technische universität dortmund

Computational Intelligence

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Plan for Today

- Approximate Reasoning
- Fuzzy Control

Approximative Reasoning

So far:

- p: IF *X* is A THEN *Y* is B

  → R(x, y) = Imp(A(x), B(y)) — rule as relation; fuzzy implication

- rule: IF *X* is A THEN *Y* is B
  fact: *X* is A‘
  conclusion: *Y* is B‘

  → B‘(y) = $\sup_{x \in X}$ t(A‘(x), R(x, y)) — composition rule of inference

Thus:

- B‘(y) = $\sup_{x \in X}$ t(A‘(x), Imp(A(x), B(y)))

given	: fuzzy rule
input	: fuzzy set A‘
output	: fuzzy set B‘

Approximative Reasoning

special case:

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

$$B'(y) = \sup_{x \in X} t(\, A'(x), \text{Imp}(\, A(x), B(y)\,)\,)$$

$$= \begin{cases} \sup_{x \neq x_0} t(\, 0, \text{Imp}(\, A(x), B(y)\,)\,) & \text{for } x \neq x_0 \\ t(\, 1, \text{Imp}(\, A(x_0), B(y)\,)\,) & \text{for } x = x_0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x \neq x_0 & \text{since } t(0, a) = 0 \\ \text{Imp}(\, A(x_0), B(y)\,) & \text{for } x = x_0 & \text{since } t(a, 1) = a \end{cases}$$

[A1]

Lemma:

a) $t(a, 1) = a$

b) $t(a, b) \le \min \{ a, b \}$

c) $t(0, a) = 0$

Proof:

by a)

ad a) Identical to axiom 1 of t-norms.

ad b) From monotonicity (axiom 2) follows for $b \le 1$, that $t(a, b) \le t(a, 1) = a$. Commutativity (axiom 3) and monotonicity lead in case of $a \le 1$ to $t(a, b) = t(b, a) \le t(b, 1) = b$. Thus, $t(a, b)$ is less than or equal to a as well as b, which in turn implies $t(a, b) \le \min\{ a, b \}$.

ad c) From b) follows $0 \le t(0, a) \le \min \{ 0, a \} = 0$ and therefore $t(0, a) = 0$. ■

Multiple rules:

IF X is A_1, THEN Y is B_1 → $R_1(x, y) = Imp_1(A_1(x), B_1(y))$
IF X is A_2, THEN Y is B_2 → $R_2(x, y) = Imp_2(A_2(x), B_2(y))$
IF X is A_3, THEN Y is B_3 → $R_3(x, y) = Imp_3(A_3(x), B_3(y))$
… …
IF X is A_n, THEN Y is B_n → $R_n(x, y) = Imp_n(A_n(x), B_n(y))$
X is A'

Y is B'

Multiple rules for <u>fuzzy input</u>: A'(x) is given

$B_1'(y) = \sup_{x \in X} t(A'(x), R_1(x, y))$
…
$B_n'(y) = \sup_{x \in X} t(A'(x), R_n(x, y))$

aggregation of rules or local inferences necessary!

aggregate! ⇒ $B'(y) =$ **aggr**$\{ B_1'(y), \ldots, B_n'(y) \}$, where **aggr** = { min, max

Why?

Axioms of Aggregation

[cf. Fung/Fu 1975; quoted from W. Cholewa: Fuzzy Sets & Systems 17:249-258, 1985]

Let A, A_1, A_2, ... be fuzzy sets over X. The aggregate is denoted by $A_1 \oplus A_2$.

