technische universität dortmund

Computational Intelligence

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Computational Intelligence

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Plan for Today

- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Definition

A function $c: [0,1] \rightarrow [0,1]$ is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall\, a, b \in [0,1]: a \leq b \;\Rightarrow\; c(a) \geq c(b)$. — monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall\, a \in [0,1]: c(c(a)) = a$ — involutive

Examples:

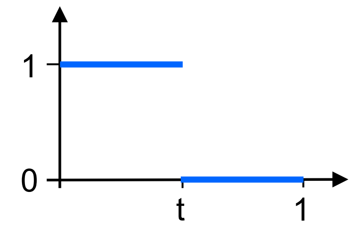
a) standard fuzzy complement $c(a) = 1 - a$

ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$
ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)

ad (A3): ☑
ad (A4): $1 - (1 - a) = a$

Fuzzy Complement: Examples

b) $c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$



ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t

☑

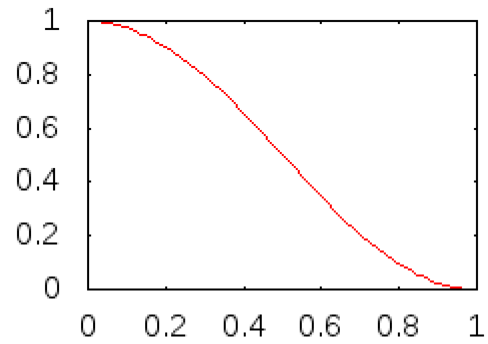
ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(¼)) = c(1) = 0 \neq ¼$ for $t = ½$

Fuzzy Complement: Examples

c) $c(a) = \dfrac{1 + \cos(\pi\, a)}{2}$



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c'(a) = -\frac{1}{2}\pi \sin(\pi\, a) < 0$ since $\sin(\pi\, a) > 0$ for $a \in (0,1)$

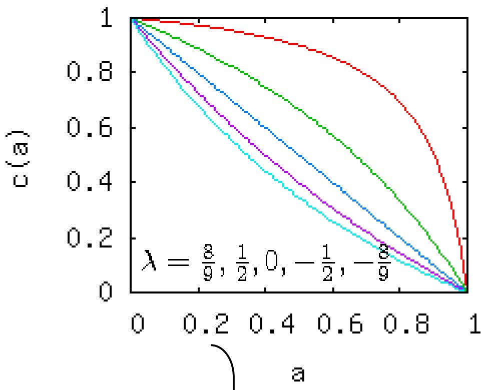
☑

ad (A3): is continuous as a composition of continuous functions;
alternative argument: derivative exists, see c'(a) in (A2)

ad (A4): not valid → counter example

$$c\left(c\left(\frac{1}{3}\right)\right) = c\left(\frac{3}{4}\right) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \neq \frac{1}{3}$$

d) c(a) = $\frac{1-a}{1+\lambda a}$ for $\lambda > -1$ **Sugeno class**



ad (A1): c(0) = 1 and c(1) = 0

ad (A2): $c(a) \geq c(b) \Leftrightarrow \frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b} \Leftrightarrow$

$$(1-a)(1+\lambda b) \geq (1-b)(1+\lambda a) \Leftrightarrow$$

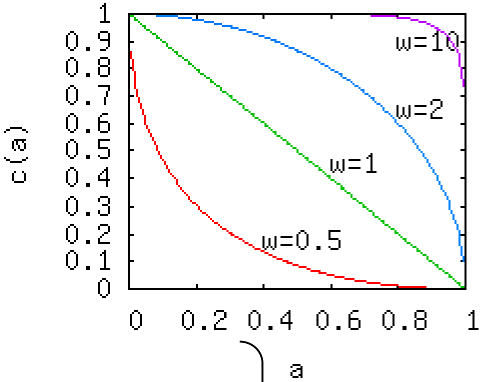
$$b(\lambda+1) \geq a(\lambda+1) \Leftrightarrow b \geq a$$

☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right) = \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda \frac{1-a}{1+\lambda a}} = \frac{a(\lambda+1)}{\lambda+1} = a$

☑



e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$

$a^w \leq b^w \Leftrightarrow a \leq b$ ☑

ad (A3): is continuous as a composition of continuous functions

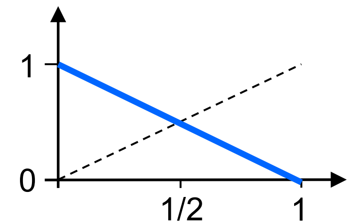
ad (A4):
$$c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$$
$$= (1-(1-a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$$
☑

Theorem

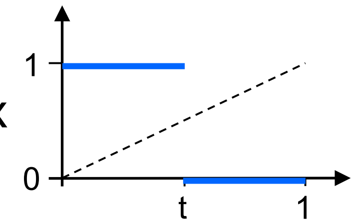
If function $c:[0,1] \to [0,1]$ satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a^* with $c(a^*) = a^*$.

