$

crisp case:
functional
relationship



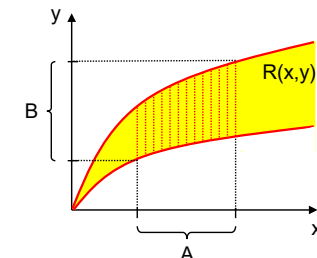
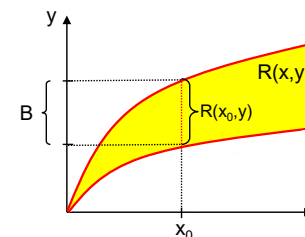
toward inference from fuzzy statements:

- let relationship between x and y be a relation R on $\mathcal{X} \times \mathcal{Y}$

$$\text{IF } X = x_0 \text{ THEN } Y \in B = \{y \in \mathcal{Y} : (x_0, y) \in R\}$$

- IF $X \in A$ THEN $Y \in B = \{y \in \mathcal{Y} : (x, y) \in R, x \in A\}$

crisp case:
relational
relationship



toward inference from fuzzy statements:

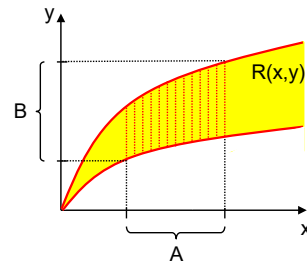
IF $X \in A$ THEN $Y \in B = \{y \in \mathcal{Y} : (x, y) \in R, x \in A\}$

also expressible via characteristic functions of sets A, B, R:

$$B(y) = 1 \text{ iff } \exists x: A(x) = 1 \text{ and } R(x, y) = 1$$

$$\Leftrightarrow \exists x: \min\{A(x), R(x, y)\} = 1$$

$$\Leftrightarrow \max_{x \in \mathcal{X}} \min\{A(x), R(x, y)\} = 1$$



$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min\{A(x), R(x, y)\}$$

inference from fuzzy statements

Now: A' , B' fuzzy sets over $\mathcal{X}$ resp. $\mathcal{Y}$

Assume: $R(x, y)$ and $A'(x)$ are given.

Idea: Generalize characteristic function of $B(y)$ to membership function $B'(y)$

$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min\{A(x), R(x, y)\}$$

characteristic functions

$$\forall y \in \mathcal{Y}: B'(y) = \sup_{x \in \mathcal{X}} \min\{A'(x), R(x, y)\}$$

membership functions

composition rule of inference (in matrix form): $B^T = A \circ R$

Note:
 $A'(x)$ is **not** the derivative of $A(x)$!
It is the membership function
of fuzzy set A' .

inference from fuzzy statements

- conventional:
modus ponens

$$\begin{array}{l} a \Rightarrow b \\ a \\ \hline b \end{array}$$

- fuzzy:
generalized modus ponens (GMP)

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ X \text{ is } A' \\ \hline Y \text{ is } B' \end{array}$$

e.g.: IF *heating* is hot, THEN *energy consumption* is high
heating is warm
energy consumption is normal

example: GMP

consider

$$A: \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline 0.5 & 1.0 & 0.6 \\ \hline \end{array}$$

$$B: \begin{array}{|c|c|} \hline y_1 & y_2 \\ \hline 1.0 & 0.4 \\ \hline \end{array}$$

$$\Rightarrow R: \begin{array}{|c|c|c|} \hline & y_1 & y_2 \\ \hline x_1 & 1.0 & 0.9 \\ \hline x_2 & 1.0 & 0.4 \\ \hline x_3 & 1.0 & 0.8 \\ \hline \end{array}$$

with the rule: IF X is A THEN Y is B

using

$$\text{Imp}(a, b) = \min\{1, 1 - a + b\}$$

given fact

$$A': \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline 0.6 & 0.9 & 0.7 \\ \hline \end{array}$$

thus: $A' \circ R = B'$

with max-min-composition

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

inference from fuzzy statements

- conventional:
modus tollens

$$\frac{a \Rightarrow b}{\bar{b}} \\ \hline \bar{a}$$

- fuzzy:
generalized modus tollens (GMT)

$$\frac{\text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ Y \text{ is } B'}{X \text{ is } A'}$$

e.g.: $\frac{\text{IF heating is hot, THEN energy consumption is high} \\ \text{energy consumption is normal}}{\text{heating is warm}}$

