technische universität
dortmund

Computational Intelligence

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- Fuzzy Sets
  - Basic Definitions and Results for Standard Operations
  - Algebraic Difference between Fuzzy and Crisp Sets

Observation:

Communication between people is not precise but somehow fuzzy and vague.

“If the water is too hot then add a little bit of cold water.“

Despite these shortcomings in human language we are able

- to process fuzzy / uncertain information and
- to accomplish complex tasks!

Goal:

Development of formal framework to process fuzzy statements in computer.

Consider the statement: "The water is hot."

Which temperature defines "hot"?

A single temperature T = 95° C?

No! Rather, an interval of temperatures: $T \in [\,70, 120\,]$!

But who defines the limits of the intervals?

Some people regard temperatures > 60° C as hot, others already T > 50° C!

Idea: All people might agree that a temperature in the set [70, 120] defines a hot temperature!

If T = 65°C not all people regard this as hot. It does not belong to [70,120].

But it is hot to some degree.
Or: T = 65°C belongs to set of hot temperatures to some degree!

⇒ **Can be the concept for capturing fuzziness!** ⇒ **Formalize this concept!**

Definition

A map F: X → [0,1] $\subset \mathbb{R}$ that assigns its ***degree of membership*** F(x) to each x $\in$ X is termed a **fuzzy set**.

Remark:

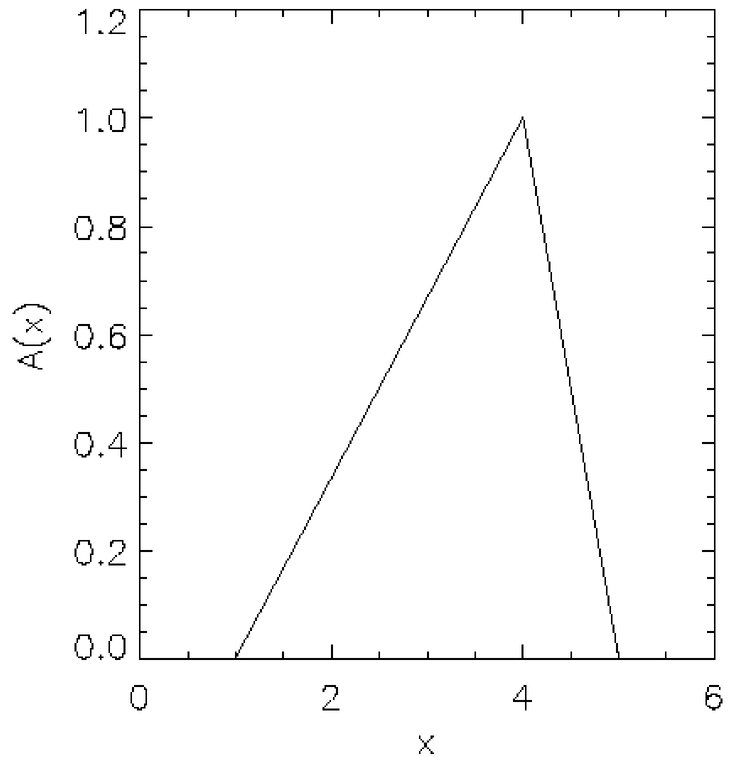
A fuzzy set F is actually a map F(x). Shorthand notation is simply F.

Same point of view possible for traditional ("***crisp***") sets:

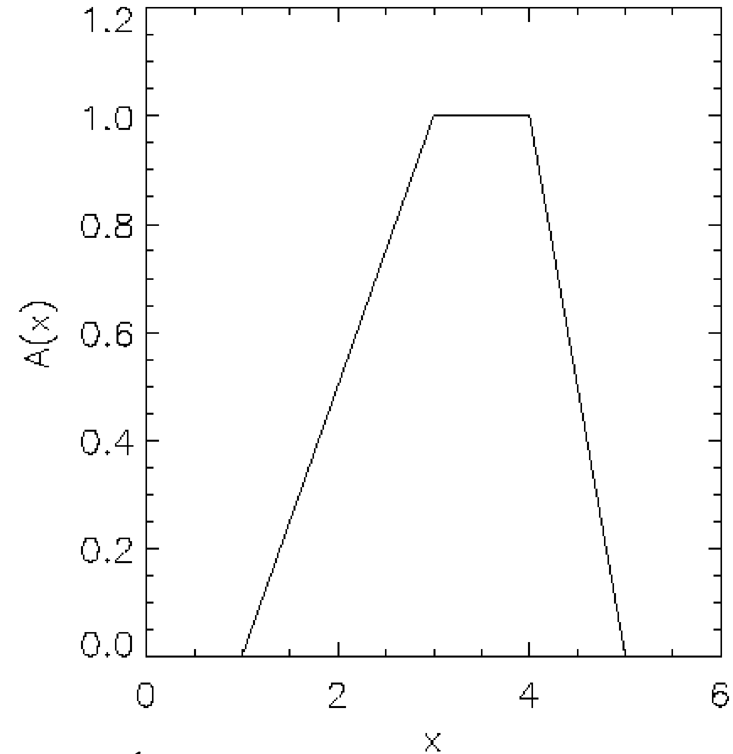
$$A(x) := 1_{[x \in A]} := 1_A(x) := \begin{cases} 1 & \text{, if } x \in A \\ 0 & \text{, if } x \notin A \end{cases}$$