Proof:

one fixed point → see example (a) → intersection with bisectrix



no fixed point → see example (b) → no intersection with bisectrix



assume ∃ n > 1 fixed points, for example a^* and b^* with $a^* < b^*$

$\Rightarrow c(a^*) = a^*$ and $c(b^*) = b^*$ (fixed points)

$\Rightarrow c(a^*) < c(b^*)$ with $a^* < b^*$ impossible if $c(\cdot)$ is monotone decreasing

$\Rightarrow$ contradiction to axiom (A2) ■

Theorem

If function $c:[0,1] \to [0,1]$ satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a^* with $c(a^*) = a^*$.

Proof:

Intermediate value theorem →

If $c(\cdot)$ continuous (A3) and $c(0) \geq c(1)$ (A1/A2)

then $\forall\ v \in [c(1), c(0)] = [0,1]: \exists\ a \in [0,1]: c(a) = v$.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points! ■

Examples:

(a) $c(a) = 1 - a$ ⇒ $a = 1 - a$ ⇒ $a^* = ½$

(b) $c(a) = (1 - a^w)^{1/w}$ ⇒ $a = (1 - a^w)^{1/w}$ ⇒ $a^* = (½)^{1/w}$

Theorem

c: $[0,1] \to [0,1]$ is involutive fuzzy complement iff

$\exists$continuous function g: $[0,1] \to \mathbb{R}$ with

- $g(0) = 0$
- strictly monotone increasing
- $\forall\, a \in [0,1]$: $c(a) = g^{(-1)}(\, g(1) - g(a)\,)$. ■

defines an **increasing generator**

$g^{(-1)}(x)$ pseudo-inverse

$= g^{-1}(\,\min\{\, g(1), x\,\}\,)$

Examples

a) $g(x) = x \quad \Rightarrow g^{(-1)}(x) = x \quad \Rightarrow c(a) = 1 - a$ (Standard)

b) $g(x) = x^w \quad \Rightarrow g^{(-1)}(x) = x^{1/w} \quad \Rightarrow c(a) = (1 - a^w)^{1/w}$ (Yager class, w > 0)

c) $g(x) = \log(x+1) \Rightarrow \underbrace{g^{(-1)}(x) = e^x - 1}_{?} \Rightarrow c(a) = \exp(\,\log(2) - \log(a+1)\,) - 1$

$= \dfrac{1-a}{1+a}$ (Sugeno class. $\lambda = 1$)

→ make sure that pseudoinverse is equal to inverse, here!

$$g(x) = \log(x+1) \quad \rightarrow \quad g^{-1}(x) = e^x - 1$$ ☑ (inverse)

$$g^{(-1)}(x) = g^{-1}(\min\{g(1), \underset{?}{\underset{\downarrow}{x}}\})$$ (pseudoinverse)

$$c(a) = g^{(-1)}(\underbrace{g(1) - g(a)}_{= x})$$

$\min\{g(1), g(1) - a\} = g(1) - g(a) \le g(1)$ since $0 \le g(a) \le \log 2$ for $a \in [0,1]$

therefore,

$$c(a) = g^{(-1)}(g(1) - g(a)) = g^{-1}(g(1) - g(a))$$ ☑

Examples

d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda\, a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda\, a} > 0$ for $a \in [0, 1]$
- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda\, a)-1}{\lambda}$, thus

$$
\begin{aligned}
c(a) &= g^{-1}\left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda\, a)}{\lambda}\right) \\
&= \frac{\exp(\log(1+\lambda) - \log(1+\lambda\, a)) - 1}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda\, a} - 1\right) = \frac{1-a}{1+\lambda\, a} \quad \text{(Sugeno Complement)}
\end{aligned}
$$

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃continuous function f: [0,1] → $\mathbb{R}$ with

