

# Computational Intelligence

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- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

**Considered so far:**

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

⇒ Compatible with operators for crisp sets

with membership functions with values in  $\mathbb{B} = \{ 0, 1 \}$

∃ Non-standard operators? ⇒ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

### Definition

A function  $c: [0,1] \rightarrow [0,1]$  is a **fuzzy complement** iff

(A1)  $c(0) = 1$  and  $c(1) = 0$ .

(A2)  $\forall a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$ .

monotone decreasing

“nice to have”:

(A3)  $c(\cdot)$  is continuous.

(A4)  $\forall a \in [0,1]: c(c(a)) = a$

involution

### Examples:

a) standard fuzzy complement  $c(a) = 1 - a$

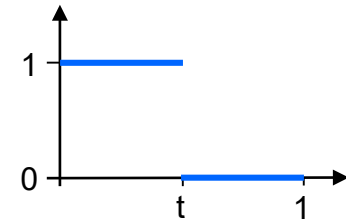
ad (A1):  $c(0) = 1 - 0 = 1$  and  $c(1) = 1 - 1 = 0$

ad (A2):  $c'(a) = -1 < 0$  (monotone decreasing)

ad (A3):

ad (A4):  $1 - (1 - a) = a$

$$b) c(a) = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{otherwise} \end{cases} \quad \text{for some } t \in (0, 1)$$



ad (A1):  $c(0) = 1$  since  $0 < t$  and  $c(1) = 0$  since  $t < 1$ .

ad (A2): monotone (actually: constant) from 0 to  $t$  and  $t$  to 1, decreasing at  $t$

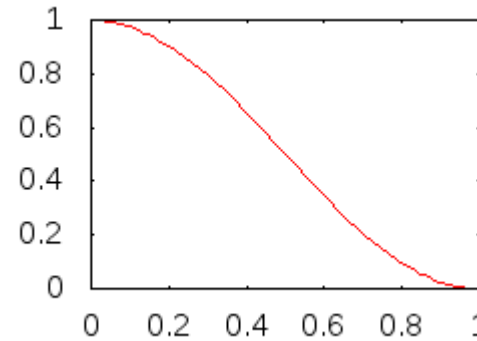


ad (A3): **not valid**  $\rightarrow$  discontinuity at  $t$

ad (A4): **not valid**  $\rightarrow$  counter example

$$c(c(1/4)) = c(1) = 0 \neq 1/4 \text{ for } t = 1/2$$

c)  $c(a) = \frac{1 + \cos(\pi a)}{2}$



ad (A1):  $c(0) = 1$  and  $c(1) = 0$

ad (A2):  $c'(a) = -\frac{1}{2} \pi \sin(\pi a) < 0$  since  $\sin(\pi a) > 0$  for  $a \in (0,1)$

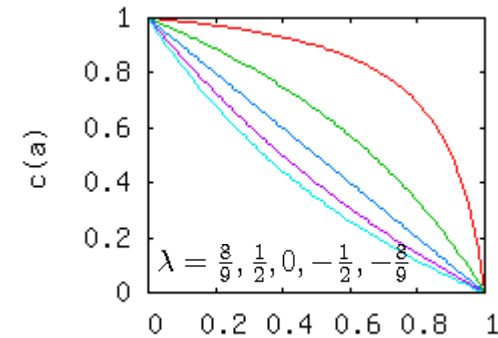


ad (A3): is continuous as a composition of continuous functions;  
*alternative argument: derivative exists, see  $c'(a)$  in (A2)*

ad (A4): **not valid** → counter example

$$c\left(c\left(\frac{1}{3}\right)\right) = c\left(\frac{3}{4}\right) = \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right) \neq \frac{1}{3}$$

d)  $c(a) = \frac{1-a}{1+\lambda a}$  for  $\lambda > -1$       **Sugeno class**



ad (A1):  $c(0) = 1$  and  $c(1) = 0$

ad (A2):  $c(a) \geq c(b) \Leftrightarrow \frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b} \Leftrightarrow$   
 $(1-a)(1+\lambda b) \geq (1-b)(1+\lambda a) \Leftrightarrow$   
 $b(\lambda+1) \geq a(\lambda+1) \Leftrightarrow b \geq a$