example: GMT

consider

$$A: \begin{array}{|c|c|c|} \hline x_1 & x_2 & x_3 \\ \hline 0.5 & 1.0 & 0.6 \\ \hline \end{array}$$

$$B: \begin{array}{|c|c|} \hline y_1 & y_2 \\ \hline 1.0 & 0.4 \\ \hline \end{array}$$

with the rule: IF X is A THEN Y is B

using $\text{Imp}(a,b) = \min\{1, 1-a+b\}$

given fact

$$B': \begin{array}{|c|c|} \hline y_1 & y_2 \\ \hline 0.9 & 0.7 \\ \hline \end{array}$$

$$\downarrow$$

$$\mathbf{R'}: \begin{array}{|c|c|c|c|} \hline & x_1 & x_2 & x_3 \\ \hline y_1 & 1.0 & 1.0 & 1.0 \\ \hline y_2 & 0.9 & 0.4 & 0.8 \\ \hline \end{array}$$

thus: $B' \circ R^{-1} = A' \quad \left(\begin{array}{cc} 0.9 & 0.7 \end{array} \right) \circ \left(\begin{array}{ccc} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{array} \right) = \left(\begin{array}{ccc} 0.9 & 0.9 & 0.9 \end{array} \right)$
with max-min-composition

■

inference from fuzzy statements

- conventional:
hypothetic syllogism

$$\frac{a \Rightarrow b \\ b \Rightarrow c}{a \Rightarrow c}$$

- fuzzy:
generalized HS

$$\frac{\text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ \text{IF } Y \text{ is } B, \text{ THEN } Z \text{ is } C}{\text{IF } X \text{ is } A, \text{ THEN } Z \text{ is } C}$$

e.g.: $\frac{\text{IF heating is hot, THEN energy consumption is high} \\ \text{IF energy consumption is high, THEN living is expensive}}{\text{IF heating is hot, THEN living is expensive}}$

example: GHS

let fuzzy sets A(x), B(y), C(z) be given

 $\Rightarrow$ determine the three relations according to Imp function

$$\begin{aligned} R_1(x,y) &= \text{Imp}(A(x), B(y)) \\ R_2(y,z) &= \text{Imp}(B(y), C(z)) \\ R_3(x,z) &= \text{Imp}(A(x), C(z)) \end{aligned}$$

and express them as matrices R_1, R_2, R_3 **We say:**GHS is valid if $R_1 \circ R_2 = R_3$

So, ... what makes sense for $\text{Imp}(\cdot, \cdot)$?

$\text{Imp}(a, b)$ ought to express fuzzy version of implication ($a \Rightarrow b$)

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b$

But how can we calculate with fuzzy “boolean” expressions?

request: must be compatible to crisp version (and more) for $a, b \in \{0, 1\}$

a	b	$a \wedge b$	$t(a, b)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

a	b	$a \vee b$	$s(a, b)$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

a	$\bar{a}$	$c(a)$
0	1	1
1	0	0

So, ... what makes sense for $\text{Imp}(\cdot, \cdot)$?

1st approach: S implications

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b$

fuzzy: $\text{Imp}(a, b) = s(c(a), b)$

2nd approach: R implications

conventional: $a \Rightarrow b$ identical to $\max\{x \in \{0, 1\} : a \wedge x \leq b\}$

fuzzy: $\text{Imp}(a, b) = \max\{x \in [0, 1] : t(a, x) \leq b\}$

3rd approach: QL implications

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b \equiv \bar{a} \vee (a \wedge b)$ law of absorption

fuzzy: $\text{Imp}(a, b) = s(c(a), t(a, b))$ (dual tripl ?)

example: S implication $\text{Imp}(a, b) = s(c_s(a), b)$ (c_s : std. complement)

1. Kleene-Dienes implication

$$s(a, b) = \max\{a, b\} \quad (\text{standard}) \quad \text{Imp}(a, b) = \max\{1-a, b\}$$

2. Reichenbach implication

$$s(a, b) = a + b - ab \quad (\text{algebraic sum}) \quad \text{Imp}(a, b) = 1 - a + ab$$

3. Łukasiewicz implication

$$s(a, b) = \min\{1, a + b\} \quad (\text{bounded sum}) \quad \text{Imp}(a, b) = \min\{1, 1 - a + b\}$$

example: R implicationen $\text{Imp}(a, b) = \max\{x \in [0, 1] : t(a, x) \leq b\}$

