

Computational Intelligence

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Computational Intelligence

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- Fuzzy sets
 - Axioms of fuzzy complement, t- and s-norms
 - Generators
 - Dual tripels

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Definition

A function $c: [0,1] \rightarrow [0,1]$ is a **fuzzy complement** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$.

monotone decreasing

“nice to have”:

(A3) $c(\cdot)$ is continuous.

(A4) $\forall a \in [0,1]: c(c(a)) = a$

involution

Examples:

a) standard fuzzy complement $c(a) = 1 - a$

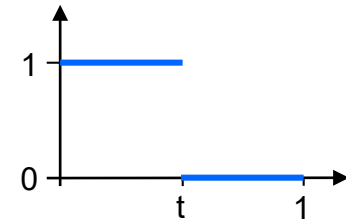
ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$

ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)

ad (A3): ☒

ad (A4): $1 - (1 - a) = a$

$$b) \ c(a) = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{otherwise} \end{cases} \quad \text{for some } t \in (0, 1)$$



ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t

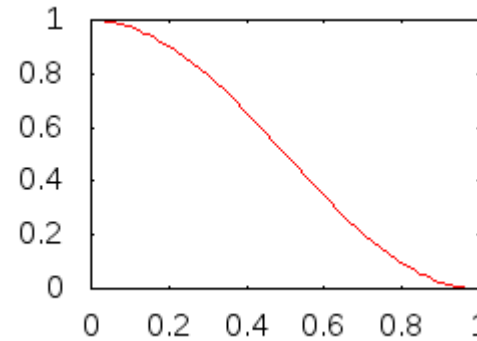


ad (A3): **not valid** $\rightarrow$ discontinuity at t

ad (A4): **not valid** $\rightarrow$ counter example

$$c(c(1/4)) = c(1) = 0 \neq 1/4 \text{ for } t = 1/2$$

c) $c(a) = \frac{1 + \cos(\pi a)}{2}$



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c'(a) = -\frac{1}{2} \pi \sin(\pi a) < 0$ since $\sin(\pi a) > 0$ for $a \in (0,1)$

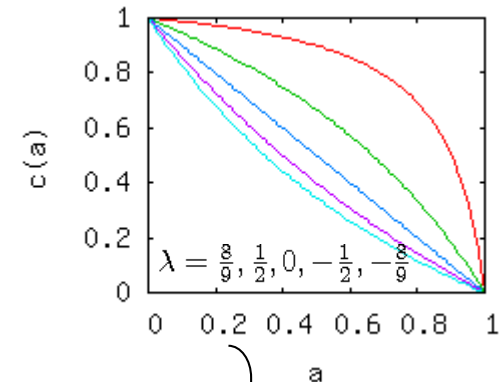


ad (A3): is continuous as a composition of continuous functions;
alternative argument: derivative exists, see $c'(a)$ in (A2)

ad (A4): **not valid** → counter example

$$c\left(c\left(\frac{1}{3}\right)\right) = c\left(\frac{3}{4}\right) = \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right) \neq \frac{1}{3}$$

d) $c(a) = \frac{1-a}{1+\lambda a}$ for $\lambda > -1$ **Sugeno class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c(a) \geq c(b) \Leftrightarrow \frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b} \Leftrightarrow$
 $(1-a)(1+\lambda b) \geq (1-b)(1+\lambda a) \Leftrightarrow$
 $b(\lambda+1) \geq a(\lambda+1) \Leftrightarrow b \geq a$



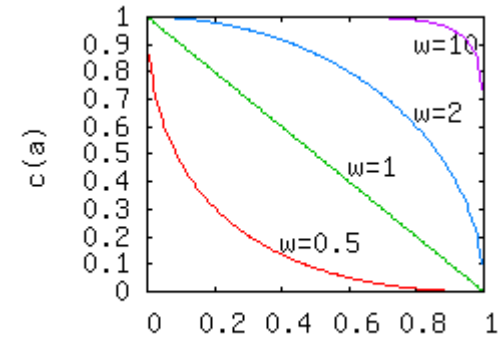
ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right) = \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda \frac{1-a}{1+\lambda a}} = \frac{a(\lambda+1)}{\lambda+1} = a$



e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$

Yager class



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$
 $a^w \leq b^w \Leftrightarrow a \leq b$



ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1 - a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1 - a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$
 $= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$

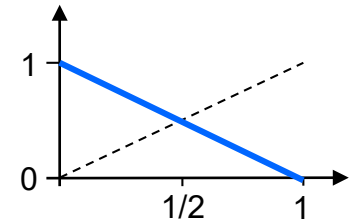


Theorem

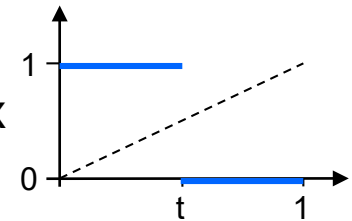
If function $c:[0,1] \rightarrow [0,1]$ satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a^* with $c(a^*) = a^*$.

Proof:

one fixed point $\rightarrow$ see example (a) $\rightarrow$ intersection with bisectrix



no fixed point $\rightarrow$ see example (b) $\rightarrow$ no intersection with bisectrix



assume $\exists n > 1$ fixed points, for example a^* and b^* with $a^* < b^*$

$\Rightarrow c(a^*) = a^*$ and $c(b^*) = b^*$ (fixed points)

$\Rightarrow c(a^*) < c(b^*)$ with $a^* < b^*$ impossible if $c(\cdot)$ is monotone decreasing

$\Rightarrow$ contradiction to axiom (A2)

■

Theorem

If function $c:[0,1] \rightarrow [0,1]$ satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a^* with $c(a^*) = a^*$.

Proof:

Intermediate value theorem $\rightarrow$

If $c(\cdot)$ continuous (A3) and $c(0) \geq c(1)$ (A1/A2)

then $\forall v \in [c(1), c(0)] = [0,1]: \exists a \in [0,1]: c(a) = v$.

$\Rightarrow$ there must be an intersection with bisectrix

$\Rightarrow$ a fixed point exists and by previous theorem there are no other fixed points! ■

Examples:

$$(a) \quad c(a) = 1 - a \quad \Rightarrow a = 1 - a \quad \Rightarrow a^* = \frac{1}{2}$$

$$(b) \quad c(a) = (1 - a^w)^{1/w} \quad \Rightarrow a = (1 - a^w)^{1/w} \quad \Rightarrow a^* = (\frac{1}{2})^{1/w}$$

Theorem

$c: [0,1] \rightarrow [0,1]$ is involutive fuzzy complement iff

$\exists$ continuous function $g: [0,1] \rightarrow \mathbb{R}$ with

- $g(0) = 0$
- strictly monotone increasing
- $\forall a \in [0,1]: c(a) = g^{(-1)}(g(1) - g(a))$. ■

defines an
increasing generator

$g^{(-1)}(x)$ pseudo-inverse
 $= g^{-1}(\min\{g(1), x\})$

Examples

$$a) \quad g(x) = x \quad \Rightarrow \quad g^{(-1)}(x) = x \quad \Rightarrow \quad c(a) = 1 - a \quad (\text{Standard})$$

$$b) \quad g(x) = x^w \quad \Rightarrow \quad g^{(-1)}(x) = x^{1/w} \quad \Rightarrow \quad c(a) = (1 - a^w)^{1/w} \quad (\text{Yager class, } w > 0)$$

$$c) \quad g(x) = \log(x+1) \Rightarrow \underbrace{g^{(-1)}(x) = e^x - 1}_{?} \Rightarrow c(a) = \exp(\log(2) - \log(a+1)) - 1 = \frac{1-a}{1+a} \quad (\text{Sugeno class. } \lambda = 1)$$

→ make sure that pseudoinverse is equal to inverse, here!