characteristic / indicator function of (crisp) set A

⇒ membership function interpreted as generalization of characteristic function triangle function



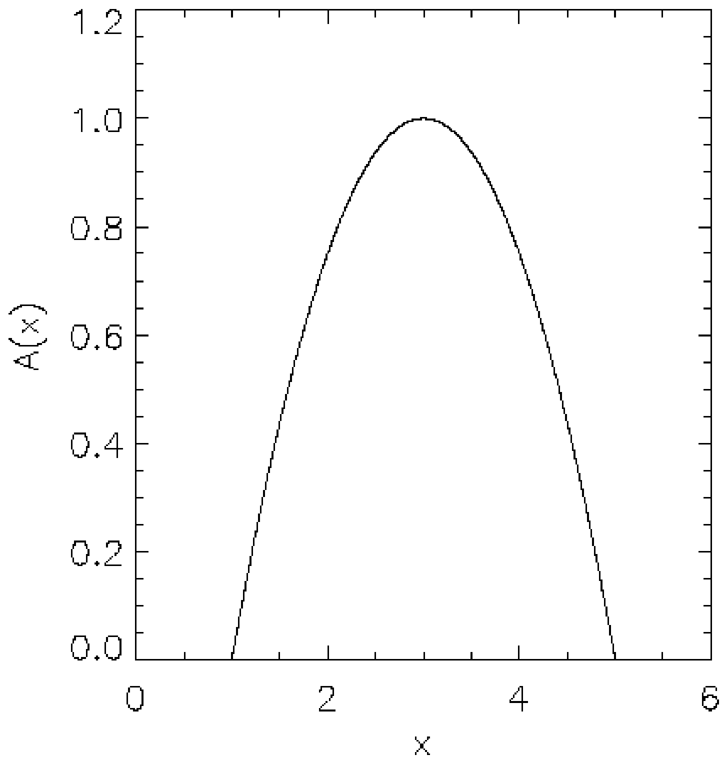
trapezoidal



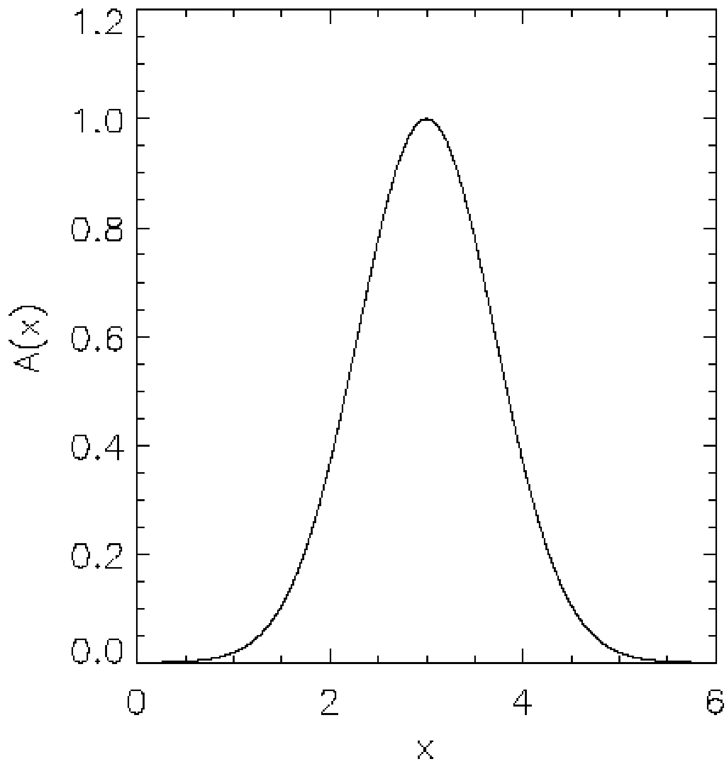
$$A(x) = \begin{cases} \frac{1}{3}(x-1) & \text{if } 1 \le x < 4 \\ 5 - x & \text{if } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$A(x) = \begin{cases} \frac{1}{2}(x-1) & \text{if } 1 \le x < 3 \\ 1 & \text{if } 3 \le x < 4 \\ 5 - x & \text{if } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

