

# Computational Intelligence

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Computational Intelligence

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- Fuzzy Sets
  - Basic Definitions and Results for Standard Operations
  - Algebraic Difference between Fuzzy and Crisp Sets

## Observation:

Communication between people is not precise but somehow fuzzy and vague.

“If the water is too hot then add a little bit of cold water.”

Despite these shortcomings in human language we are able

- to process fuzzy / uncertain information and
- to accomplish complex tasks!

## Goal:

Development of formal framework to process fuzzy statements in computer.

Consider the statement: “The water is hot.”

Which temperature defines “hot”?

A single temperature  $T = 95^\circ \text{C}$ ?

No! Rather, an interval of temperatures:  $T \in [70, 120]$  !

But who defines the limits of the intervals?

Some people regard temperatures  $> 60^\circ \text{C}$  as hot, others already  $T > 50^\circ \text{C}$ !

**Idea:** All people might agree that a temperature in the set  $[70, 120]$  defines a hot temperature!

If  $T = 65^\circ \text{C}$  not all people regard this as hot. It does not belong to  $[70, 120]$ .

But it is hot to some degree.

Or:  $T = 65^\circ \text{C}$  belongs to set of hot temperatures to some degree!

⇒ **Can be the concept for capturing fuzziness!** ⇒ **Formalize this concept!**

**Definition**

A map  $F: X \rightarrow [0,1] \subset \mathbb{R}$  that assigns its **degree of membership**  $F(x)$  to each  $x \in X$  is termed a **fuzzy set**.

**Remark:**

A fuzzy set  $F$  is actually a map  $F(x)$ . Shorthand notation is simply  $F$ .

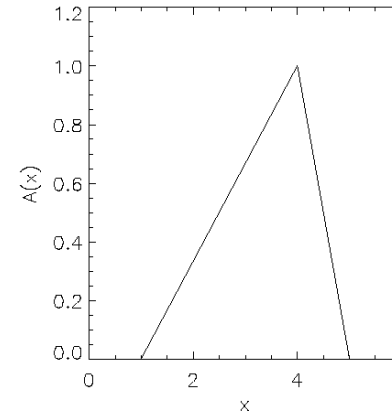
Same point of view possible for traditional (“**crisp**”) sets:

$$A(x) := 1_{[x \in A]} := 1_A(x) := \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases}$$

characteristic / indicator function of (crisp) set  $A$

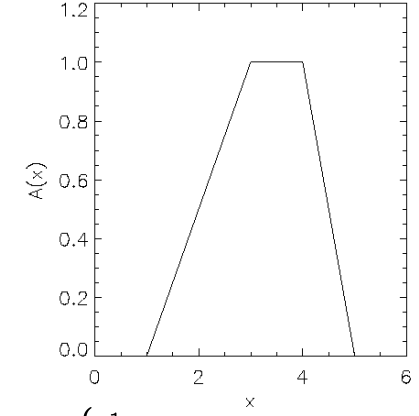
⇒ membership function interpreted as generalization of characteristic function

triangle function



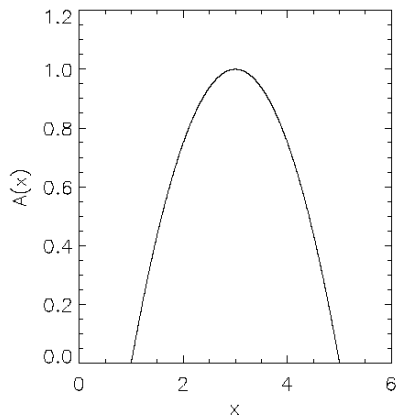
$$A(x) = \begin{cases} \frac{1}{3}(x - 1) & \text{if } 1 \leq x < 4 \\ 5 - x & \text{if } 4 \leq x < 5 \\ 0 & \text{otherwise} \end{cases}$$