} a

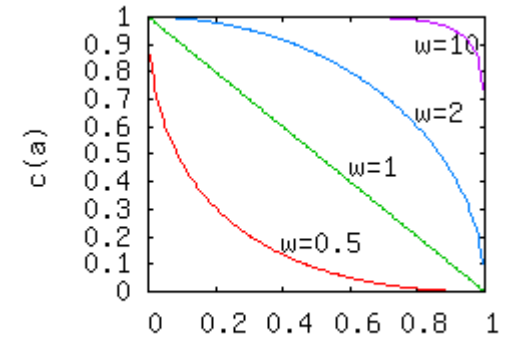
ad (A3): is continuous as a composition of continuous functions

ad (A4):  $c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right) = \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda \frac{1-a}{1+\lambda a}} = \frac{a(\lambda+1)}{\lambda+1} = a$

}

e)  $c(a) = (1 - a^w)^{1/w}$  for  $w > 0$

**Yager class**



ad (A1):  $c(0) = 1$  and  $c(1) = 0$

ad (A2):  $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$   
 $a^w \leq b^w \Leftrightarrow a \leq b$

} a

ad (A3): is continuous as a composition of continuous functions

ad (A4):  $c(c(a)) = c\left(\left(1 - a^w\right)^{\frac{1}{w}}\right) = \left(1 - \left[\left(1 - a^w\right)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$   
 $= \left(1 - (1 - a^w)\right)^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$

}

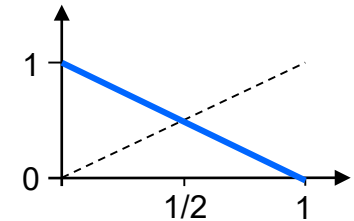


### Theorem

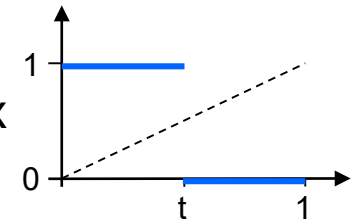
If function  $c: [0,1] \rightarrow [0,1]$  satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point  $a^*$  with  $c(a^*) = a^*$ .

### Proof:

one fixed point  $\rightarrow$  see example (a)  $\rightarrow$  intersection with bisectrix



no fixed point  $\rightarrow$  see example (b)  $\rightarrow$  no intersection with bisectrix



assume  $\exists n > 1$  fixed points, for example  $a^*$  and  $b^*$  with  $a^* < b^*$

$\Rightarrow c(a^*) = a^*$  and  $c(b^*) = b^*$  (fixed points)

$\Rightarrow c(a^*) < c(b^*)$  with  $a^* < b^*$  impossible if  $c(\cdot)$  is monotone decreasing

$\Rightarrow$  contradiction to axiom (A2) ■

### Theorem

If function  $c: [0,1] \rightarrow [0,1]$  satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point  $a^*$  with  $c(a^*) = a^*$ .

### *Proof:*

Intermediate value theorem  $\rightarrow$

If  $c(\cdot)$  continuous (A3) and  $c(0) \geq c(1)$  (A1/A2)

then  $\forall v \in [c(1), c(0)] = [0,1]: \exists a \in [0,1]: c(a) = v$ .