$$g(x) = \log(x + 1) \quad \rightarrow \quad g^{-1}(x) = e^x - 1 \quad \checkmark \quad (\text{inverse})$$

$$g^{(-1)}(x) = g^{-1}(\min\{g(1), x\}) \quad (\text{pseudoinverse})$$

$$c(a) = g^{(-1)}(\underbrace{g(1) - g(a)}_{= x})$$

$\downarrow$
 $?$

$$\min\{g(1), g(1) - a\} = g(1) - g(a) \leq g(1) \quad \text{since } 0 \leq g(a) \leq \log 2 \text{ for } a \in [0, 1]$$

therefore,

$$c(a) = g^{(-1)}(g(1) - g(a)) = g^{-1}(g(1) - g(a)) \quad \checkmark$$

Examples

d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda a} > 0$ for $a \in [0, 1]$
- inverse function on $[0, 1]$ is $g^{-1}(a) = \frac{\exp(\lambda a) - 1}{\lambda}$, thus

$$\begin{aligned}
 c(a) &= g^{-1}\left(\frac{\log(1 + \lambda)}{\lambda} - \frac{\log(1 + \lambda a)}{\lambda}\right) \\
 &= \frac{\exp(\log(1 + \lambda) - \log(1 + \lambda a)) - 1}{\lambda} \\
 &= \frac{1}{\lambda} \left(\frac{1 + \lambda}{1 + \lambda a} - 1 \right) = \frac{1 - a}{1 + \lambda a} \quad (\text{Sugeno Complement})
 \end{aligned}$$

Theorem

$c: [0,1] \rightarrow [0,1]$ is involutive fuzzy complement iff

$\exists$ continuous function $f: [0,1] \rightarrow \mathbb{R}$ with

- $f(1) = 0$
- strictly monotone decreasing
- $\forall a \in [0,1]: c(a) = f^{(-1)}(f(0) - f(a))$. ■

defines a
decreasing generator

$f^{(-1)}(x)$ pseudo-inverse
 $= f^{-1}(\min\{f(0), x\})$

Examples

$$\text{a) } f(x) = k - k \cdot x \quad (k \geq 1) \quad f^{(-1)}(x) = 1 - x/k \quad c(a) = 1 - \frac{k - (k - ka)}{k} = 1 - a$$

$$\text{b) } f(x) = 1 - x^w \quad f^{(-1)}(x) = (1 - x)^{1/w} \quad c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w} \quad (\text{Yager})$$

Definition

A function $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a **fuzzy intersection** or **t-norm** iff $\forall a,b,d \in [0,1]$

(A1) $t(a, 1) = a$ (boundary condition)

(A2) $b \leq d \Rightarrow t(a, b) \leq t(a, d)$ (monotonicity)

(A3) $t(a,b) = t(b, a)$ (commutative)

(A4) $t(a, t(b, d)) = t(t(a, b), d)$ (associative) ■

“nice to have”

(A5) $t(a, b)$ is continuous (continuity)

(A6) $t(a, a) < a$ for $0 < a < 1$ (subidempotent)

(A7) $a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow t(a_1, b_1) < t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with $t(a,a) = a$ for all $a \in [0,1]$.

- If $0 \leq a \leq b \leq 1$ then

$$\begin{array}{ccccccc}
 a & = & t(a,a) & \leq & t(a,b) & \leq & t(a, 1) = a \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{by assumption} & & \text{by monotonicity} & & \text{by boundary condition}
 \end{array}$$

and hence $t(a,b) = a$.

- If $0 \leq b \leq a \leq 1$ then

$$\begin{array}{ccccccc}
 b & = & t(b,b) & \leq & t(b,a) & \leq & t(b, 1) = b \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{by assumption} & & \text{by monotonicity} & & \text{by boundary condition}
 \end{array}$$

and hence $t(a,b) = t(b,a) = b$.

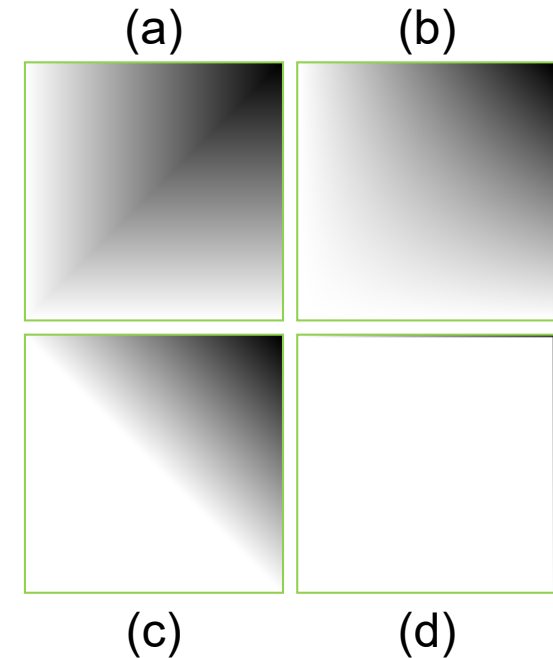
$\uparrow$
by commutativity

$t(a,b) = \min(a,b)$
is the only
possible solution!

q.e.d.