(A1) $\exists$ function $\circ$: $[0,1] \times [0,1] \to [0,1]$ with $(A_1 \oplus A_2)(x) = A_1(x) \circ A_2(x)$ $\forall x \in X$

(A2) $\forall A$: $A \oplus A = A$

(A3) $\forall$ i, j : $A_i \oplus A_j = A_j \oplus A_i$

(A4) For $m \ge 3$: $A_1 \oplus \ldots \oplus A_m = (A_1 \oplus \ldots \oplus A_{m-1}) \oplus A_m$

(A5) $\forall$ i, j, k : $A_i \oplus (A_j \oplus A_k) = (A_i \oplus A_j) \oplus A_k$

(A6) Let $A_1 = A \oplus A_3$ and $A_2 = A \oplus A_4$. If $A_3(x) > A_4(x)$ then $A_1(x) > A_2(x)$ $\forall x \in X$

Theorem

If Axioms (A1) – (A6) hold, then only three types of aggregation are possible:
1. $a \circ b = \min(a, b)$
2. $a \circ b = \max(a, b)$
3. $a \circ b = \min(a, b)$ for $a, b \ge \theta$; $= \max(a, b)$ for $a, b \le \theta$; $= \theta$ otherwise $(0 < \theta < 1)$

<u>FITA</u>: "<u>**F**irst **i**nference, **t**hen **a**ggregate!</u>"

1. Each rule of the form IF *X* is A_k THEN *Y* is B_k must be transformed by an appropriate fuzzy implication $Imp_k(\cdot,\cdot)$ to a relation R_k :
$R_k(x, y) = Imp_k(A_k(x), B_k(y))$.
2. Determine $B_k'(y) = R_k(x, y) \circ A'(x)$ for all k = 1, …, n (local inference).
3. Aggregate to $B'(y) = \beta(B_1'(y), \ldots, B_n'(y))$.

<u>FATI</u>: "<u>**F**irst **a**ggregate, **t**hen **i**nference!</u>"

1. Each rule of the form IF *X* ist A_k THEN *Y* ist B_k must be transformed by an appropriate fuzzy implication $Imp_k(\cdot, \cdot)$ to a relation R_k :
$R_k(x, y) = Imp_k(A_k(x), B_k(y))$.
2. Aggregate $R_1, \ldots, R_n$ to a **superrelation** with aggregating function $\alpha(\cdot)$:
$R(x, y) = \alpha(R_1(x, y), \ldots, R_n(x, y))$.
3. Determine $B'(y) = R(x, y) \circ A'(x)$ w.r.t. superrelation (inference).

1. **Which principle is better? FITA or FATI?**

2. **Equivalence of FITA and FATI ?**

FITA: $B'(y) = \beta(\, B_1'(y), \ldots, B_n'(y)\,)$

$= \beta(\, R_1(x, y) \circ A'(x), \ldots, R_n(x, y) \circ A'(x)\,)$

FATI: $B'(y) = R(x, y) \circ A'(x)$

$= \alpha(\, R_1(x, y), \ldots, R_n(x, y)\,) \circ A'(x)$

→ general case: no further analysis without simplifying assumptions …

special case:

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

On the equivalence of FITA and FATI:

FITA: $B'(y) = \beta(\, B_1'(y), \ldots, B_n'(y)\,)$

$= \beta(\, Imp_1(A_1(x_0), B_1(y)\,), \ldots, Imp_n(A_n(x_0), B_n(y)\,)\,)$

FATI: $B'(y) = R(x, y) \circ A'(x)$

$= \sup_{x \in X} t(\, A'(x), R(x, y)\,)$ (from now: special case)

$= R(x_0, y)$

$= \alpha(\, Imp_1(\, A_1(x_0), B_1(y)\,), \ldots, Imp_n(\, A_n(x_0), B_n(y)\,)\,)$

FATI = FITA if sup-t-composition with same t-norm, **$\alpha(\cdot) = \beta(\cdot)$,** same $Imp_i()$, and …

- **AND-connected premises**

IF $X_1 = A_{11}$ AND $X_2 = A_{12}$ AND … AND $X_m = A_{1m}$ THEN $Y = B_1$
…
IF $X_n = A_{n1}$ AND $X_2 = A_{n2}$ AND … AND $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$A_k(x_1,\ldots, x_m) = \min \{ A_{k1}(x_1), A_{k2}(x_2), \ldots, A_{km}(x_m) \}$ or in general: t-norm

