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dortmund

Computational Intelligence

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Prof. Dr. Günter Rudolph

Computational Intelligence

Fakultät für Informatik

TU Dortmund

Plan for Today

- Fuzzy relations
- Fuzzy logic
  - Linguistic variables and terms
  - Inference from fuzzy statements

relations with conventional sets $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!

$\Rightarrow$ all set operations remain valid!

crisp membership function (of x to relation R)

$$R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \ldots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition

Fuzzy relation = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$ ■

→ each tuple (x_1, ..., x_n) has a degree of membership to relation

→ degree of membership expresses
strength of relationship between elements of tuple

appropriate representation: n-dimensional membership matrix

example: Let X = { Bejing, New York, Dortmund } and Y = { New York, Paris }.

relation R = "very far away"

membership matrix →

relation R	New York	Paris
Bejing	1.0	0.9
New York	0.0	0.7
Dortmund	0.6	0.3

Definition

Let R(X, Y) be a fuzzy relation with membership matrix $\mathbf{R}$. The ***inverse fuzzy relation*** to R(X,Y), denoted R^{-1}(Y, X), is a relation on Y x X with membership matrix $\mathbf{R}'$. ■

Remark: $\mathbf{R}'$ is the transpose of membership matrix $\mathbf{R}$.

Evidently: $(R^{-1})^{-1} = R$ since $(\mathbf{R}')' = \mathbf{R}$

Definition

Let P(X, Y) and Q(Y, Z) be fuzzy relations. The operation ∘ on two relations, denoted P(X, Y) ∘ Q(Y, Z), is termed ***max-min-composition*** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$

■

Theorem

a) max-min composition on relations is associative.

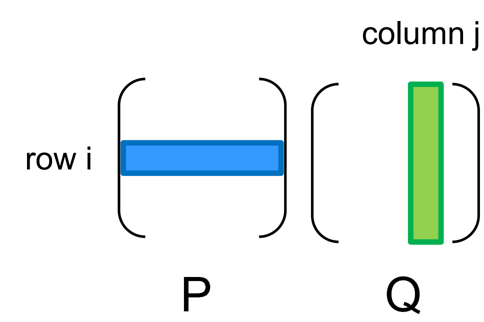
b) max-min composition on relations is not commutative.

c) $(P(X,Y) \circ Q(Y,Z))^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y, X)$.

membership matrix of max-min composition
determinable via “fuzzy matrix multiplication”: $R = P \circ Q$

fuzzy matrix multiplication $$r_{ij} = \max_k \min\{p_{ik}, q_{kj}\}$$

crisp matrix multiplication $$r_{ij} = \sum_k p_{ik} \cdot q_{kj}$$



further methods for realizing compositions of relations:

max-prod composition

$$(P \odot Q)(x,z) = \max_{y \in \mathcal{Y}} \{ P(x,y) \cdot Q(y,z) \}$$

<u>generalization:</u> sup-t composition

$$(P \circ Q)(x,z) = \sup_{y \in \mathcal{Y}} \{ t(P(x,y), Q(y,z)) \},$$ where t(.,.) is a t-norm

e.g.:	t(a,b) = min{a, b}	$\Rightarrow$ max-min-composition
	t(a,b) = a · b	$\Rightarrow$ max-prod-composition

Binary fuzzy relations on X x X : properties

- **reflexive** $\Leftrightarrow \forall\, x \in X : R(x,x) = 1$
- **irreflexive** $\Leftrightarrow \exists\, x \in X : R(x,x) < 1$
- **antireflexive** $\Leftrightarrow \forall\, x \in X : R(x,x) < 1$

- **symmetric** $\Leftrightarrow \forall\, (x,y) \in X \times X : R(x,y) = R(y,x)$
- **asymmetric** $\Leftrightarrow \exists\, (x,y) \in X \times X : R(x,y) \neq R(y,x)$
- **antisymmetric** $\Leftrightarrow \forall\, (x,y) \in X \times X : R(x,y) \neq R(y,x)$

- **transitive** $\Leftrightarrow \forall\, (x,z) \in X \times X : R(x,z) \geq \max\limits_{y \in X} \min \{ R(x,y), R(y,z) \}$
- **intransitive** $\Leftrightarrow \exists\, (x,z) \in X \times X : R(x,z) < \max\limits_{y \in X} \min \{ R(x,y), R(y,z) \}$
- **antitransitive** $\Leftrightarrow \forall\, (x,z) \in X \times X : R(x,z) < \max\limits_{y \in X} \min \{ R(x,y), R(y,z) \}$

actually, here: max-min-transitivity (→ in general: sup-t-transitivity)

binary fuzzy relation on X x X: <u>example</u>

Let **X** be a subset of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to“.

