

Computational Intelligence

Winter Term 2025/26

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- Fuzzy Sets
 - Basic Definitions and Results for Standard Operations
 - Algebraic Difference between Fuzzy and Crisp Sets

Observation:

Communication between people is not precise but somehow fuzzy and vague.

“If the water is too hot then add a little bit of cold water.”

Despite these shortcomings in human language we are able

- to process fuzzy / uncertain information and
- to accomplish complex tasks!

Goal:

Development of formal framework to process fuzzy statements in computer.

Consider the statement: “The water is hot.”

Which temperature defines “hot”?

A single temperature $T = 95^\circ \text{C}$?

No! Rather, an interval of temperatures: $T \in [70, 120]$!

But who defines the limits of the intervals?

Some people regard temperatures $> 60^\circ \text{C}$ as hot, others already $T > 50^\circ \text{C}$!

Idea: All people might agree that a temperature in the set $[70, 120]$ defines a hot temperature!

If $T = 65^\circ \text{C}$ not all people regard this as hot. It does not belong to $[70, 120]$.

But it is hot to some degree.

Or: $T = 65^\circ \text{C}$ belongs to set of hot temperatures to some degree!

⇒ **Can be the concept for capturing fuzziness!** ⇒ **Formalize this concept!**

Definition

A map $F: X \rightarrow [0,1] \subset \mathbb{R}$ that assigns its **degree of membership** $F(x)$ to each $x \in X$ is termed a **fuzzy set**.

Remark:

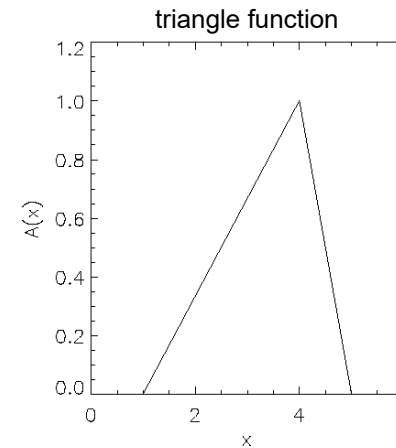
A fuzzy set F is actually a map $F(x)$. Shorthand notation is simply F .

Same point of view possible for traditional ("**crisp**") sets:

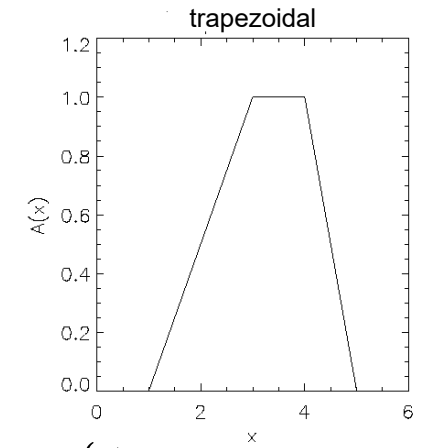
$$A(x) := 1_{[x \in A]} := 1_A(x) := \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases}$$

characteristic / indicator function of (crisp) set A

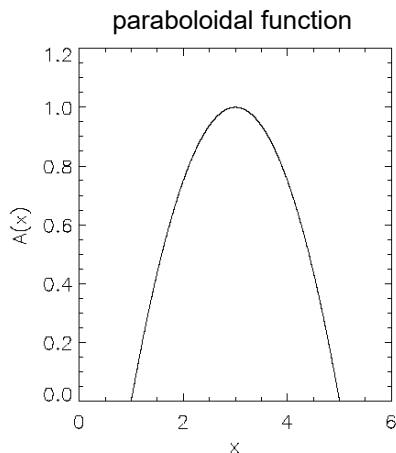
$\Rightarrow$ membership function interpreted as generalization of characteristic function



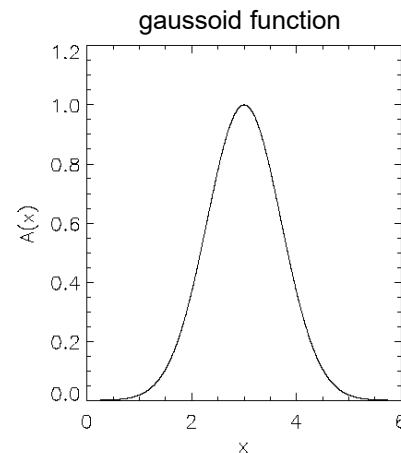
$$A(x) = \begin{cases} \frac{1}{3}(x-1) & \text{if } 1 \leq x < 4 \\ 5-x & \text{if } 4 \leq x < 5 \\ 0 & \text{otherwise} \end{cases}$$