- **OR-connected premises**

IF $X_1 = A_{11}$ OR $X_2 = A_{12}$ OR … OR $X_m = A_{1m}$ THEN $Y = B_1$
…
IF $X_n = A_{n1}$ OR $X_2 = A_{n2}$ OR … OR $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$A_k(x_1,\ldots, x_m) = \max \{ A_{k1}(x_1), A_{k2}(x_2), \ldots, A_{km}(x_m) \}$ or in general: s-norm

important:

- if rules of the form **IF *X* is A THEN *Y* is B** interpreted as logical implication

  ⇒ $R(x, y) = Imp(\, A(x), B(y)\,)$ makes sense

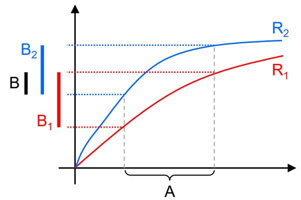
- we obtain: $B'(y) = \sup_{x \in X} t(\, A'(x), R(x, y)\,)$

interpretation of output set B'(y):

- $B'_k(y)$ is the set of values that are possible under the particular rule k
- each rule leads to a different restriction of the values that are possible
- must determine set of values that are possible for **all** rules

⇒ resulting fuzzy sets $B'_k(y)$ obtained from single rules must be mutually intersected!

⇒ aggregation via $B'(y) = \mathbf{min} \{ B_1'(y), \ldots, B_n'(y) \}$



important:

- if rules of the form **IF *X* is A THEN *Y* is B** are not interpreted as logical implications, then the function Fct(•) in

  R(x, y) = Fct(A(x), B(y))

  can be chosen as required for desired interpretation.
- frequent choice (especially in fuzzy control):
  - R(x, y) = min { A(x), B(y) } Mamdani – "implication"
  - R(x, y) = A(x) · B(y) Larsen – "implication"

⇒ of course, they are no implications but specific t-norms!

⇒ thus, if relation R(x, y) is given, then the *composition rule of inference*

$$B'(y) = A'(x) \circ R(x, y) = \sup_{x \in X} \min \{ A'(x), R(x, y) \}$$

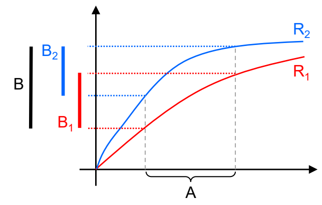
still can lead to a conclusion via fuzzy logic.

interpretation of output set B'(y):

- $B'_k(y)$ is the set of values that are possible under the particular rule k
- technical system must work for all values that are possible
- each rule may extend the set of the values that are possible

⇒ resulting fuzzy sets $B'_k(y)$ obtained from single rules must be mutually united!

⇒ aggregation via $B'(y) = \mathbf{max} \{ B_1'(y), \ldots, B_n'(y) \}$



example: [JM96, S. 244ff.]

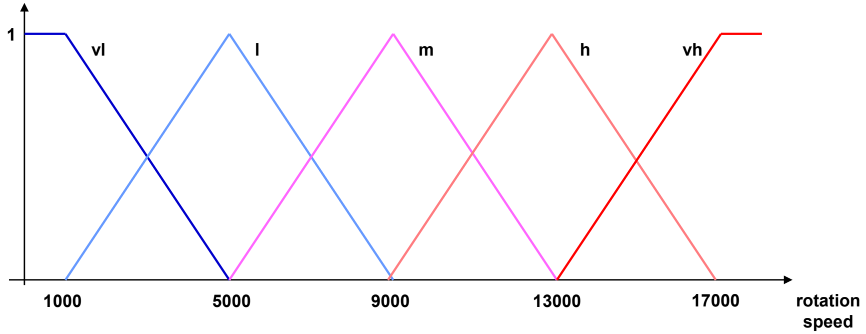
industrial drill machine → control of cooling supply

modelling

linguistic variable : **rotation speed**

linguistic terms : ***very low***, ***low***, ***medium***, ***high***, ***very high***

ground set : X with 0 ≤ x ≤ 18000 [rpm]



example: (continued)