- R(x,x) = 1, since every city is certainly very close to itself.

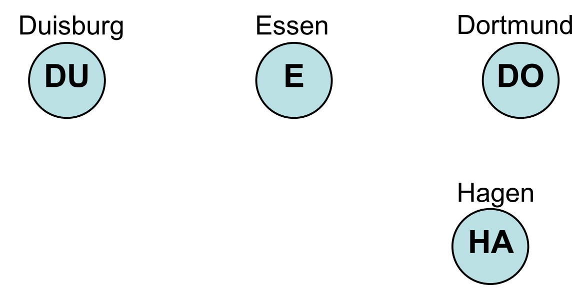
  ⇒ **reflexive**

- R(x,y) = R(y,x): if city x is very close to city y, then also vice versa.

  ⇒ **symmetric**

-

R	DU	E	DO	HA
DU	1	0.7	0.5	0.4
E	0.7	1	0.8	0.8
DO	0.5	0.8	1	0.9
HA	0.4	0.8	0.9	1



$R(DO,DU) = 0.5 < \max_y \min\{R(DO,y), R(y, DU)\} = 0.7$

$R(E\ \ ,DO) = 0.8 \geq \max_y \min\{R(E\ \ ,y), R(y, DO)\} = 0.8$

⇒ **intransitive**

crisp:
relation R is equivalence relation , R reflexive, symmetric, transitive

fuzzy:
relation R is similarity relation , R reflexive, symmetric, (max-min-) transitive

examples:

- *equivalence relation*: farm animals
  cattle, pigs, chicken, …
  R(cow, ox) = 1 but R(cow, hen) = 0

- *similarity relation*: farm animals
  cattle, pigs, chicken, horse, donkey, mule, …
  R(mule, (male) donkey) = 0.5 and R(mule, (female) horse) = 0.5

What about **hybrids**?

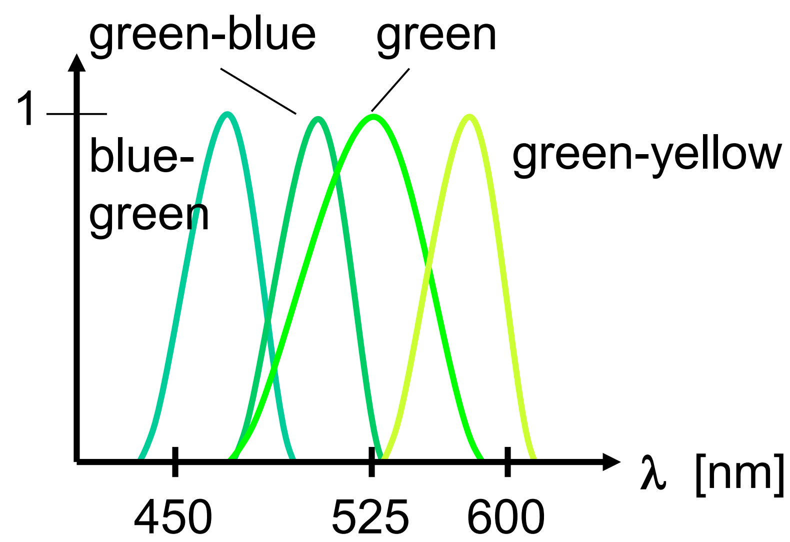
linguistic variable:

variable that can attain several values of lingustic / verbal nature

e.g.: **color** can attain values **red**, **green**, **blue**, **yellow**, …

values (red, green, …) of linguistic variable are called **linguistic terms**

linguistic terms are associated with fuzzy sets



fuzzy proposition

p: *temperature* is *high*

linguistic variable (LV) linguistic term (LT)