trapezoidal



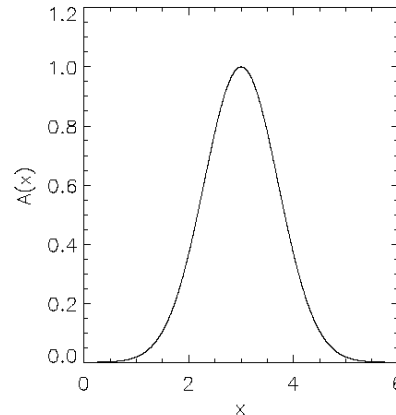
$$A(x) = \begin{cases} \frac{1}{2}(x - 1) & \text{if } 1 \leq x < 3 \\ 1 & \text{if } 3 \leq x < 4 \\ 5 - x & \text{if } 4 \leq x < 5 \\ 0 & \text{otherwise} \end{cases}$$

paraboloidal function



$$A(x) = \begin{cases} -\frac{(x-1)(x-5)}{4} & \text{if } 1 \leq x < 5 \\ 0 & \text{otherwise} \end{cases}$$

gaussoid function



$$A(x) = \exp\left(-\frac{(x-3)^2}{2}\right)$$

**Definition**

A fuzzy set  $F$  over the crisp set  $X$  is termed

- a) **empty** if  $F(x) = 0$  for all  $x \in X$ ,
- b) **universal** if  $F(x) = 1$  for all  $x \in X$ .

Empty fuzzy set is denoted by  $\emptyset$ . Universal set is denoted by  $\mathbb{U}$ . ■

**Definition**

Let  $A$  and  $B$  be fuzzy sets over the crisp set  $X$ .

- a)  $A$  and  $B$  are termed **equal**, denoted  $A = B$ , if  $A(x) = B(x)$  for all  $x \in X$ .
- b)  $A$  is a **subset** of  $B$ , denoted  $A \subseteq B$ , if  $A(x) \leq B(x)$  for all  $x \in X$ .
- c)  $A$  is a **strict subset** of  $B$ , denoted  $A \subset B$ , if  $A \subseteq B$  and  $\exists x \in X: A(x) < B(x)$ . ■

**Remark:** A strict subset is also called a **proper** subset.

**Theorem**

Let A, B and C be fuzzy sets over the crisp set X. The following relations are valid:

- a) reflexivity :  $A \subseteq A$ .
- b) antisymmetry :  $A \subseteq B$  and  $B \subseteq A \Rightarrow A = B$ .
- c) transitivity :  $A \subseteq B$  and  $B \subseteq C \Rightarrow A \subseteq C$ .

**Proof:** (via reduction to definitions and exploiting operations on crisp sets)

ad a)  $\forall x \in X: A(x) \leq A(x)$ .

ad b)  $\forall x \in X: A(x) \leq B(x)$  and  $B(x) \leq A(x) \Rightarrow A(x) = B(x)$ .

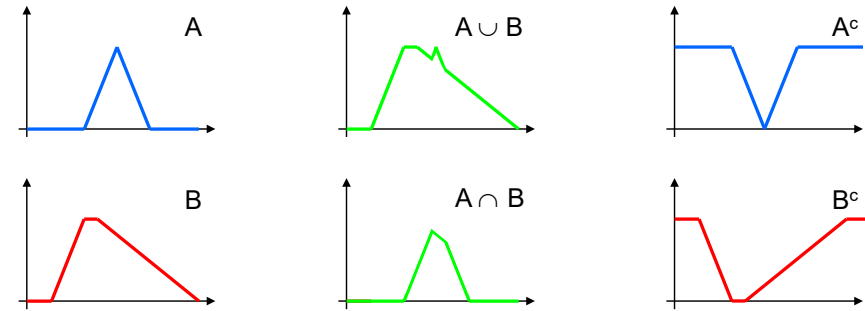
ad c)  $\forall x \in X: A(x) \leq B(x)$  and  $B(x) \leq C(x) \Rightarrow A(x) \leq C(x)$ . **q.e.d.**