paraboloidal function



gaussoid function



$$A(x) = \begin{cases} -\frac{(x-1)(x-5)}{4} & \text{if } 1 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$A(x) = \exp\left(-\frac{(x-3)^2}{2}\right)$$

Definition

A fuzzy set F over the crisp set X is termed

a) ***empty*** if $F(x) = 0$ for all $x \in X$,

b) ***universal*** if $F(x) = 1$ for all $x \in X$.

Empty fuzzy set is denoted by $\mathbb{O}$. Universal set is denoted by $\mathbb{U}$. ■

Definition

Let A and B be fuzzy sets over the crisp set X.

a) A and B are termed ***equal***, denoted $A = B$, if $A(x) = B(x)$ for all $x \in X$.

b) A is a ***subset*** of B, denoted $A \subseteq B$, if $A(x) \leq B(x)$ for all $x \in X$.

c) A is a ***strict subset*** of B, denoted $A \subset B$, if $A \subseteq B$ and $\exists\, x \in X: A(x) < B(x)$. ■

Remark: A strict subset is also called a ***proper*** subset.

Theorem

Let A, B and C be fuzzy sets over the crisp set X. The following relations are valid:

a) reflexivity : $A \subseteq A$.

b) antisymmetry : $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

c) transitivity : $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

Proof: (via reduction to definitions and exploiting operations on crisp sets)

ad a) $\forall x \in X: A(x) \leq A(x)$.

ad b) $\forall x \in X: A(x) \leq B(x)$ and $B(x) \leq A(x) \Rightarrow A(x) = B(x)$.

ad c) $\forall x \in X: A(x) \leq B(x)$ and $B(x) \leq C(x) \Rightarrow A(x) \leq C(x)$. **q.e.d.**

Remark: Same relations valid for crisp sets. No Surprise! Why?