- LV may be associated with several LT : *high, medium, low, …*
- *high, medium, low* temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given **concrete crisp** temperature value v is interpreted as equal to the degree of membership *high*(v) of the fuzzy set *high*

fuzzy proposition

p: *V* is F

linguistic variable (LV)

linguistic term (LT)

actually:

p: *V* is F(v)

and

T(p) = F(v) for a concrete crisp value v

trueness(p)

establishes connection between *degree of membership* of a fuzzy set and the *degree of trueness* of a fuzzy proposition

fuzzy proposition

p: IF *heating* is *hot,* THEN *energy consumption* is *high*

LV LT LV LT

expresses <u>relation</u> between

a) temperature of heating and

b) quantity of energy consumption

relation

p: (*heating, energy consumption*) $\in$ R

fuzzy proposition

p: IF *X* is A, THEN *Y* is B

LV (for *X*), LT (for A), LV (for *Y*), LT (for B)

How can we determine / express degree of trueness T(p) ?

- For crisp, given values x, y we know A(x) and B(y)
- A(x) and B(y) must be processed to single value via relation R
- R(x, y) = function(A(x), B(y)) is fuzzy set over X x Y
- as before: interprete T(p) as degree of membership R(x,y)

fuzzy proposition

p: IF X is A, THEN Y is B

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over X x Y

$\forall\ (x,y) \in X \times Y$: $R(x, y) = Imp(\ A(x), B(y)\ )$

What is Imp(·,·) ?

$\Rightarrow$ „appropriate“ fuzzy implication $[0,1] \times [0,1] \rightarrow [0,1]$

assumption: we know an „appropriate“ Imp(a,b).

How can we determine the *degree of trueness* T(p) ?

example: (discrete case)

let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets

A:

x_1	x_2
0.5	0.9

B:

y_1	y_2	y_3
0.1	0.8	0.4

$\Rightarrow$

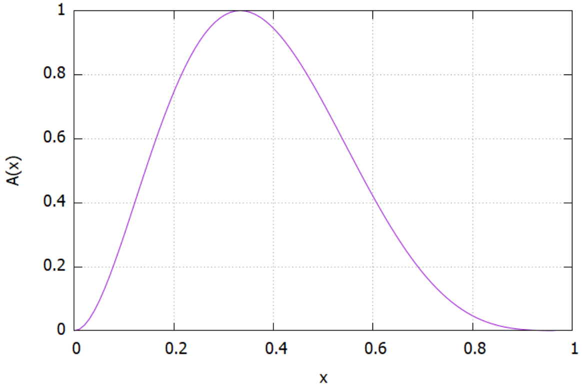
R	y_1	y_2	y_3
x_1	0.6	1.0	0.9
x_2	0.2	0.9	0.5

z.B.
$R(x_2, y_1) = \text{Imp}(A(x_2), B(y_1)) = \text{Imp}(0.9, 0.1) =$
$\min\{1.0, 1.0 - 0.9 + 0.1\} = 0.2$

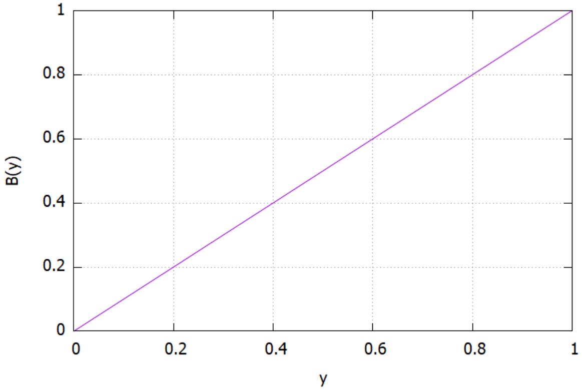
and T(p) for (x_2, y_1) is $R(x_2, y_1) = 0.2$ ■

example: (continuous case)