**Remark:** Same relations valid for crisp sets. No Surprise! Why?

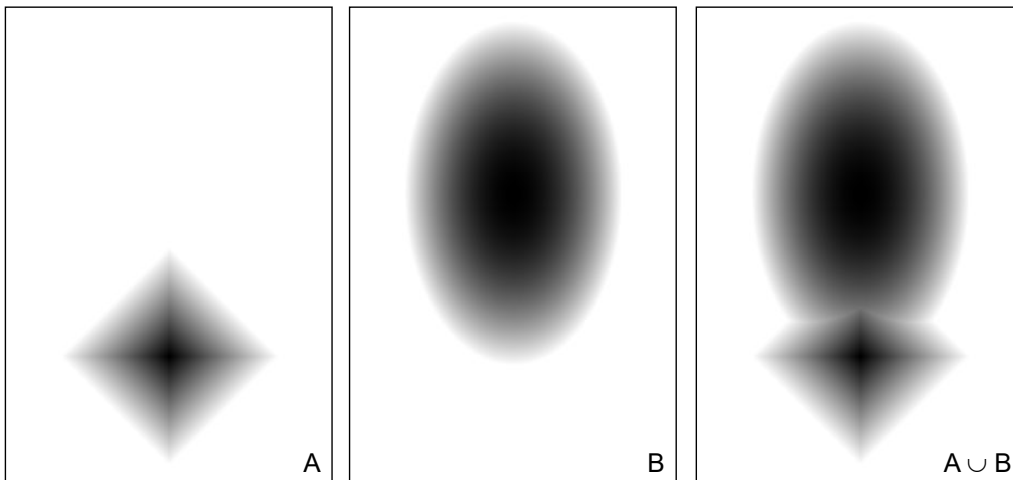
**Definition**

Let A and B be fuzzy sets over the crisp set X. The set C is the

- a) **union** of A and B, denoted  $C = A \cup B$ , if  $C(x) = \max\{A(x), B(x)\}$  for all  $x \in X$ ;
- b) **intersection** of A and B, denoted  $C = A \cap B$ , if  $C(x) = \min\{A(x), B(x)\}$  for all  $x \in X$ ;
- c) **complement** of A, denoted  $C = A^c$ , if  $C(x) = 1 - A(x)$  for all  $x \in X$ . ■