$$A(x) = \begin{cases} \frac{1}{2}(x-1) & \text{if } 1 \leq x < 3 \\ 1 & \text{if } 3 \leq x < 4 \\ 5-x & \text{if } 4 \leq x < 5 \\ 0 & \text{otherwise} \end{cases}$$



$$A(x) = \begin{cases} -\frac{(x-1)(x-5)}{4} & \text{if } 1 \leq x < 5 \\ 0 & \text{otherwise} \end{cases}$$



$$A(x) = \exp\left(-\frac{(x-3)^2}{2}\right)$$

Definition

A fuzzy set F over the crisp set X is termed

- a) **empty** if $F(x) = 0$ for all $x \in X$,
- b) **universal** if $F(x) = 1$ for all $x \in X$.

Empty fuzzy set is denoted by $\emptyset$. Universal set is denoted by $\mathbb{U}$. ■

Definition

Let A and B be fuzzy sets over the crisp set X .

- a) A and B are termed **equal**, denoted $A = B$, if $A(x) = B(x)$ for all $x \in X$.
- b) A is a **subset** of B , denoted $A \subseteq B$, if $A(x) \leq B(x)$ for all $x \in X$.
- c) A is a **strict subset** of B , denoted $A \subset B$, if $A \subseteq B$ and $\exists x \in X: A(x) < B(x)$. ■

Remark: A strict subset is also called a **proper** subset.

Theorem

Let A , B and C be fuzzy sets over the crisp set X . The following relations are valid:

- a) reflexivity : $A \subseteq A$.
- b) antisymmetry : $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.
- c) transitivity : $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

Proof: (via reduction to definitions and exploiting operations on crisp sets)

ad a) $\forall x \in X: A(x) \leq A(x)$.

ad b) $\forall x \in X: A(x) \leq B(x)$ and $B(x) \leq A(x) \Rightarrow A(x) = B(x)$.

ad c) $\forall x \in X: A(x) \leq B(x)$ and $B(x) \leq C(x) \Rightarrow A(x) \leq C(x)$.

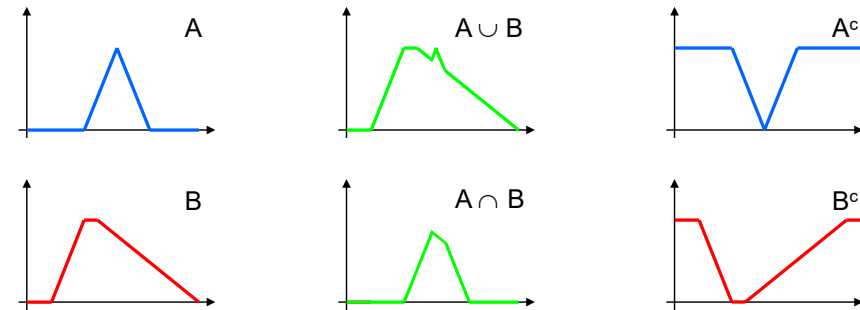
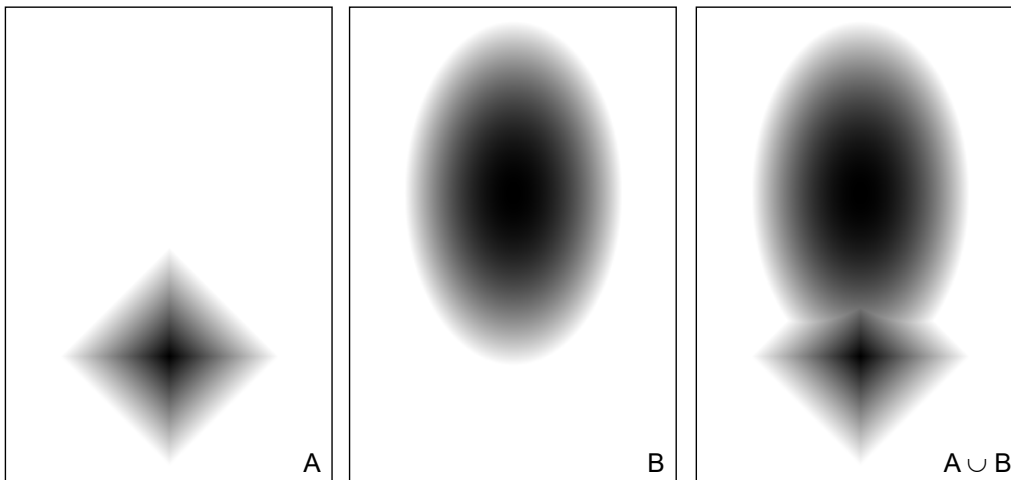
q.e.d.