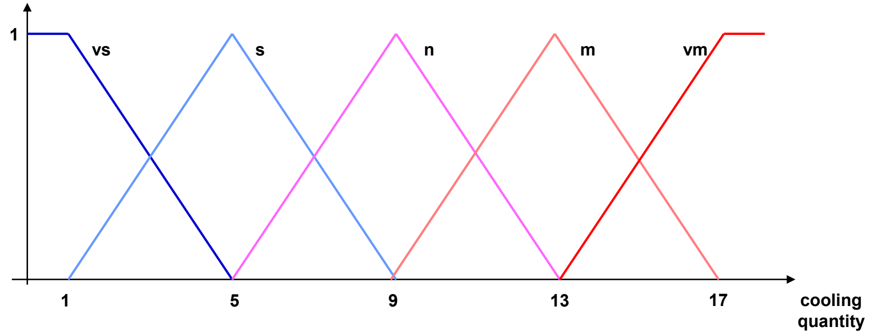
industrial drill machine → control of cooling supply

modelling

linguistic variable : **cooling quantity**

linguistic terms : ***very small***, ***small***, ***normal***, ***much***, ***very much***

ground set : Y with 0 ≤ y ≤ 18 [liter / time unit]



example: (continued)

industrial drill machine → control of cooling supply

rule base

IF rotation speed IS	***very low***	**THEN cooling quantity IS**	***very small***
	low		***small***
	medium		***normal***
	high		***much***
	very high		***very much***

↑ sets S_{vl}, S_l, S_m, S_h, S_{vh} "rotation speed"

↑ sets C_{vs}, C_s, C_n, C_m, C_{vm} "cooling quantity"

example: (continued)

industrial drill machine → control of cooling supply

1. **input**: crisp value x_0 = 10 000 min^{-1} (not a fuzzy set!)

   → **fuzzyfication** = determine membership for each fuzzy set over X

   → yields S' = (0, 0, ¾, ¼, 0) via $x \mapsto (S_{vl}(x_0), S_l(x_0), S_m(x_0), S_h(x_0), S_{vh}(x_0))$

2. FITA: local **inference** ⇒ note: Imp(0,a) = 1 (axiom 3)

   S_{vl}: $C'_{vs}(y)$ = Imp(0 , $C_{vs}(y)$)

   S_l: $C'_s(y)$ = Imp(0, $C_s(y)$)

   S_m: $C'_n(y)$ = Imp(¾, $C_n(y)$)

   S_h: $C'_m(y)$ = Imp(¼ , $C_m(y)$)

   S_{vh}: $C'_{vm}(y)$ = Imp(0 , $C_{vm}(y)$)

Must we replace logical Imp() by technical relation?

example: (continued)

industrial drill machine → control of cooling supply

in case of control task typically **no logic-based interpretation**:

→ max-aggregation and

→ relation R(x,y) not interpreted as implication.

often: R(x,y) = min(A(x), B(y)) „Mamdani controller"

2. FITA: local **inference**

   S_{vl}: $C'_{vs}(y)$ = min(0, $C_{vs}(y)$) = 0

   S_l: $C'_s(y)$ = min(0, $C_s(y)$) = 0

   S_m: $C'_n(y)$ = min(¾, $C_n(y)$) ≥ 0

   S_h: $C'_m(y)$ = min(¼, $C_m(y)$) ≥ 0

   S_{vh}: $C'_{vm}(y)$ = min(0, $C_{vm}(y)$) = 0

   ⇒ since min(0,a) = 0 and max-aggr. we only need to consider C_n and C_m

example: (continued)

industrial drill machine → control of cooling supply

3. **aggregation**:

   $C'(y)$ = **aggr** { $C'_n(y)$, $C'_m(y)$ } = max { min(¾, $C_n(y)$), min(¼, $C_m(y)$) }