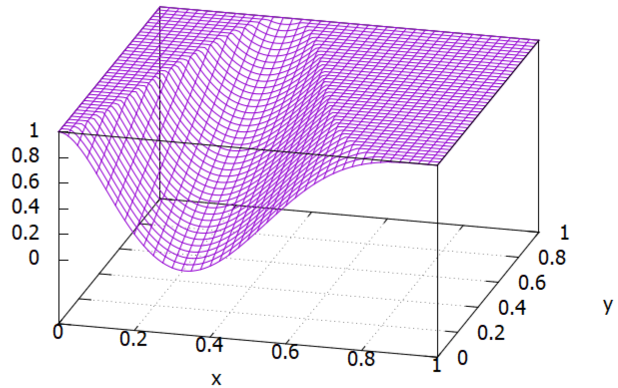
let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets



$$A(x) = \frac{729}{16}\, x^2\, (1-x)^4 \text{ for } x \in [0,1]$$



$$B(y) = y \text{ for } y \in [0,1]$$



$$\Rightarrow\quad R(x,y) = \min\{1, 1 - A(x) + B(y)\} = \min\{1, 1 - \frac{789}{16}\, x^2\, (1-x)^4 + y\}$$

toward inference from fuzzy statements:

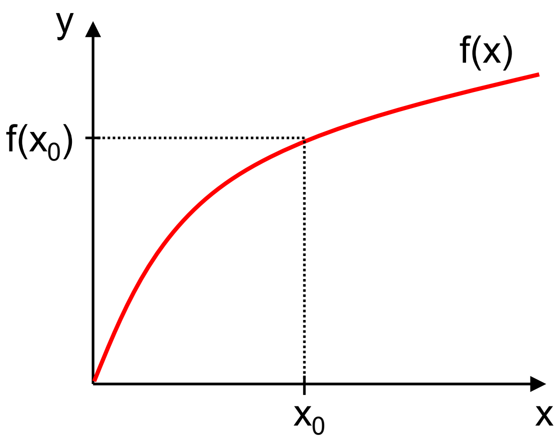
- let R = { (x, y) : y = f(x) } for a function f: $\mathbb{R} \to \mathbb{R}$

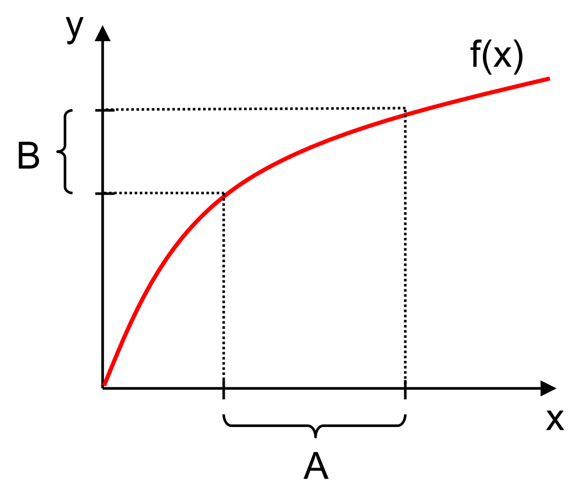
  IF X = { x_0 } THEN Y = { $f(x_0)$ }

- IF X $\in$ A THEN Y $\in$ B = { y $\in \mathcal{Y}$: y = f(x), x $\in$ A }

crisp case:

functional relationship





toward inference from fuzzy statements:

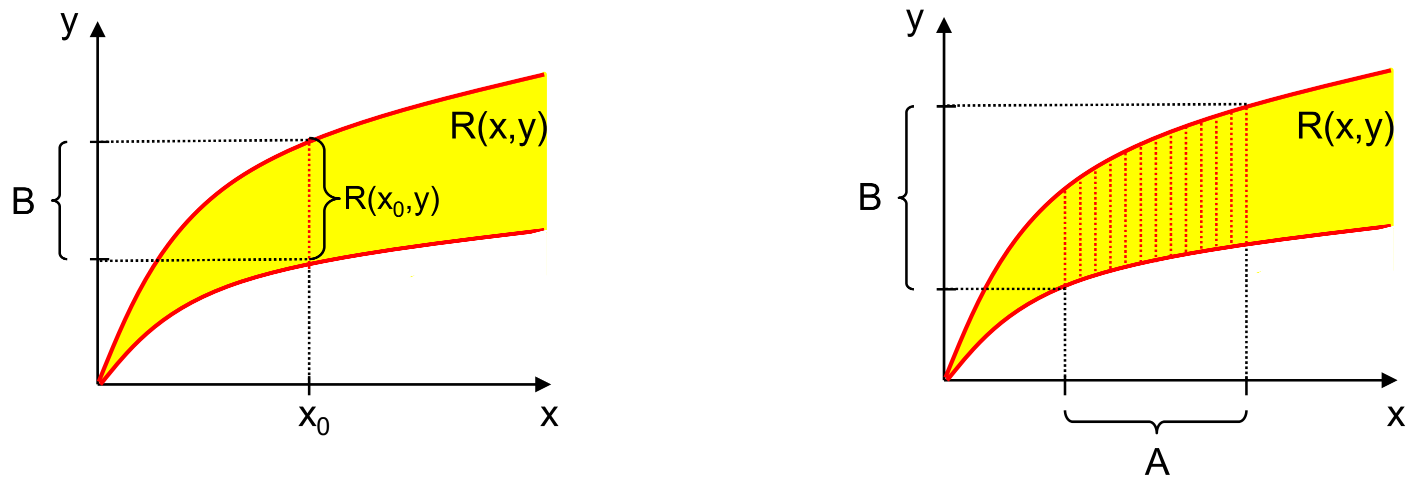
- let relationship between x and y be a relation R on $\mathcal{X} \times \mathcal{Y}$