Remark: Same relations valid for crisp sets. No Surprise! Why?

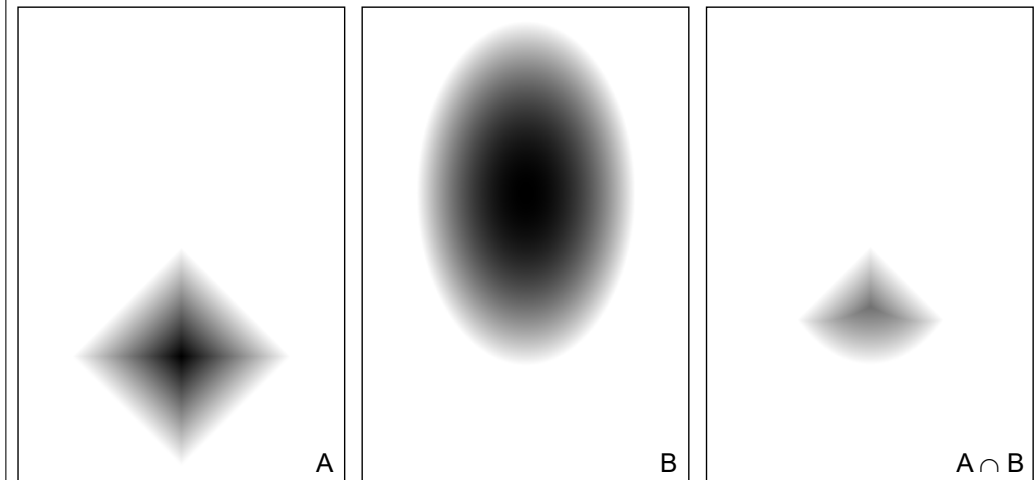
Definition

Let A and B be fuzzy sets over the crisp set X . The set C is the

- a) **union** of A and B , denoted $C = A \cup B$, if $C(x) = \max\{A(x), B(x)\}$ for all $x \in X$;
- b) **intersection** of A and B , denoted $C = A \cap B$, if $C(x) = \min\{A(x), B(x)\}$ for all $x \in X$;
- c) **complement** of A , denoted $C = A^c$, if $C(x) = 1 - A(x)$ for all $x \in X$. ■