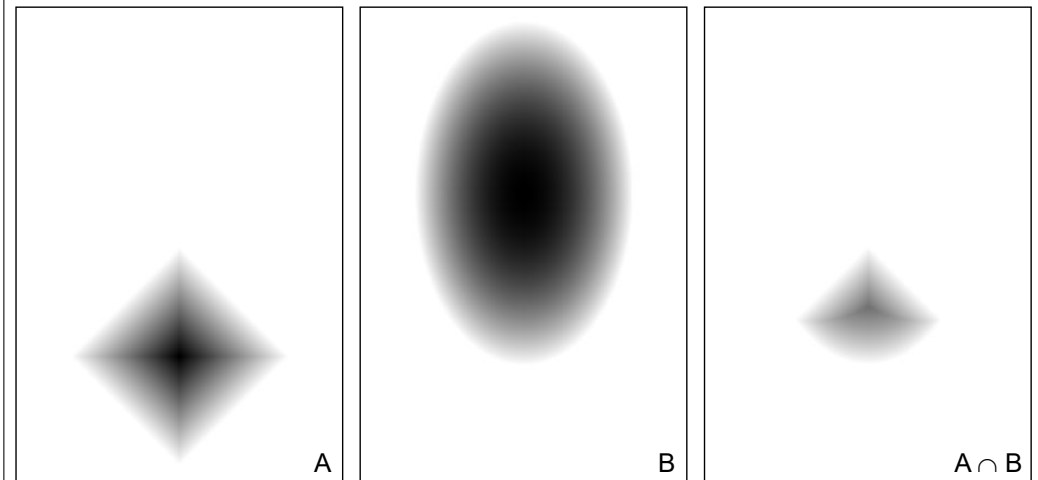


**standard fuzzy union**



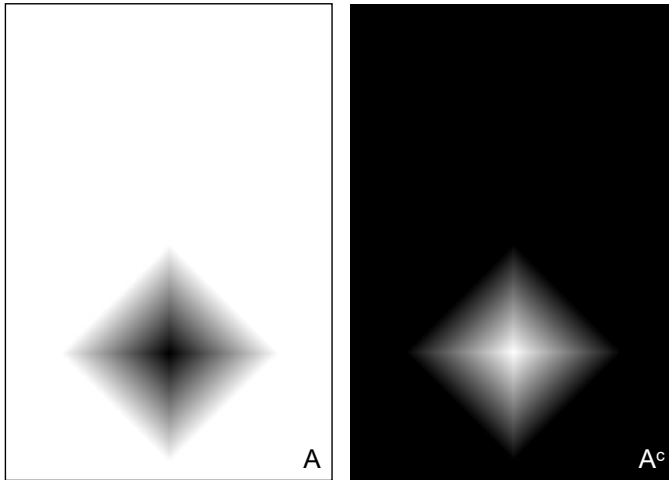
**interpretation:** membership = 0 is white, = 1 is black, in between is gray

**standard fuzzy intersection**



**interpretation:** membership = 0 is white, = 1 is black, in between is gray

standard fuzzy complement

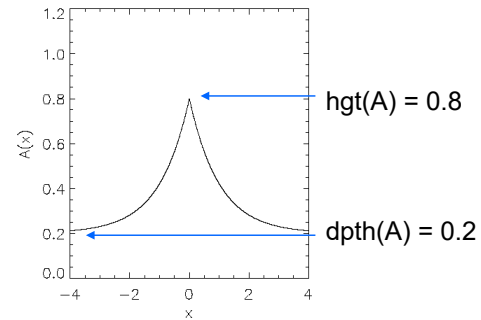


interpretation: membership = 0 is white, = 1 is black, in between is gray

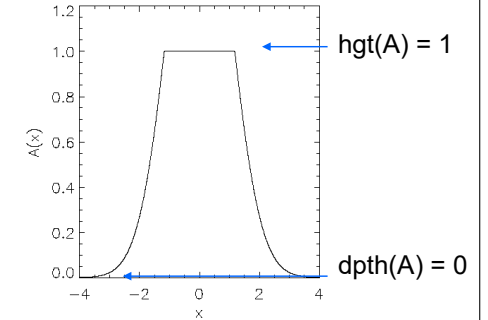
Definition

The fuzzy set A over the crisp set X has

- a) **height**  $\text{hgt}(A) = \sup\{A(x) : x \in X\}$ ,
- b) **depth**  $\text{dpth}(A) = \inf\{A(x) : x \in X\}$ .



$$A(x) = \frac{1}{5} + \frac{3}{5} \exp(-|x|)$$

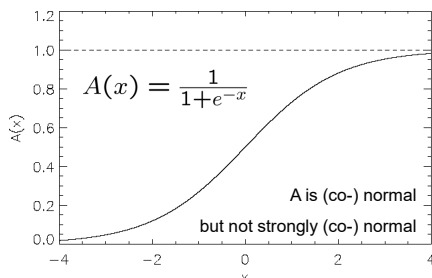


$$A(x) = \min\left\{1, 2 \exp\left(-\frac{x^2}{2}\right)\right\}$$

Definition

The fuzzy set A over the crisp set X is

- a) **normal** if  $\text{hgt}(A) = 1$
- b) **strongly normal** if  $\exists x \in X: A(x) = 1$
- c) **co-normal** if  $\text{dpth}(A) = 0$
- d) **strongly co-normal** if  $\exists x \in X: A(x) = 0$
- e) **subnormal** if  $0 < A(x) < 1$  for all  $x \in X$ .