  IF X = x_0 THEN Y $\in$ B = $\{ y \in \mathcal{Y} : (x_0, y) \in R \}$

- IF X $\in$ A THEN Y $\in$ B = $\{ y \in \mathcal{Y} : (x, y) \in R, \; x \in A \}$

crisp case:

relational relationship



toward inference from fuzzy statements:

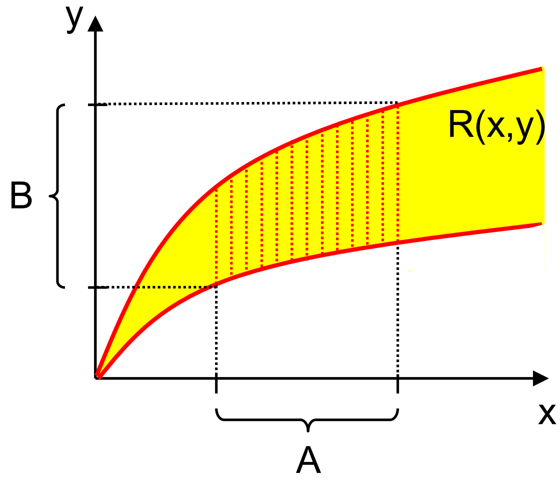
IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y} : (x, y) \in R, \ x \in A \}$

also expressible via characteristic functions of sets A, B, R:

$B(y) = 1$ iff $\exists x$: $A(x) = 1$ and $R(x, y) = 1$

$\Leftrightarrow \ \exists x$: $\min\{ A(x), R(x, y) \} = 1$

$\Leftrightarrow \ \max_{x \in \mathcal{X}} \min\{ A(x), R(x, y) \} = 1$



$$\forall y \in \mathcal{Y} : B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$$

inference from fuzzy statements

Note:
A'(x) is **not** the derivative of A(x)!
It is the membership function
of fuzzy set A'.

Now: A', B' fuzzy sets over $\mathcal{X}$ resp. $\mathcal{Y}$

Assume: R(x,y) and A'(x) are given.

Idea: Generalize characteristic function of B(y) to membership function B'(y)

$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$$

characteristic functions

$$\downarrow$$

$$\forall y \in \mathcal{Y}: B'(y) = \sup_{x \in \mathcal{X}} \min \{ A'(x), R(x, y) \}$$

membership functions

composition rule of inference (in matrix form): $\mathbf{B^T = A \circ R}$

inference from fuzzy statements

- conventional:
  modus ponens

$$\frac{\begin{array}{l} a \Rightarrow b \\ a \end{array}}{b}$$

- fuzzy:
  generalized modus ponens (GMP)

$$\frac{\begin{array}{l} \text{IF } X \text{ is A, THEN } Y \text{ is B} \\ X \text{ is A'} \end{array}}{Y \text{ is B'}}$$