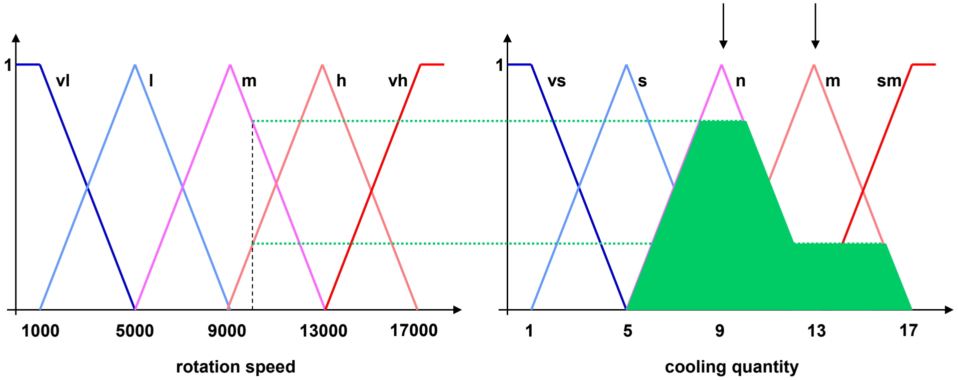
Remark:
This approach can be applied with every t-norm and max-aggregation
⇒ $C'(y)$ = max { t(¾, $C_n(y)$), t(¼, $C_m(y)$) }

→ graphical illustration

example: (continued)

industrial drill machine → control of cooling supply

$C'(y) = \max \{ \min \{ ¾, C_n(y) \}, \min \{ ¼, C_m(y) \} \}$, x_0 = 10 000 [rpm]



open and closed loop control:

affect the dynamical behavior of a system
in a desired manner

- **open loop control**

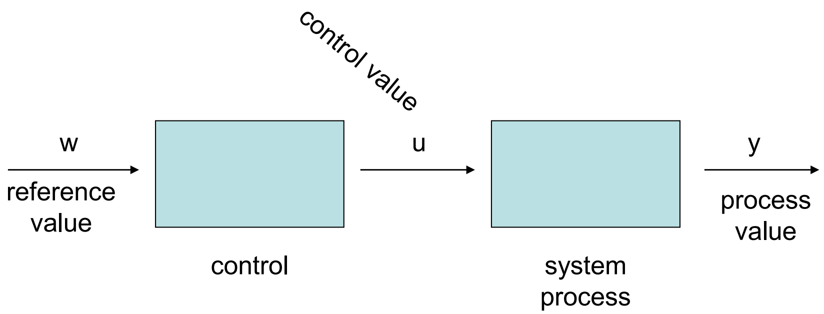
  control is aware of reference values and has a model of the system
  ⇒ control values can be adjusted,
  such that process value of system is equal to reference value

  problem: noise! ⇒ deviation from reference value not detected

- **closed loop control**

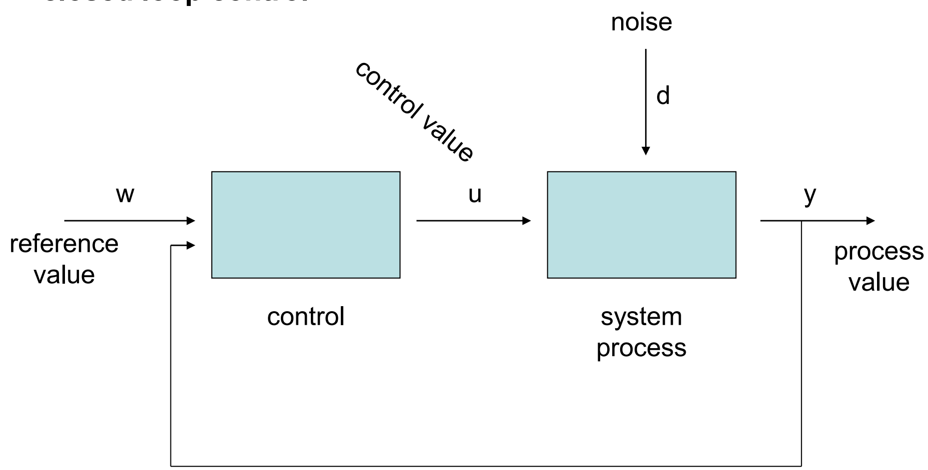
  now: detection of deviations from reference value possible
  (by means of measurements / sensors)
  and new control values can take into account the amount of deviation

open loop control



assumption: undisturbed operation ⇒ process value = reference value

closed loop control



control deviation = reference value – process value

required:

model of system / process

→ as differential equations or difference equations (DEs)

→ well developed theory available

so, why fuzzy control?