- f(1) = 0
- strictly monotone decreasing
- ∀ a ∈ [0,1]: c(a) = $f^{(-1)}$(f(0) – f(a)). ■

} defines a **decreasing generator**

$f^{(-1)}(x)$ pseudo-inverse

= $f^{-1}(\min\{ f(0), x \})$

Examples

a) f(x) = $k - k \cdot x$ ($k \geq 1$) $f^{(-1)}(x) = 1 - x/k$ c(a) = $1 - \frac{k-(k-ka)}{k} = 1 - a$

b) f(x) = $1 - x^w$ $f^{(-1)}(x) = (1 - x)^{1/w}$ c(a) = $f^{-1}(a^w) = (1 - a^w)^{1/w}$ (Yager)

Definition

A function $t:[0,1] \times [0,1] \to [0,1]$ is a ***fuzzy intersection*** or ***t-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $t(a, 1) = a$ (boundary condition)

(A2) $b \le d \Rightarrow t(a, b) \le t(a, d)$ (monotonicity)

(A3) $t(a,b) = t(b, a)$ (commutative)

(A4) $t(a, t(b, d)) = t(t(a, b), d)$ (associative) ■

"nice to have"

(A5) $t(a, b)$ is continuous (continuity)

(A6) $t(a, a) < a$ for $0 < a < 1$ (subidempotent)

(A7) $a_1 < a_2$ and $b_1 \le b_2 \Rightarrow t(a_1, b_1) < t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with t(a,a) = a for all a ∈ [0,1].

- If $0 \le a \le b \le 1$ then

$$a \underset{\text{by assumption}}{=} t(a,a) \underset{\text{by monotonicity}}{\le} t(a,b) \underset{\text{by monotonicity}}{\le} t(a,1) \underset{\text{by boundary condition}}{=} a$$

  and hence t(a,b) = a.

- If $0 \le b \le a \le 1$ then

$$b \underset{\text{by assumption}}{=} t(b,b) \underset{\text{by monotonicity}}{\le} t(b,a) \underset{\text{by monotonicity}}{\le} t(b,1) \underset{\text{by boundary condition}}{=} b$$

  and hence $t(a,b) \underset{\text{by commutativity}}{=} t(b,a) = b$.

t(a,b) = min(a,b)

is the only possible solution!

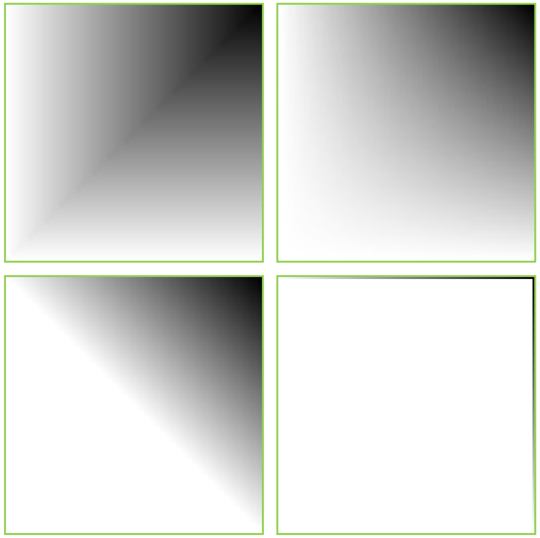
q.e.d.

Fuzzy Intersection: t-norm

Examples:

Name	Function
(a) Standard	t(a, b) = min { a, b }
(b) Algebraic Product	t(a, b) = a · b
(c) Bounded Difference	t(a, b) = max { 0, a + b – 1 }
(d) Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$

(a) (b)



(c) (d)

Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): t(a, 1) = a · 1 = a ☑

ad (A2): a · b ≤ a · d ⇔ b ≤ d ☑

ad (A3): t(a, b) = a · b = b · a = t(b, a) ☑

ad (A4): a · (b · d) = (a · b) · d ☑

Theorem

Function t: [0,1] x [0,1] → [0,1] is a t-norm ,

∃decreasing generator f:[0,1] → $\mathbb{R}$ with t(a, b) = f^{-1}(min{ f(0), f(a) + f(b) }). ■