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
heating is warm

---

energy consumption is normal

example: GMP

consider A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

$\Rightarrow$

R	y_1	y_2
x_1	1.0	0.9
x_2	1.0	0.4
x_3	1.0	0.8

with the rule: IF *X* is A THEN *Y* is B using Imp(a,b) = min{1, 1-a+b }

given fact A‘:

x_1	x_2	x_3
0.6	0.9	0.7

thus: A‘ ∘ R = B‘

with max-min-composition

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

■

inference from fuzzy statements

- conventional:
  modus tollens

$$\begin{array}{l} a \Rightarrow b \\ \overline{b} \\ \hline \overline{a} \end{array}$$

- fuzzy:
  generalized modus tollens (GMT)

$$\begin{array}{l} \text{IF } X \text{ is A, THEN } Y \text{ is B} \\ Y \text{ is B'} \\ \hline X \text{ is A'} \end{array}$$

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
energy consumption is normal

---

heating is warm

example: GMT

consider

A:

x_1	x_2	x_3
0.5	1.0	0.6

B:

y_1	y_2
1.0	0.4

with the rule: IF *X* is A THEN *Y* is B   using Imp(a,b) = min{1, 1-a+b }

$\Downarrow$

given fact

B':

y_1	y_2
0.9	0.7

R'	x_1	x_2	x_3
y_1	1.0	1.0	1.0
y_2	0.9	0.4	0.8

thus: $B' \circ R^{-1} = A'$

$$\begin{pmatrix} 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.9 & 0.9 \end{pmatrix}$$

with max-min-composition

■

inference from fuzzy statements

- conventional:
  hypothetic syllogism

$$\frac{\begin{array}{l} a \Rightarrow b \\ b \Rightarrow c \end{array}}{a \Rightarrow c}$$

- fuzzy:
  generalized HS

$$\frac{\begin{array}{l} \text{IF } X \text{ is A, THEN } Y \text{ is B} \\ \text{IF } Y \text{ is B, THEN } Z \text{ is C} \end{array}}{\text{IF X is A, THEN Z is C}}$$

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
IF *energy consumption* is high, THEN *living* is expensive

IF *heating* is hot, THEN *living* is expensive

example: GHS

let fuzzy sets A(x), B(y), C(z) be given

$\Rightarrow$ determine the three relations according to Imp function

$R_1(x,y) = \text{Imp}(A(x),B(y))$
$R_2(y,z) = \text{Imp}(B(y),C(z))$
$R_3(x,z) = \text{Imp}(A(x),C(z))$

and express them as matrices R_1, R_2, R_3

We say:

GHS is valid if $R_1 \circ R_2 = R_3$

So, ... what makes sense for Imp(·,·) ?

Imp(a,b) ought to express fuzzy version of implication ($a \Rightarrow b$)

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b$

But how can we calculate with fuzzy "boolean" expressions?

request: must be compatible to crisp version (and more) for $a,b \in \{ 0, 1 \}$

a	b	$a \wedge b$	t(a,b)
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

a	b	$a \vee b$	s(a,b)
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

a	$\overline{a}$	c(a)
0	1	1
1	0	0

So, ... what makes sense for Imp(·,·) ?

1st approach: S implications

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b$

fuzzy: $\text{Imp}(a, b) = s(\, c(a), b)$

2nd approach: R implications

conventional: $a \Rightarrow b$ identical to $\max\{\, x \in \{0,1\}: \; a \wedge x \le b \,\}$

fuzzy: $\text{Imp}(a, b) = \max\{\, x \in [0,1] : t(a, x) \le b \,\}$

3rd approach: QL implications

conventional: $a \Rightarrow b$ identical to $\overline{a} \vee b \equiv \overline{a} \vee (a \wedge b)$ law of absorption

fuzzy: $\text{Imp}(a, b) = s(\, c(a), t(a, b)\,)$ (dual tripel ?)

example: S implication $\mathrm{Imp}(a, b) = s(\, c_s(a), b)$ (c_s : std. complement)

1. Kleene-Dienes implication

   $s(a, b) = \max\{ a, b \}$ (standard) $\mathrm{Imp}(a,b) = \max\{ 1-a, b \}$

2. Reichenbach implication

   $s(a, b) = a + b – ab$ (algebraic sum) $\mathrm{Imp}(a, b) = 1 – a + ab$

3. Łukasiewicz implication

   $s(a, b) = \min\{ 1, a + b \}$ (bounded sum) $\mathrm{Imp}(a, b) = \min\{ 1, 1 – a + b \}$

example: R implicationen Imp(a, b) = max{ x ∈[0,1] : t(a, x) ≤ b }

1. Gödel implication

   t(a, b) = min{ a, b } (std.) $\text{Imp}(a, b) = \begin{cases} 1 & \text{, if } a \leq b \\ b & \text{, else} \end{cases}$