Examples:

Name	Function
(a) Standard	$t(a, b) = \min \{ a, b \}$
(b) Algebraic Product	$t(a, b) = a \cdot b$
(c) Bounded Difference	$t(a, b) = \max \{ 0, a + b - 1 \}$
(d) Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$



Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): $t(a, 1) = a \cdot 1 = a$ ☒

ad (A3): $t(a, b) = a \cdot b = b \cdot a = t(b, a)$ ☒

ad (A2): $a \cdot b \leq a \cdot d \Leftrightarrow b \leq d$ ☒

ad (A4): $a \cdot (b \cdot d) = (a \cdot b) \cdot d$ ☒

Theorem

Function $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm ,

$\exists$ decreasing generator $f: [0,1] \rightarrow \mathbb{R}$ with $t(a, b) = f^{-1}(\min\{ f(0), f(a) + f(b) \})$. ■

Example:

$f(x) = 1/x - 1$ is decreasing generator since

- $f(x)$ is continuous ☑
- $f(1) = 1/1 - 1 = 0$ ☑
- $f'(x) = -1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \frac{1}{x+1}$; $f(0) = \infty \Rightarrow \min\{ f(0), f(a) + f(b) \} = f(a) + f(b)$

$$\Rightarrow t(a, b) = f^{-1} \left(\frac{1}{a} + \frac{1}{b} - 2 \right) = \frac{1}{\frac{1}{a} + \frac{1}{b} - 1} = \frac{ab}{a+b-a-b}$$

Definition

A function $s:[0,1] \times [0,1] \rightarrow [0,1]$ is a **fuzzy union** or **s-norm** iff $\forall a,b,d \in [0,1]$

(A1) $s(a, 0) = a$ (boundary condition)

(A2) $b \leq d \Rightarrow s(a, b) \leq s(a, d)$ (monotonicity)

(A3) $s(a, b) = s(b, a)$ (commutative)

(A4) $s(a, s(b, d)) = s(s(a, b), d)$ (associative) ■

“nice to have”

(A5) $s(a, b)$ is continuous (continuity)

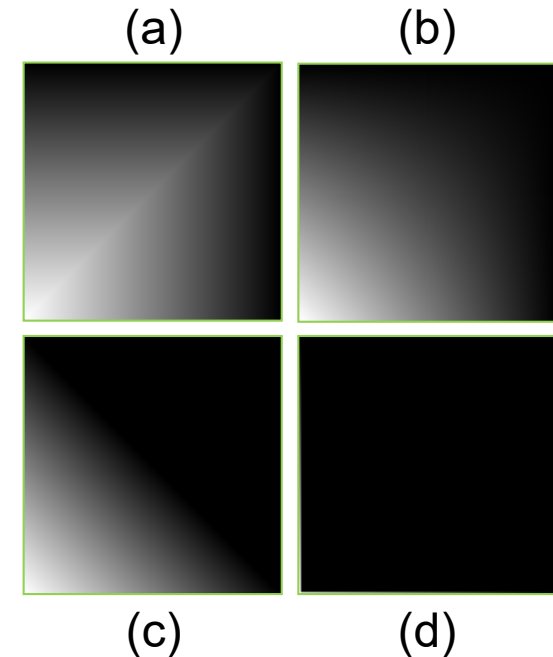
(A6) $s(a, a) > a$ for $0 < a < 1$ (superidempotent)

(A7) $a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow s(a_1, b_1) < s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	$s(a, b) = \max \{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min \{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$



Is algebraic sum an s-norm? Check the 4 axioms!

ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$ ☒

ad (A3): ☒

ad (A2): $a + b - a \cdot b \leq a + d - a \cdot d \Leftrightarrow b(1 - a) \leq d(1 - a) \Leftrightarrow b \leq d$ ☒

ad (A4): ☒

Theorem

Function $s: [0,1] \times [0,1] \rightarrow [0,1]$ is a s-norm $\Leftrightarrow$

$\exists$ increasing generator $g: [0,1] \rightarrow \mathbb{R}$ with $s(a, b) = g^{-1}(\min\{g(1), g(a) + g(b)\})$. ■

Example:

$g(x) = -\log(1 - x)$ is increasing generator since

- $g(x)$ is continuous ✓
- $g(0) = -\log(1 - 0) = 0$ ✓
- $g'(x) = 1/(1 - x) > 0$ (monotone increasing) ✓

inverse function is $g^{-1}(x) = 1 - \exp(-x)$; $g(1) = \infty \Rightarrow \min\{g(1), g(a) + g(b)\} = g(a) + g(b)$

$$\begin{aligned} \Rightarrow s(a, b) &= g^{-1}(-\log(1 - a) - \log(1 - b)) \\ &= 1 - \exp(\log(1 - a) + \log(1 - b)) \\ &= 1 - (1 - a)(1 - b) = a + b - ab \quad (\text{algebraic sum}) \end{aligned}$$

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition

A pair of t-norm $t(\cdot, \cdot)$ and s-norm $s(\cdot, \cdot)$ is said to be **dual with regard to the fuzzy complement** $c(\cdot)$ iff

- $c(t(a, b)) = s(c(a), c(b))$
- $c(s(a, b)) = t(c(a), c(b))$

for all $a, b \in [0, 1]$. ■

Definition

Let (c, s, t) be a triplet of fuzzy complement $c(\cdot)$, s- and t-norm.

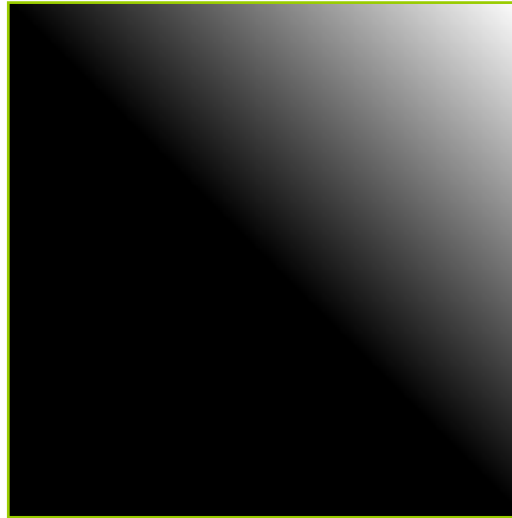
If t and s are dual to c then the triplet (c, s, t) is called a **dual triplet**. ■

Examples of dual triplets

t-norm	s-norm	complement
$\min \{ a, b \}$	$\max \{ a, b \}$	$1 - a$
$a \cdot b$	$a + b - a \cdot b$	$1 - a$
$\max \{ 0, a + b - 1 \}$	$\min \{ 1, a + b \}$	$1 - a$



$c(t(a, b))$



$s(c(a), c(b))$



Dual Triple:

- bounded difference
- bounded sum
- standard complement

$\Rightarrow$ left image = right image

Non-Dual Triple:

- algebraic product
- bounded sum
- standard complement

$\Rightarrow$ left image $\neq$ right image

Why are dual triples so important?

⇒ allow equivalence transformations of fuzzy set expressions

⇒ required to transform into some equivalent normal form (standardized input)

⇒ e.g. two stages: intersection of unions $\bigcap_{i=1}^n (A_i \cup B_i)$

or union of intersections $\bigcup_{i=1}^n (A_i \cap B_i)$

Example:

$$A \cup (B \cap (C \cap D)^c) =$$

← not in normal form

$$A \cup (B \cap (C^c \cup D^c)) =$$

← equivalent if DeMorgan's law valid (dual triples!)

$$A \cup (B \cap C^c) \cup (B \cap D^c)$$

← equivalent (distributive lattice!)