Remark:  
How to normalize a non-normal fuzzy set A?

$$A^*(x) = \frac{A(x)}{\text{hgt}(A)}$$

Definition

The **cardinality**  $\text{card}(A)$  of a fuzzy set A over the crisp set X is

$$\text{card}(A) := \begin{cases} \sum_{x \in X} A(x) & , \text{ if } X \text{ countable} \\ \int_X A(x) dx & , \text{ if } X \subseteq \mathbb{R}^n \end{cases}$$

Examples:

a)  $A(x) = q^x$  with  $q \in (0,1), x \in \mathbb{N}_0 \Rightarrow \text{card}(A) = \sum_{x \in X} A(x) = \sum_{x=0}^{\infty} q^x = \frac{1}{1-q} < \infty$

b)  $A(x) = 1/x$  with  $x \in \mathbb{N} \Rightarrow \text{card}(A) = \sum_{x \in X} A(x) = \sum_{x=1}^{\infty} \frac{1}{x} = \infty$

c)  $A(x) = \exp(-|x|)$  with  $x \in \mathbb{R} \Rightarrow \text{card}(A) = \int_{x \in X} A(x) dx = \int_{x=-\infty}^{\infty} \exp(-|x|) dx = 2$

**Theorem**

For fuzzy sets A, B and C over a crisp set X the standard union operation is

- a) **commutative** :  $A \cup B = B \cup A$   
 b) **associative** :  $A \cup (B \cup C) = (A \cup B) \cup C$   
 c) **idempotent** :  $A \cup A = A$   
 d) **monotone** :  $A \subseteq B \Rightarrow (A \cup C) \subseteq (B \cup C)$ .

**Proof:** (via reduction to definitions)

ad a)  $A \cup B = \max \{ A(x), B(x) \} = \max \{ B(x), A(x) \} = B \cup A$ .

ad b)  $A \cup (B \cup C) = \max \{ A(x), \max \{ B(x), C(x) \} \} = \max \{ A(x), B(x), C(x) \}$   
 $= \max \{ \max \{ A(x), B(x) \}, C(x) \} = (A \cup B) \cup C$ .

ad c)  $A \cup A = \max \{ A(x), A(x) \} = A(x) = A$ .

ad d)  $A \cup C = \max \{ A(x), C(x) \} \leq \max \{ B(x), C(x) \} = B \cup C$  since  $A(x) \leq B(x)$ . **q.e.d.**

**Theorem**

For fuzzy sets A, B and C over a crisp set X the standard intersection operation is

- a) **commutative** :  $A \cap B = B \cap A$   
 b) **associative** :  $A \cap (B \cap C) = (A \cap B) \cap C$   
 c) **idempotent** :  $A \cap A = A$   
 d) **monotone** :  $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$ .

**Proof:** (analogous to proof for standard union operation) ■

**Theorem**

For fuzzy sets A, B and C over a crisp set X there are the distributive laws

- a)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$   
 b)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ .

**Proof:**

ad a)  $\max \{ A(x), \min \{ B(x), C(x) \} \} = \begin{cases} \max \{ A(x), B(x) \} & \text{if } B(x) \leq C(x) \\ \max \{ A(x), C(x) \} & \text{otherwise} \end{cases}$

If  $B(x) \leq C(x)$  then  $\max \{ A(x), B(x) \} \leq \max \{ A(x), C(x) \}$ .

Otherwise  $\max \{ A(x), C(x) \} \leq \max \{ A(x), B(x) \}$ .

$\Rightarrow$  result is always the smaller max-expression

$\Rightarrow$  result is  $\min \{ \max \{ A(x), B(x) \}, \max \{ A(x), C(x) \} \} = (A \cup B) \cap (A \cup C)$ .

ad b) analogous. ■

**Theorem**

If A is a fuzzy set over a crisp set X then

- a)  $A \cup \mathbb{0} = A$   
 b)  $A \cup \mathbb{1} = \mathbb{1}$   
 c)  $A \cap \mathbb{0} = \mathbb{0}$   
 d)  $A \cap \mathbb{1} = A$ .