Example:

f(x) = 1/x – 1 is decreasing generator since

- f(x) is continuous ☑
- f(1) = 1/1 – 1 = 0 ☑
- f'(x) = $-1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \frac{1}{x+1}$; f(0) = ∞ ⇒ min{ f(0), f(a) + f(b) } = f(a) + f(b)

$$\Rightarrow t(a, b) = f^{-1}\left(\frac{1}{a} + \frac{1}{b} - 2\right) = \frac{1}{\frac{1}{a}+\frac{1}{b}-1} = \frac{ab}{a+b-a\,b}$$

Definition

A function $s:[0,1] \times [0,1] \rightarrow [0,1]$ is a ***fuzzy union*** or ***s-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $s(a, 0) = a$ (boundary condition)

(A2) $b \le d \Rightarrow s(a, b) \le s(a, d)$ (monotonicity)

(A3) $s(a, b) = s(b, a)$ (commutative)

(A4) $s(a, s(b, d)) = s(s(a, b), d)$ (associative) ■

"nice to have"

(A5) $s(a, b)$ is continuous (continuity)

(A6) $s(a, a) > a$ for $0 < a < 1$ (superidempotent)

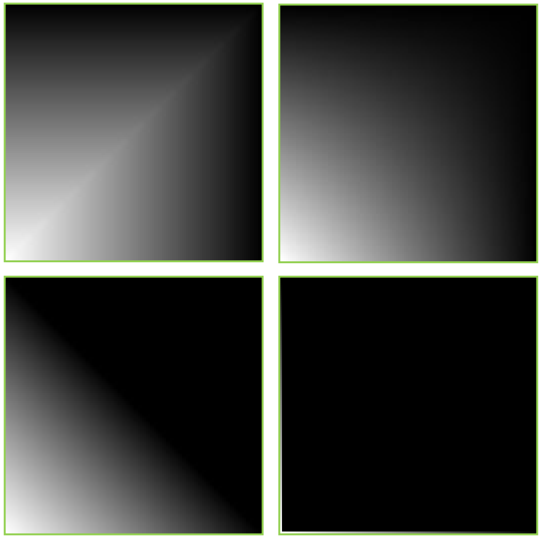
(A7) $a_1 < a_2$ and $b_1 \le b_2 \Rightarrow s(a_1, b_1) < s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	$s(a, b) = \max \{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min \{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$

(a) (b)



(c) (d)

Is algebraic sum an s-norm? Check the 4 axioms!

ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$ ☑

ad (A2): $a + b - a \cdot b \le a + d - a \cdot d \Leftrightarrow b\,(1 - a) \le d\,(1 - a) \Leftrightarrow b \le d$ ☑

ad (A3): ☑

ad (A4): ☑

Theorem

Function s: [0,1] x [0,1] → [0,1] is a s-norm ⇔

∃ increasing generator g:[0,1] → $\mathbb{R}$ with s(a, b) = g^{-1}(min{ g(1), g(a) + g(b) }). ■

Example:

g(x) = –log(1 – x) is increasing generator since

- g(x) is continuous ☑
- g(0) = –log(1 – 0) = 0 ☑
- g'(x) = 1/(1 – x) > 0 (monotone increasing) ☑

inverse function is $g^{-1}(x) = 1 - \exp(-x)$; g(1) = ∞ ⇒ min{g(1), g(a) + g(b)} = g(a) + g(b)

$$\Rightarrow s(a, b) = g^{-1}(-\log(1-a) - \log(1-b))$$
$$= 1 - \exp(\log(1-a) + \log(1-b))$$
$$= 1 - (1-a)(1-b) = a + b - a\,b \quad \text{(algebraic sum)}$$

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan‘s laws

Definition

A pair of t-norm t(·, ·) and s-norm s(·, ·) is said to be ***dual with regard to the fuzzy complement*** c(·) iff

- c(t(a, b)) = s(c(a), c(b))
- c(s(a, b)) = t(c(a), c(b))

for all a, b $\in$ [0,1]. ■

Definition

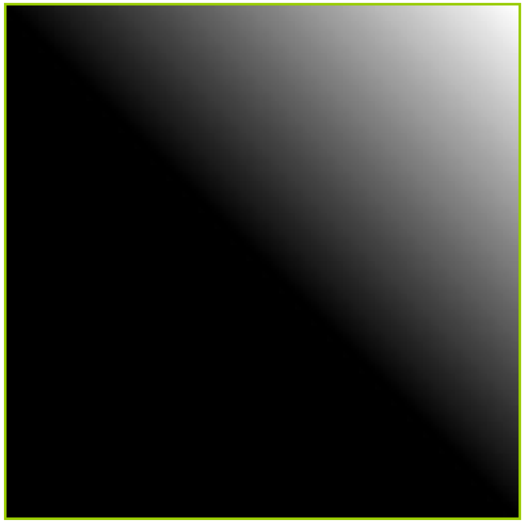
Let (c, s, t) be a tripel of fuzzy complement c(·), s- and t-norm.