$\Rightarrow$  there must be an intersection with bisectrix

$\Rightarrow$  a fixed point exists and by previous theorem there are no other fixed points! ■

### Examples:

$$(a) \quad c(a) = 1 - a \qquad \Rightarrow a = 1 - a \qquad \Rightarrow a^* = \frac{1}{2}$$

$$(b) \quad c(a) = (1 - a^w)^{1/w} \qquad \Rightarrow a = (1 - a^w)^{1/w} \qquad \Rightarrow a^* = (\frac{1}{2})^{1/w}$$

### Theorem

$c: [0,1] \rightarrow [0,1]$  is involutive fuzzy complement iff

$\exists$  continuous function  $g: [0,1] \rightarrow \mathbb{R}$  with

- $g(0) = 0$
- strictly monotone increasing
- $\forall a \in [0,1]: c(a) = g^{(-1)}( g(1) - g(a) )$ . ■

defines an  
**increasing generator**

$g^{(-1)}(x)$  pseudo-inverse  
 $= g^{-1}( \min\{ g(1), x \} )$

### Examples

a)  $g(x) = x \quad \Rightarrow \quad g^{(-1)}(x) = x \quad \Rightarrow \quad c(a) = 1 - a \quad \text{(Standard)}$

b)  $g(x) = x^w \quad \Rightarrow \quad g^{(-1)}(x) = x^{1/w} \quad \Rightarrow \quad c(a) = (1 - a^w)^{1/w} \quad \text{(Yager class, } w > 0)$

c)  $g(x) = \log(x+1) \Rightarrow \underbrace{g^{(-1)}(x) = e^x - 1}_{?} \Rightarrow c(a) = \exp( \log(2) - \log(a+1) ) - 1$   
 $= \frac{1 - a}{1 + a} \quad \text{(Sugeno class. } \lambda = 1)$

→ make sure that pseudoinverse is equal to inverse, here!

$$g(x) = \log(x + 1) \quad \rightarrow \quad g^{-1}(x) = e^x - 1 \quad \checkmark \quad \text{(inverse)}$$

$$g^{(-1)}(x) = g^{-1}(\min\{g(1), x\}) \quad \text{(pseudoinverse)}$$

↓  
?

$$c(a) = g^{(-1)}(\underbrace{g(1) - g(a)}_{= x})$$

$$\min\{g(1), g(1) - a\} = g(1) - g(a) \leq g(1) \quad \text{since } 0 \leq g(a) \leq \log 2 \text{ for } a \in [0, 1]$$

therefore,

$$c(a) = g^{(-1)}(g(1) - g(a)) = g^{-1}(g(1) - g(a)) \quad \checkmark$$

**Examples**

d)  $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda a)$  for  $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since  $g'(a) = \frac{1}{1+\lambda a} > 0$  for  $a \in [0, 1]$
- inverse function on  $[0, 1]$  is  $g^{-1}(a) = \frac{\exp(\lambda a) - 1}{\lambda}$ , thus

$$\begin{aligned} c(a) &= g^{-1} \left( \frac{\log(1 + \lambda)}{\lambda} - \frac{\log(1 + \lambda a)}{\lambda} \right) \\ &= \frac{\exp(\log(1 + \lambda) - \log(1 + \lambda a)) - 1}{\lambda} \\ &= \frac{1}{\lambda} \left( \frac{1 + \lambda}{1 + \lambda a} - 1 \right) = \frac{1 - a}{1 + \lambda a} \quad (\text{Sugeno Complement}) \end{aligned}$$

### Theorem

$c: [0,1] \rightarrow [0,1]$  is involutive fuzzy complement iff

$\exists$  continuous function  $f: [0,1] \rightarrow \mathbb{R}$  with

- $f(1) = 0$
- strictly monotone decreasing
- $\forall a \in [0,1]: c(a) = f^{(-1)}( f(0) - f(a) )$ . ■

defines a  
**decreasing generator**

$f^{(-1)}(x)$  pseudo-inverse  
 $= f^{-1}( \min\{ f(0), x \} )$

### Examples

$$\text{a) } f(x) = k - k \cdot x \quad (k \geq 1) \quad f^{(-1)}(x) = 1 - x/k \quad c(a) = 1 - \frac{k - (k - ka)}{k} = 1 - a$$

$$\text{b) } f(x) = 1 - x^w \quad f^{(-1)}(x) = (1 - x)^{1/w} \quad c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w} \quad (\text{Yager})$$