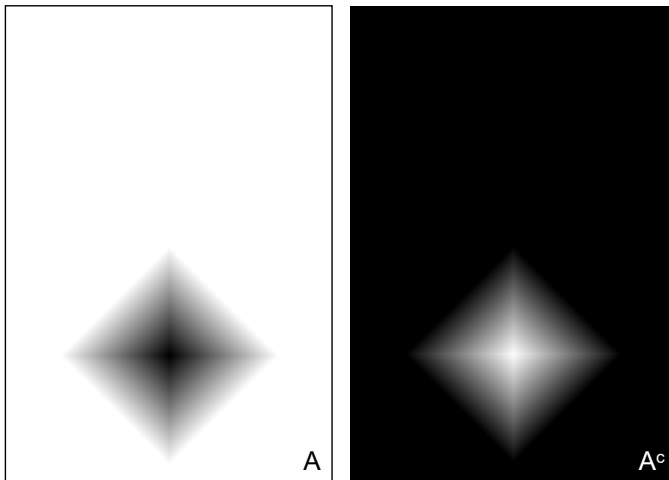
**standard fuzzy union**

interpretation: membership = 0 is white, = 1 is black, in between is gray

standard fuzzy intersection

interpretation: membership = 0 is white, = 1 is black, in between is gray

standard fuzzy complement

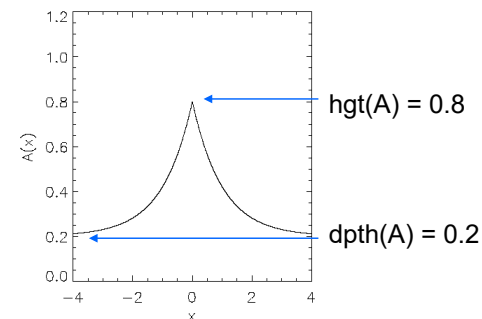


interpretation: membership = 0 is white, = 1 is black, in between is gray

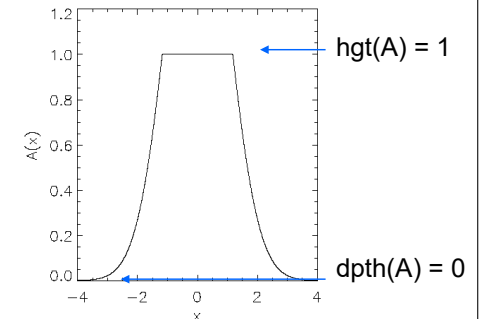
Definition

The fuzzy set A over the crisp set X has

- a) **height** $\text{hgt}(A) = \sup\{A(x) : x \in X\}$,
 b) **depth** $\text{dpth}(A) = \inf\{A(x) : x \in X\}$.



$$A(x) = \frac{1}{5} + \frac{3}{5} \exp(-|x|)$$

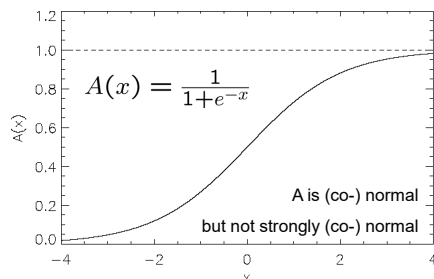


$$A(x) = \min\left\{1, 2 \exp\left(-\frac{x^2}{2}\right)\right\}$$

Definition

The fuzzy set A over the crisp set X is

- a) **normal** if $\text{hgt}(A) = 1$
 b) **strongly normal** if $\exists x \in X: A(x) = 1$
 c) **co-normal** if $\text{dpth}(A) = 0$
 d) **strongly co-normal** if $\exists x \in X: A(x) = 0$
 e) **subnormal** if $0 < A(x) < 1$ for all $x \in X$.



Remark:

How to normalize a non-normal fuzzy set A?

$$A^*(x) = \frac{A(x)}{\text{hgt}(A)}$$

Definition

The **cardinality** $\text{card}(A)$ of a fuzzy set A over the crisp set X is

$$\text{card}(A) := \begin{cases} \sum_{x \in X} A(x) & , \text{ if } X \text{ countable} \\ \int_X A(x) dx & , \text{ if } X \subseteq \mathbb{R}^n \end{cases}$$

Examples:

a) $A(x) = q^x$ with $q \in (0, 1)$, $x \in \mathbb{N}_0 \Rightarrow \text{card}(A) = \sum_{x \in X} A(x) = \sum_{x=0}^{\infty} q^x = \frac{1}{1-q} < \infty$

b) $A(x) = 1/x$ with $x \in \mathbb{N} \Rightarrow \text{card}(A) = \sum_{x \in X} A(x) = \sum_{x=1}^{\infty} \frac{1}{x} = \infty$

c) $A(x) = \exp(-|x|)$ with $x \in \mathbb{R} \Rightarrow \text{card}(A) = \int_X A(x) dx = \int_{-\infty}^{\infty} \exp(-|x|) dx = 2$

Theorem

For fuzzy sets A, B and C over a crisp set X the standard union operation is

- a) **commutative** : $A \cup B = B \cup A$
- b) **associative** : $A \cup (B \cup C) = (A \cup B) \cup C$
- c) **idempotent** : $A \cup A = A$
- d) **monotone** : $A \subseteq B \Rightarrow (A \cup C) \subseteq (B \cup C)$.

Proof: (via reduction to definitions)

ad a) $A \cup B = \max \{ A(x), B(x) \} = \max \{ B(x), A(x) \} = B \cup A$.

ad b) $A \cup (B \cup C) = \max \{ A(x), \max \{ B(x), C(x) \} \} = \max \{ A(x), B(x), C(x) \}$
 $= \max \{ \max \{ A(x), B(x) \}, C(x) \} = (A \cup B) \cup C$.

ad c) $A \cup A = \max \{ A(x), A(x) \} = A(x) = A$.

ad d) $A \cup C = \max \{ A(x), C(x) \} \leq \max \{ B(x), C(x) \} = B \cup C$ since $A(x) \leq B(x)$. **q.e.d.**

Theorem

For fuzzy sets A, B and C over a crisp set X the standard intersection operation is

- a) **commutative** : $A \cap B = B \cap A$
- b) **associative** : $A \cap (B \cap C) = (A \cap B) \cap C$
- c) **idempotent** : $A \cap A = A$
- d) **monotone** : $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$.