1. Gödel implication

$$t(a, b) = \min\{a, b\} \quad (\text{std.}) \quad \text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ b & , \text{ else } \end{cases}$$

2. Goguen implication

$$t(a, b) = ab \quad (\text{algeb. product}) \quad \text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ \frac{b}{a} & , \text{ else } \end{cases}$$

3. Łukasiewicz implication

$$t(a, b) = \max\{0, a + b - 1\} \quad (\text{bounded diff.}) \quad \text{Imp}(a, b) = \min\{1, 1 - a + b\}$$

example: QL implication $\text{Imp}(a, b) = s(c(a), t(a, b))$

1. Zadeh implication

$$\begin{aligned} t(a, b) &= \min\{a, b\} & (\text{std.}) \\ s(a, b) &= \max\{a, b\} & (\text{std.}) \end{aligned} \quad \text{Imp}(a, b) = \max\{1 - a, \min\{a, b\}\}$$

2. „NN“ implication ☺ (Klir/Yuan 1994)

$$\begin{aligned} t(a, b) &= ab & (\text{algebr. prod.}) \\ s(a, b) &= a + b - ab & (\text{algebr. sum}) \end{aligned} \quad \text{Imp}(a, b) = 1 - a + a^2b$$

3. Kleene-Dienes implication

$$\begin{aligned} t(a, b) &= \max\{0, a + b - 1\} & (\text{bounded diff.}) \\ s(a, b) &= \min\{1, a + b\} & (\text{bounded sum}) \end{aligned} \quad \text{Imp}(a, b) = \max\{1 - a, b\}$$

axioms for fuzzy implications

- | | |
|--|--------------------------|
| 1. $a \leq b$ implies $\text{Imp}(a, x) \geq \text{Imp}(b, x)$ | monotone in 1st argument |
| 2. $a \leq b$ implies $\text{Imp}(x, a) \leq \text{Imp}(x, b)$ | monotone in 2nd argument |
| 3. $\text{Imp}(0, a) = 1$ | dominance of falseness |
| 4. $\text{Imp}(1, b) = b$ | neutrality of trueness |
| 5. $\text{Imp}(a, a) = 1$ | identity |
| 6. $\text{Imp}(a, \text{Imp}(b, x)) = \text{Imp}(b, \text{Imp}(a, x))$ | exchange property |
| 7. $\text{Imp}(a, b) = 1$ iff $a \leq b$ | boundary condition |
| 8. $\text{Imp}(a, b) = \text{Imp}(c(b), c(a))$ | contraposition |
| 9. $\text{Imp}(\cdot, \cdot)$ is continuous | continuity |

Caution!

Not all S-, R-, QL- implications obey all axioms for fuzzy implications!

Implication	Valid Axioms
Kleene-Dienes	1 2 3 4 – 6 – 8 9
Reichenbach	1 2 3 4 – 6 – 8 9
Łukasiewicz	1 2 3 4 5 6 7 8 9 ←
Gödel	1 2 3 4 5 6 7 – –
Goguen	1 2 3 4 5 6 7 – 9
Zadeh	1 2 3 4 – – – – 9
Klir-Yuan	– 2 3 4 – – – – 9

characterization of fuzzy implication

Theorem:

$\text{Imp}: [0, 1] \times [0, 1] \rightarrow [0, 1]$ satisfies axioms 1 - 9 for fuzzy implications for a certain fuzzy complement $c(\cdot) \Leftrightarrow$

$\exists$ strictly monotone increasing, continuous function $f: [0, 1] \rightarrow [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0, 1]: \text{Imp}(a, b) = f^{-1}(\min\{f(1) - f(a) + f(b), f(1)\})$
- $\forall a \in [0, 1]: c(a) = f^{-1}(f(1) - f(a))$

Proof: Smets & Magrez (1987), p. 337f. ■

examples: (in tutorial)

choosing an „appropriate“ fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem.“

guideline:

GMP, GMT, GHS should be compatible with MP, MT, HS

for fuzzy implication in calculations with relations:

$$B(y) = \sup \{ t(A(x), \text{Imp}(A(x), B(y))) : x \in X \}$$

example:

Gödel implication for t-norm = bounded difference