**Definition**

A function  $t: [0, 1] \times [0, 1] \rightarrow [0, 1]$  is a **fuzzy intersection** or **t-norm** iff  $\forall a, b, d \in [0, 1]$

(A1)  $t(a, 1) = a$  (boundary condition)

(A2)  $b \leq d \Rightarrow t(a, b) \leq t(a, d)$  (monotonicity)

(A3)  $t(a, b) = t(b, a)$  (commutative)

(A4)  $t(a, t(b, d)) = t(t(a, b), d)$  (associative) ■

**“nice to have”**

(A5)  $t(a, b)$  is continuous (continuity)

(A6)  $t(a, a) < a$  for  $0 < a < 1$  (subidempotent)

(A7)  $a_1 < a_2$  and  $b_1 \leq b_2 \Rightarrow t(a_1, b_1) < t(a_2, b_2)$  (strict monotonicity)

**Note:** the only idempotent t-norm is the standard fuzzy intersection

### Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

### Proof:

Assume there exists a t-norm with  $t(a,a) = a$  for all  $a \in [0,1]$ .

- If  $0 \leq a \leq b \leq 1$  then

$$\begin{array}{ccccccc}
 a & = & t(a,a) & \leq & t(a,b) & \leq & t(a, 1) = a \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{by assumption} & & \text{by monotonicity} & & \text{by boundary condition} & & 
 \end{array}$$

and hence  $t(a,b) = a$ .

- If  $0 \leq b \leq a \leq 1$  then

$$\begin{array}{ccccccc}
 b & = & t(b,b) & \leq & t(b,a) & \leq & t(b, 1) = b \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{by assumption} & & \text{by monotonicity} & & \text{by boundary condition} & & 
 \end{array}$$

and hence  $t(a,b) = t(b,a) = b$ .

$\uparrow$   
by commutativity

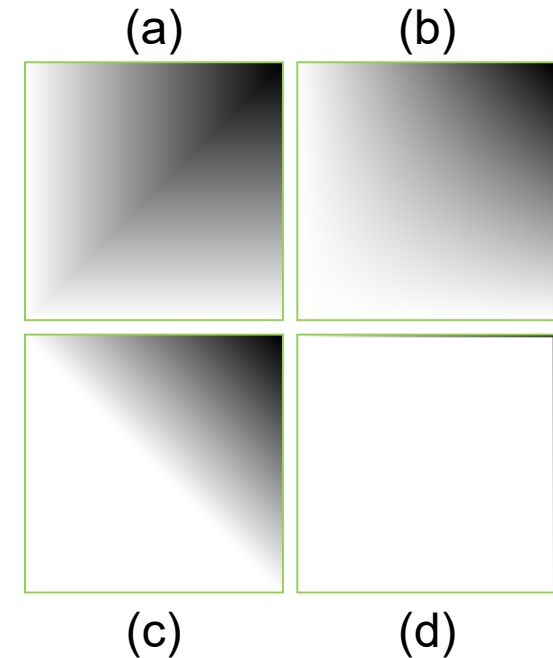
$t(a,b) = \min(a,b)$   
is the only  
possible solution!

**q.e.d.**



### Examples:

Name	Function
(a) Standard	$t(a, b) = \min \{ a, b \}$
(b) Algebraic Product	$t(a, b) = a \cdot b$
(c) Bounded Difference	$t(a, b) = \max \{ 0, a + b - 1 \}$
(d) Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$



Is algebraic product a t-norm? Check the 4 axioms!