Proof: (analogous to proof for standard union operation) ■

Theorem

For fuzzy sets A, B and C over a crisp set X there are the distributive laws

- a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof:

ad a) $\max \{ A(x), \min \{ B(x), C(x) \} \} = \begin{cases} \max \{ A(x), B(x) \} & \text{if } B(x) \leq C(x) \\ \max \{ A(x), C(x) \} & \text{otherwise} \end{cases}$

If $B(x) \leq C(x)$ then $\max \{ A(x), B(x) \} \leq \max \{ A(x), C(x) \}$.

Otherwise $\max \{ A(x), C(x) \} \leq \max \{ A(x), B(x) \}$.

$\Rightarrow$ result is always the smaller max-expression

$\Rightarrow$ result is $\min \{ \max \{ A(x), B(x) \}, \max \{ A(x), C(x) \} \} = (A \cup B) \cap (A \cup C)$.

ad b) analogous. ■

Theorem

If A is a fuzzy set over a crisp set X then

- a) $A \cup \emptyset = A$
- b) $A \cup \mathbb{U} = \mathbb{U}$
- c) $A \cap \emptyset = \emptyset$
- d) $A \cap \mathbb{U} = A$.

Proof:

(via reduction to definitions)

ad a) $\max \{ A(x), 0 \} = A(x)$

ad b) $\max \{ A(x), 1 \} = \mathbb{U}(x) \equiv 1$

ad c) $\min \{ A(x), 0 \} = \emptyset(x) \equiv 0$

ad d) $\min \{ A(x), 1 \} = A(x)$. ■

Breakpoint:

So far we know that fuzzy sets with operations $\cap$ and $\cup$ are a distributive lattice.

If we can show the validity of

• $(A^c)^c = A$

• $A \cup A^c = \mathbb{U}$

• $A \cap A^c = \emptyset$

$\Rightarrow$ Fuzzy Sets would be Boolean Algebra! **Is it true ?**

Theorem

If A is a fuzzy set over a crisp set X then

- a) $(A^c)^c = A$
- b) $\frac{1}{2} \leq (A \cup A^c)(x) < 1$ for $A(x) \in (0,1)$
- c) $0 < (A \cap A^c)(x) \leq \frac{1}{2}$ for $A(x) \in (0,1)$

Remark:

Recall the identities

$$\min\{a, b\} = \frac{a+b-|a-b|}{2}$$

$$\max\{a, b\} = \frac{a+b+|a-b|}{2}$$

Proof:

ad a) $\forall x \in X: 1 - (1 - A(x)) = A(x).$

ad b) $\forall x \in X: \max\{A(x), 1 - A(x)\} = \frac{1}{2} + |A(x) - \frac{1}{2}| \geq \frac{1}{2}.$

Value 1 only attainable for $A(x) = 0$ or $A(x) = 1$.

ad c) $\forall x \in X: \min\{A(x), 1 - A(x)\} = \frac{1}{2} - |A(x) - \frac{1}{2}| \leq \frac{1}{2}.$

Value 0 only attainable for $A(x) = 0$ or $A(x) = 1$.

q.e.d.

Conclusion:

Fuzzy sets with $\cup$ and $\cap$ are a distributive lattice.

But in general:

- a) $A \cup A^c \neq \mathbb{U}$
 - b) $A \cap A^c \neq \mathbb{O}$
- $\Rightarrow$ Fuzzy sets with $\cup$ and $\cap$ are **not** a Boolean algebra!

Remarks:

ad a) The **law of excluded middle** does not hold!

(„Everything must either be or not be!“)

ad b) The **law of noncontradiction** does not hold!

(„Nothing can both be and not be!“)

$\Rightarrow$ Nonvalidity of these laws generate the desired fuzziness!

but: Fuzzy sets still endowed with much algebraic structure (distributive lattice)!

Theorem

If A and B are fuzzy sets over a crisp set X with standard union, intersection, and complement operations then **DeMorgan's** laws are valid:

- a) $(A \cap B)^c = A^c \cup B^c$
- b) $(A \cup B)^c = A^c \cap B^c$

Proof: (via reduction to elementary identities)

ad a) $(A \cap B)^c(x) = 1 - \min\{A(x), B(x)\} = \max\{1 - A(x), 1 - B(x)\} = A^c(x) \cup B^c(x)$

ad b) $(A \cup B)^c(x) = 1 - \max\{A(x), B(x)\} = \min\{1 - A(x), 1 - B(x)\} = A^c(x) \cap B^c(x)$

q.e.d.

Question : Why restricting result above to "standard" operations?

Conjecture : Most likely there also exist "nonstandard" operations!