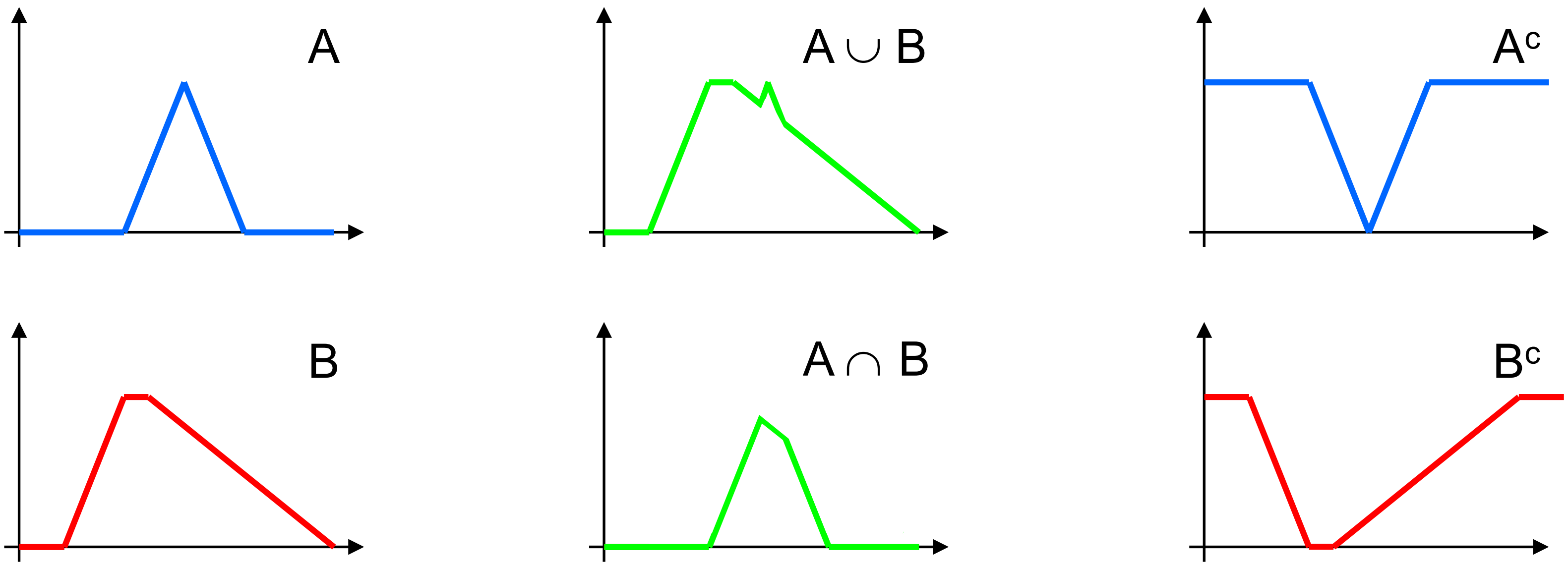
Definition

Let A and B be fuzzy sets over the crisp set X. The set C is the

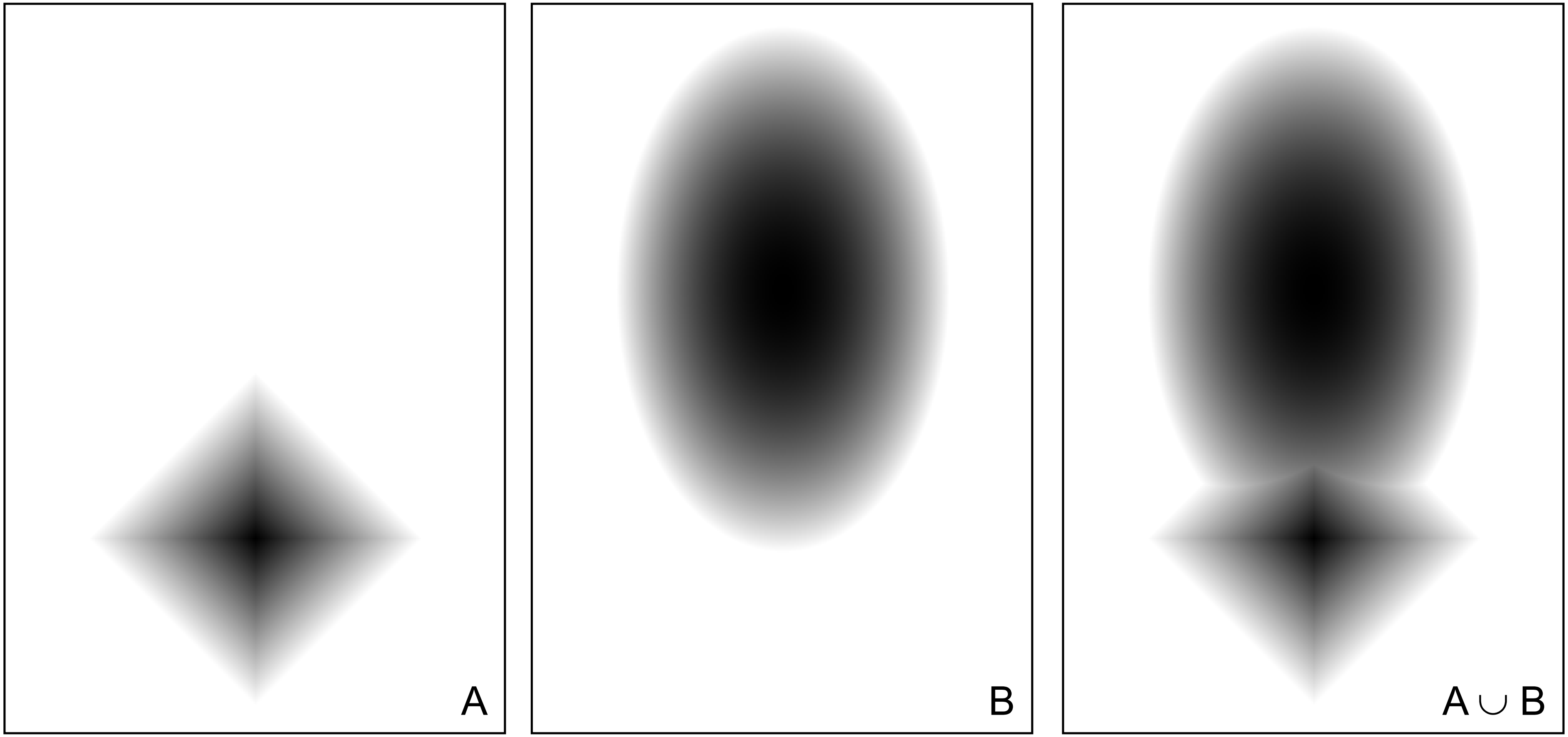
a) ***union*** of A and B, denoted $C = A \cup B$, if $C(x) = \max\{ A(x), B(x) \}$ for all $x \in X$;