- if there exists no process model in form of DEs etc. (operator/human being has realized control by hand)
- if process with high-dimensional nonlinearities → no classic methods available
- if control goals are vaguely formulated („soft“ changing gears in cars)

fuzzy description of control behavior

IF X is A_1, THEN Y is B_1
IF X is A_2, THEN Y is B_2
IF X is A_3, THEN Y is B_3
…
IF X is A_n, THEN Y is B_n
X is A‘
―――――――
Y is B‘

} similar to approximative reasoning

but fact A‘ is not a fuzzy set but a crisp input

→ actually, it is the current process value

fuzzy controller executes inference step

→ yields fuzzy output set B‘(y)

but crisp control value required for the process / system

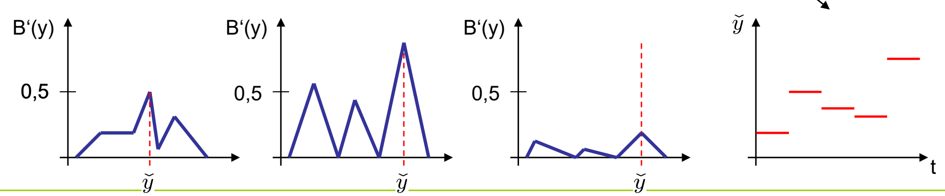
→ defuzzification (= “condense” fuzzy set to crisp value)

defuzzification

Def: rule k active ⇔ $A_k(x_0) > 0$

- maximum method
  - only active rule with largest activation level is taken into account
    - → suitable for pattern recognition / classification
    - → decision for a single alternative among finitely many alternatives
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - if used for control: discontinuous curve of output values (leaps)

$$\breve{y} = \text{argmax } B'(y)$$

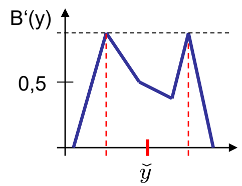


defuzzification

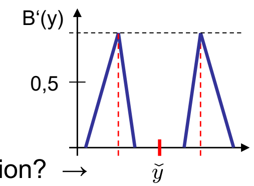
$Y^* = \{ y \in Y: B‘(y) = hgt(B‘) \}$

- maximum mean value method
  - all active rules with largest activation level are taken into account
    - → interpolations possible, but need not be useful
    - → obviously, only useful for neighboring rules with max. activation
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - if used in control: incontinuous curve of output values (leaps)

$$\breve{y} = \frac{1}{|Y^*|} \sum_{y^* \in Y^*} y^*$$



useful solution? →

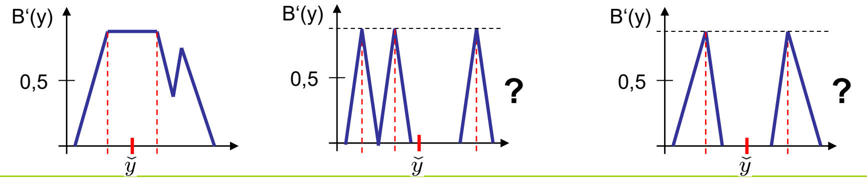


defuzzification

Y* = { y ∈ Y: B‘(y) = hgt(B‘) }

- center-of-maxima method (COM)
  - only **extreme** active rules with largest activation level are taken into account
    - → interpolations possible, but need not be useful
    - → obviously, only useful for neighboring rules with max. activation level
  - selection indepependent from activation level of rule (0.05 vs. 0.95)
  - in case of control: incontinuous curve of output values (leaps)

$$\breve{y} = \frac{\inf Y^* + \sup Y^*}{2}$$



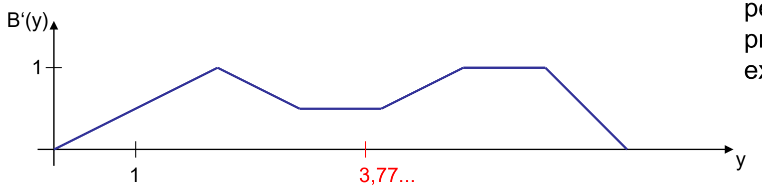
defuzzification

- Center of Gravity (COG)
  - all active rules are taken into account
    - → but numerically expensive … …only valid for HW solution, today!
    - → borders cannot appear in output (∃ work-around)
  - if only single active rule: independent from activation level
  - continuous curve for output values