ad (A1):  $t(a, 1) = a \cdot 1 = a$

ad (A3):  $t(a, b) = a \cdot b = b \cdot a = t(b, a)$

ad (A2):  $a \cdot b \leq a \cdot d \Leftrightarrow b \leq d$

ad (A4):  $a \cdot (b \cdot d) = (a \cdot b) \cdot d$

### Theorem

Function  $t: [0,1] \times [0,1] \rightarrow [0,1]$  is a t-norm ,

$\exists$  decreasing generator  $f: [0,1] \rightarrow \mathbb{R}$  with  $t(a, b) = f^{-1}( \min\{ f(0), f(a) + f(b) \} )$ . ■

### Example:

$f(x) = 1/x - 1$  is decreasing generator since

- $f(x)$  is continuous ☑
- $f(1) = 1/1 - 1 = 0$  ☑
- $f'(x) = -1/x^2 < 0$  (monotone decreasing) ☑

inverse function is  $f^{-1}(x) = \frac{1}{x+1}$  ;  $f(0) = \infty \Rightarrow \min\{ f(0), f(a) + f(b) \} = f(a) + f(b)$

$$\Rightarrow t(a, b) = f^{-1} \left( \frac{1}{a} + \frac{1}{b} - 2 \right) = \frac{1}{\frac{1}{a} + \frac{1}{b} - 1} = \frac{ab}{a+b-a-b}$$

**Definition**

A function  $s: [0,1] \times [0,1] \rightarrow [0,1]$  is a **fuzzy union** or **s-norm** iff  $\forall a,b,d \in [0,1]$

(A1)  $s(a, 0) = a$  (boundary condition)

(A2)  $b \leq d \Rightarrow s(a, b) \leq s(a, d)$  (monotonicity)

(A3)  $s(a, b) = s(b, a)$  (commutative)

(A4)  $s(a, s(b, d)) = s(s(a, b), d)$  (associative) ■

**“nice to have”**

(A5)  $s(a, b)$  is continuous (continuity)

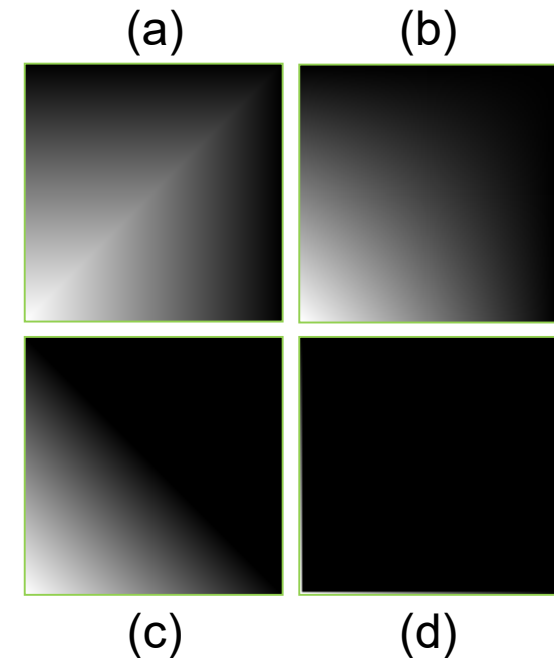
(A6)  $s(a, a) > a$  for  $0 < a < 1$  (superidempotent)

(A7)  $a_1 < a_2$  and  $b_1 \leq b_2 \Rightarrow s(a_1, b_1) < s(a_2, b_2)$  (strict monotonicity)

**Note:** the only idempotent s-norm is the standard fuzzy union

### Examples:

Name	Function
Standard	$s(a, b) = \max \{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min \{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$



Is algebraic sum an s-norm? Check the 4 axioms!

ad (A1):  $s(a, 0) = a + 0 - a \cdot 0 = a$

ad (A3):

ad (A2):  $a + b - a \cdot b \leq a + d - a \cdot d \Leftrightarrow b(1 - a) \leq d(1 - a) \Leftrightarrow b \leq d$

ad (A4):

**Theorem**

Function  $s: [0,1] \times [0,1] \rightarrow [0,1]$  is a s-norm  $\Leftrightarrow$