2. Goguen implication

   t(a, b) = ab (algeb. product) $\text{Imp}(a, b) = \begin{cases} 1 & \text{, if } a \leq b \\ \frac{b}{a} & \text{, else} \end{cases}$

3. Łukasiewicz implication

   t(a, b) = max{ 0, a + b – 1 } (bounded diff.) Imp(a, b) = min{ 1, 1 – a + b }

example: QL implication $Imp(a, b) = s(\ c(a), t(a, b)\ )$

1. Zadeh implication

   $t(a, b) = \min \{ a, b \}$ (std.) $Imp(a, b) = \max\{ 1 – a, \min\{a, b\} \}$
   $s(a,b) = \max\{ a, b \}$ (std.)

2. „NN“ implication ☺ (Klir/Yuan 1994)

   $t(a, b) = ab$ (algebr. prd.) $Imp(a, b) = 1 – a + a^2b$
   $s(a,b) = a + b – ab$ (algebr. sum)

3. Kleene-Dienes implication

   $t(a, b) = \max\{ 0, a + b – 1 \}$ (bounded diff.) $Imp(a, b) = \max\{ 1-a, b \}$
   $s(a,b) = \min \{ 1, a + b)$ (bounded sum)

axioms for fuzzy implications

1.	$a \le b$ implies $\mathrm{Imp}(a, x) \ge \mathrm{Imp}(b, x)$	monotone in 1st argument
2.	$a \le b$ implies $\mathrm{Imp}(x, a) \le \mathrm{Imp}(x, b)$	monotone in 2nd argument
3.	$\mathrm{Imp}(0, a) = 1$	dominance of falseness
4.	$\mathrm{Imp}(1, b) = b$	neutrality of trueness
5.	$\mathrm{Imp}(a, a) = 1$	identity
6.	$\mathrm{Imp}(a, \mathrm{Imp}(b, x)) = \mathrm{Imp}(b, \mathrm{Imp}(a, x))$	exchange property
7.	$\mathrm{Imp}(a, b) = 1$ iff $a \le b$	boundary condition
8.	$\mathrm{Imp}(a, b) = \mathrm{Imp}(c(b), c(a))$	contraposition
9.	$\mathrm{Imp}(\cdot,\cdot)$ is continuous	continuity

Caution!

Not all S-, R-, QL- implications obey all axioms for fuzzy implications!

Implication	**Valid Axioms**
Kleene-Dienes	1 2 3 4 – 6 – 8 9
Reichenbach	1 2 3 4 – 6 – 8 9
Łukasiewicz	1 2 3 4 5 6 7 8 9 ←
Gödel	1 2 3 4 5 6 7 – –
Goguen	1 2 3 4 5 6 7 – 9
Zadeh	1 2 3 4 – – – – 9
Klir-Yuan	– 2 3 4 – – – – 9

characterization of fuzzy implication

Theorem:
Imp: $[0,1] \times [0,1] \to [0,1]$ satisfies axioms 1 - 9 for fuzzy implications for a certain fuzzy complement $c(\cdot)$ $\Leftrightarrow$

$\exists$ strictly monotone increasing, continuous function $f: [0,1] \to [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0,1]$: $\mathrm{Imp}(a, b) = f^{-1}(\min\{ f(1) - f(a) + f(b), f(1)\})$
- $\forall a \in [0,1]$: $c(a) = f^{-1}(f(1) - f(a))$

Proof: Smets & Magrez (1987), p. 337f. ■

examples: (in tutorial)

choosing an „appropriate“ fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem.“

guideline:
GMP, GMT, GHS should be compatible with MP, MT, HS
for fuzzy implication in calculations with relations:
$B(y) = \sup \{ t(A(x), \mathrm{Imp}(A(x), B(y))) : x \in X \}$

example:
Gödel implication for t-norm = bounded difference