$$\breve{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$

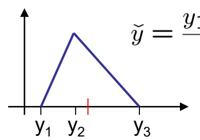
Excursion: COG

$$\breve{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$



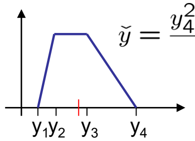
pendant in probability theory: expectation value

triangle:

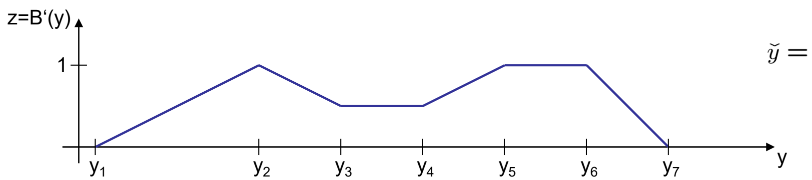


$$\breve{y} = \frac{y_1 + y_2 + y_3}{3}$$

trapezoid:



$$\breve{y} = \frac{y_4^2 + y_3^2 - y_2^2 - y_1^2 + y_3 y_4 - y_1 y_2}{3\,(y_4 + y_3 - y_2 - y_1)}$$



$$\breve{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$

assumption: fuzzy membership functions piecewise linear

output set B‘(y) represented by sequence of points (y_1, z_1), (y_2, z_2), …, (y_n, z_n)

⇒ area under B‘(y) and weighted area can be determined additively piece by piece

⇒ linear equation z = m y + b → insert (y_i, z_i) and (y_{i+1}, z_{i+1})

⇒ yields m and b for each of the n-1 linear sections

$$\Rightarrow F_i = \int_{y_i}^{y_{i+1}} (m\,y + b)\, dy = \frac{m}{2}(y_{i+1}^2 - y_i^2) + b(y_{i+1} - y_i)$$

$$\Rightarrow G_i = \int_{y_i}^{y_{i+1}} y\,(m\,y + b)\, dy = \frac{m}{3}(y_{i+1}^3 - y_i^3) + \frac{b}{2}(y_{i+1}^2 - y_i^2)$$

$$\breve{y} = \frac{\sum_i G_i}{\sum_i F_i}$$

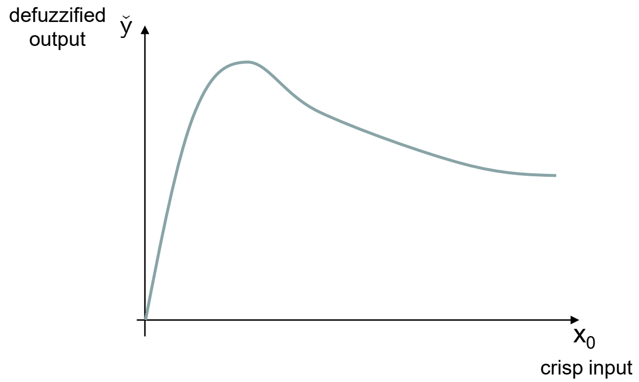
Defuzzification

- Center of Area (COA)
  - developed as an approximation of COG
  - let $\hat{y}_k$ be the COGs of the output sets $B'_k(y)$:

$$\breve{y} = \frac{\sum_k A_k(x_0) \cdot \hat{y}_k}{\sum_k A_k(x_0)}$$

how to:
assume that fuzzy sets $A_k(x)$ and $B_k(x)$ are triangles or trapezoids
let x_0 be the crisp input value
for each fuzzy rule "IF A_k is X THEN B_k is Y"
determine $B'_k(y) = R(A_k(x_0), B_k(y))$, where R(.,.) is the relation
find $\hat{y}_k$ as center of gravity of $B'_k(y)$

Putting all together:



→ map controller (in german: *Kennfeldregler*)