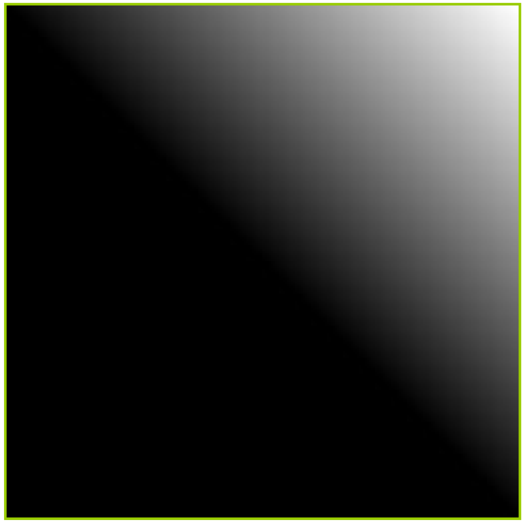
If t and s are dual to c then the tripel (c,s, t) is called a ***dual tripel***. ■

Examples of dual tripels

t-norm	s-norm	complement
min { a, b }	max { a, b }	1 – a
a · b	a + b – a · b	1 – a
max { 0, a + b – 1 }	min { 1, a + b }	1 – a

Dual Triples vs. Non-Dual Triples





c(t(a, b))

s(c(a), c(b))

Dual Triple:

- bounded difference
- bounded sum
- standard complement

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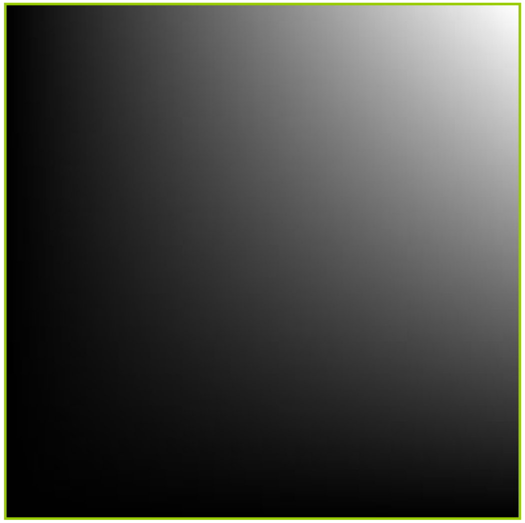

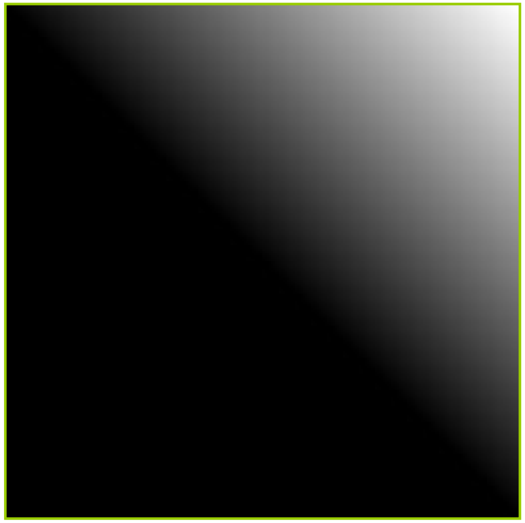

Non-Dual Triple:

- algebraic product
- bounded sum
- standard complement

⇒ left image ≠ right image

Why are dual triples so important?

⇒ allow equivalence transformations of fuzzy set expressions

⇒ required to transform into some equivalent normal form (standardized input)

⇒ e.g. two stages: intersection of unions $\bigcap_{i=1}^{n} (A_i \cup B_i)$

or union of intersections $\bigcup_{i=1}^{n} (A_i \cap B_i)$

Example:

$A \cup (B \cap (C \cap D)^c) =$	← not in normal form
$A \cup (B \cap (C^c \cup D^c)) =$	← equivalent if DeMorgan‘s law valid (dual triples!)
$A \cup (B \cap C^c) \cup (B \cap D^c)$	← equivalent (distributive lattice!)