b) ***intersection*** of A and B, denoted $C = A \cap B$, if $C(x) = \min\{ A(x), B(x) \}$ for all $x \in X$;

c) ***complement*** of A, denoted $C = A^c$, if $C(x) = 1 - A(x)$ for all $x \in X$. ■

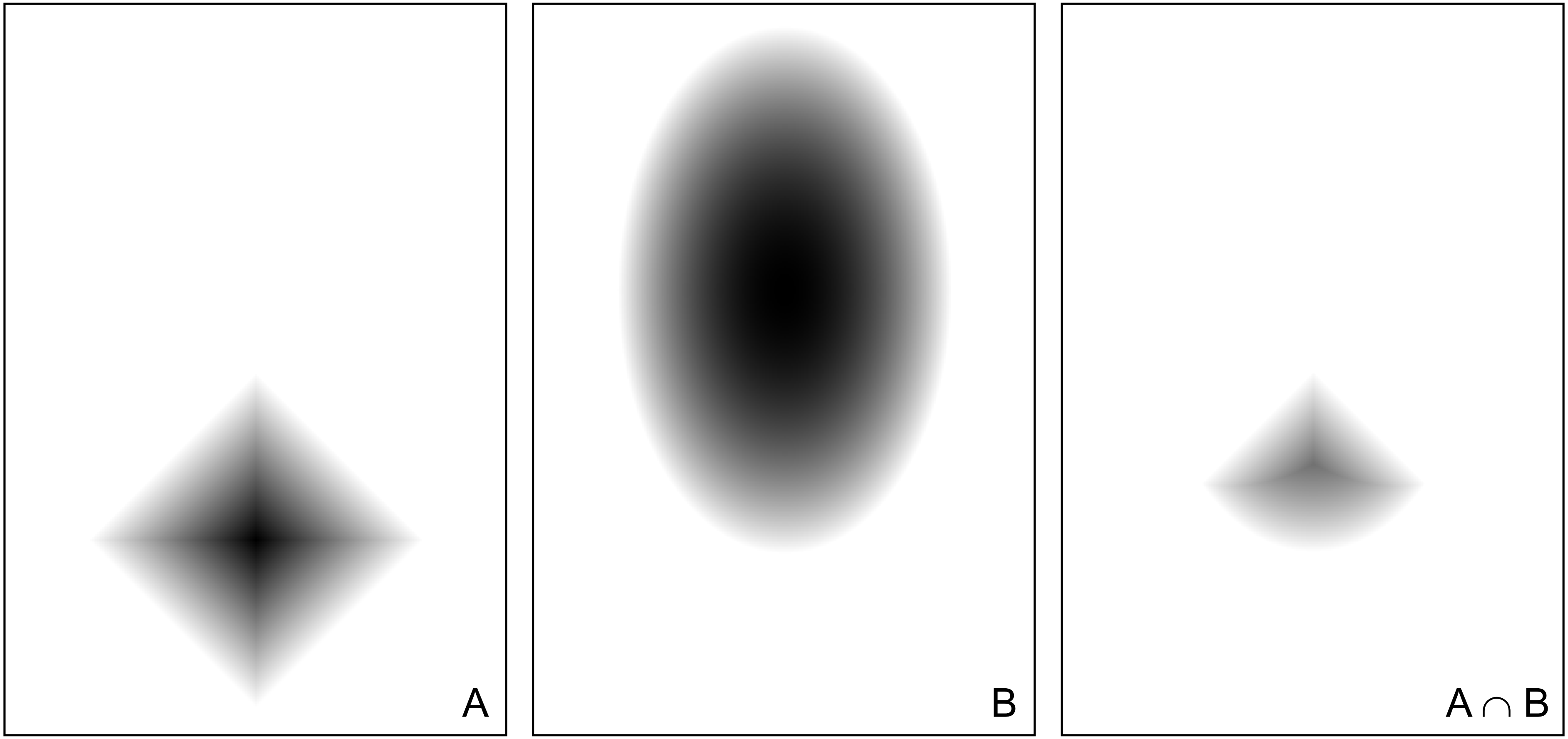


standard fuzzy union