**Proof:**

(via reduction to definitions)

ad a)  $\max \{ A(x), 0 \} = A(x)$

ad b)  $\max \{ A(x), 1 \} = \mathbb{1}(x) \equiv 1$

ad c)  $\min \{ A(x), 0 \} = \mathbb{0}(x) \equiv 0$

ad d)  $\min \{ A(x), 1 \} = A(x)$ . ■

**Breakpoint:**

So far we know that fuzzy sets with operations  $\cap$  and  $\cup$  are a distributive lattice.

If we can show the validity of

•  $(A^c)^c = A$

•  $A \cup A^c = \mathbb{1}$

•  $A \cap A^c = \mathbb{0}$

$\Rightarrow$  Fuzzy Sets would be Boolean Algebra! **Is it true ?**

**Theorem**

If A is a fuzzy set over a crisp set X then

- a)  $(A^c)^c = A$
- b)  $\frac{1}{2} \leq (A \cup A^c)(x) < 1$  for  $A(x) \in (0,1)$
- c)  $0 < (A \cap A^c)(x) \leq \frac{1}{2}$  for  $A(x) \in (0,1)$

**Remark:**

Recall the identities

$$\min\{a, b\} = \frac{a+b-|a-b|}{2}$$

$$\max\{a, b\} = \frac{a+b+|a-b|}{2}$$

**Proof:**

ad a)  $\forall x \in X: 1 - (1 - A(x)) = A(x)$ .

ad b)  $\forall x \in X: \max\{A(x), 1 - A(x)\} = \frac{1}{2} + |A(x) - \frac{1}{2}| \geq \frac{1}{2}$ .

Value 1 only attainable for  $A(x) = 0$  or  $A(x) = 1$ .

ad c)  $\forall x \in X: \min\{A(x), 1 - A(x)\} = \frac{1}{2} - |A(x) - \frac{1}{2}| \leq \frac{1}{2}$ .

Value 0 only attainable for  $A(x) = 0$  or  $A(x) = 1$ .

**q.e.d.**

**Conclusion:**

Fuzzy sets with  $\cup$  and  $\cap$  are a distributive lattice.

But in general:

- a)  $A \cup A^c \neq \mathbb{U}$
  - b)  $A \cap A^c \neq \mathbb{O}$
- }  $\Rightarrow$  Fuzzy sets with  $\cup$  and  $\cap$  are **not** a Boolean algebra!

**Remarks:**

ad a) The **law of excluded middle** does not hold!

(„Everything must either be or not be!“)

ad b) The **law of noncontradiction** does not hold!

(„Nothing can both be and not be!“)

$\Rightarrow$  Nonvalidity of these laws generate the desired fuzziness!

**but:** Fuzzy sets still endowed with much algebraic structure (distributive lattice)!

**Theorem**

If A and B are fuzzy sets over a crisp set X with standard union, intersection, and complement operations then **DeMorgan's** laws are valid:

- a)  $(A \cap B)^c = A^c \cup B^c$
- b)  $(A \cup B)^c = A^c \cap B^c$

**Proof:** (via reduction to elementary identities)

ad a)  $(A \cap B)^c(x) = 1 - \min\{A(x), B(x)\} = \max\{1 - A(x), 1 - B(x)\} = A^c(x) \cup B^c(x)$

ad b)  $(A \cup B)^c(x) = 1 - \max\{A(x), B(x)\} = \min\{1 - A(x), 1 - B(x)\} = A^c(x) \cap B^c(x)$

**q.e.d.**

**Question** : Why restricting result above to "standard" operations?

**Conjecture** : Most likely there also exist "nonstandard" operations!