$\exists$  increasing generator  $g: [0,1] \rightarrow \mathbb{R}$  with  $s(a, b) = g^{-1}(\min\{g(1), g(a) + g(b)\})$ . ■

**Example:**

$g(x) = -\log(1 - x)$  is increasing generator since

- $g(x)$  is continuous ☑
- $g(0) = -\log(1 - 0) = 0$  ☑
- $g'(x) = 1/(1 - x) > 0$  (monotone increasing) ☑

inverse function is  $g^{-1}(x) = 1 - \exp(-x)$ ;  $g(1) = \infty \Rightarrow \min\{g(1), g(a) + g(b)\} = g(a) + g(b)$

$$\begin{aligned} \Rightarrow s(a, b) &= g^{-1}(-\log(1 - a) - \log(1 - b)) \\ &= 1 - \exp(\log(1 - a) + \log(1 - b)) \\ &= 1 - (1 - a)(1 - b) = a + b - ab \quad (\text{algebraic sum}) \end{aligned}$$

### Background from classical set theory:

$\cap$  and  $\cup$  operations are dual w.r.t. complement since they obey DeMorgan's laws

#### Definition

A pair of t-norm  $t(\cdot, \cdot)$  and s-norm  $s(\cdot, \cdot)$  is said to be **dual with regard to the fuzzy complement**  $c(\cdot)$  iff

- $c(t(a, b)) = s(c(a), c(b))$
- $c(s(a, b)) = t(c(a), c(b))$

for all  $a, b \in [0, 1]$ . ■

#### Definition

Let  $(c, s, t)$  be a triple of fuzzy complement  $c(\cdot)$ , s- and t-norm.

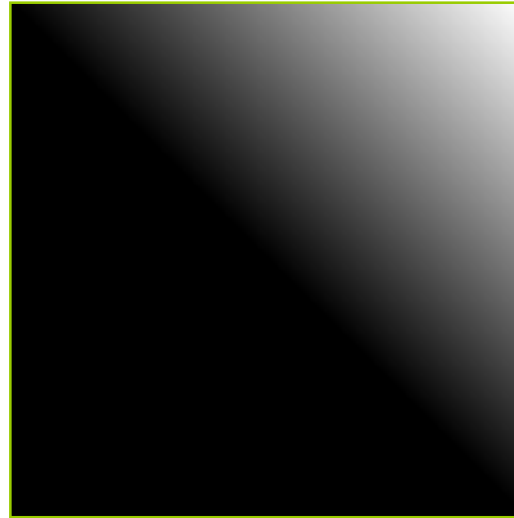
If  $t$  and  $s$  are dual to  $c$  then the triple  $(c, s, t)$  is called a **dual triple**. ■

### Examples of dual triples

t-norm	s-norm	complement
$\min \{ a, b \}$	$\max \{ a, b \}$	$1 - a$
$a \cdot b$	$a + b - a \cdot b$	$1 - a$
$\max \{ 0, a + b - 1 \}$	$\min \{ 1, a + b \}$	$1 - a$



$c(t(a, b))$



$s(c(a), c(b))$



Dual Triple:

- bounded difference
- bounded sum
- standard complement

⇒ left image = right image

Non-Dual Triple:

- algebraic product
- bounded sum
- standard complement

⇒ left image ≠ right image

## Why are dual triples so important?

⇒ allow equivalence transformations of fuzzy set expressions

⇒ required to transform into some equivalent normal form (standardized input)

⇒ e.g. two stages: intersection of unions

$$\bigcap_{i=1}^n (A_i \cup B_i)$$

or union of intersections

$$\bigcup_{i=1}^n (A_i \cap B_i)$$

### Example:

$$A \cup (B \cap (C \cap D)^c) =$$

← not in normal form

$$A \cup (B \cap (C^c \cup D^c)) =$$

← equivalent if DeMorgan's law valid (dual triples!)

$$A \cup (B \cap C^c) \cup (B \cap D^c)$$

← equivalent (distributive lattice!)