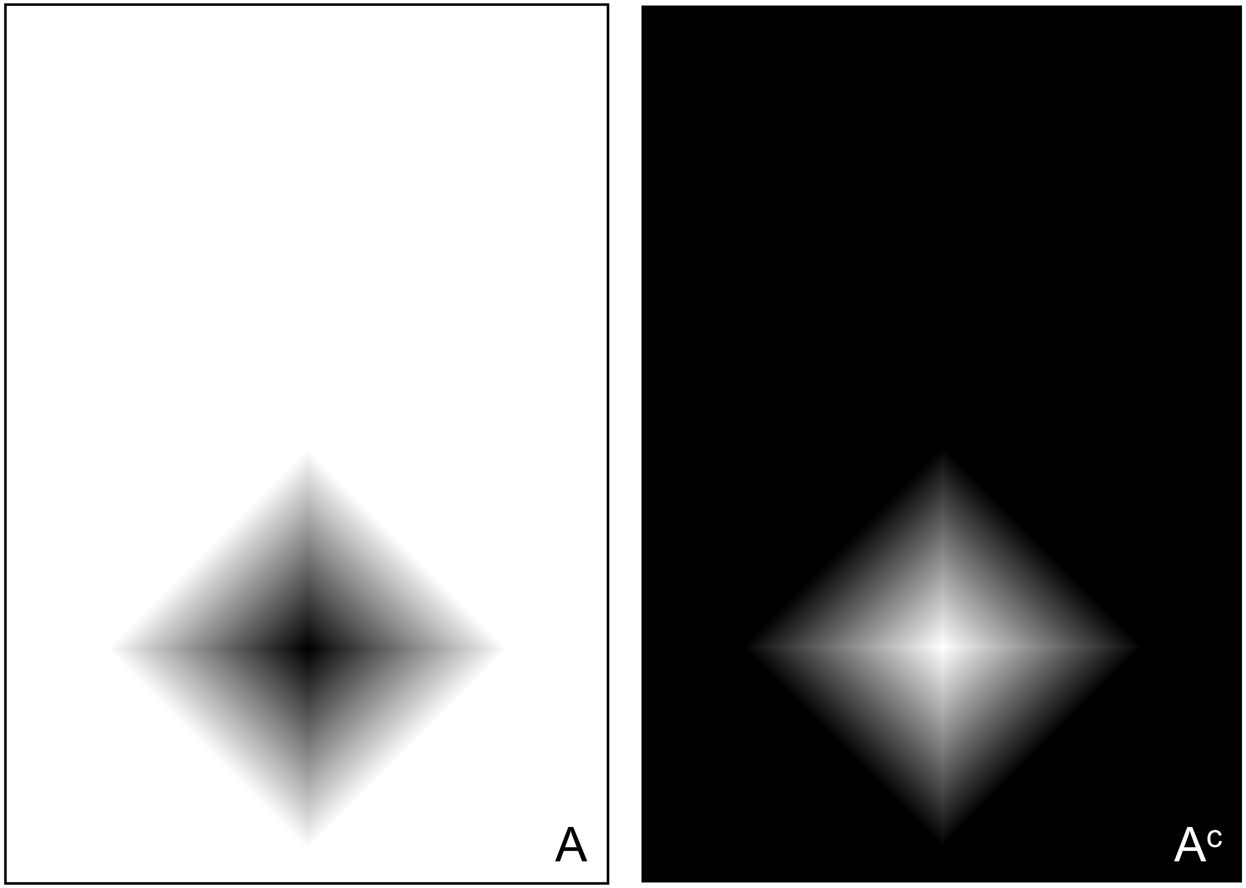
interpretation: membership = 0 is white, = 1 is black, in between is gray

standard fuzzy intersection



interpretation: membership = 0 is white, = 1 is black, in between is gray

standard fuzzy complement



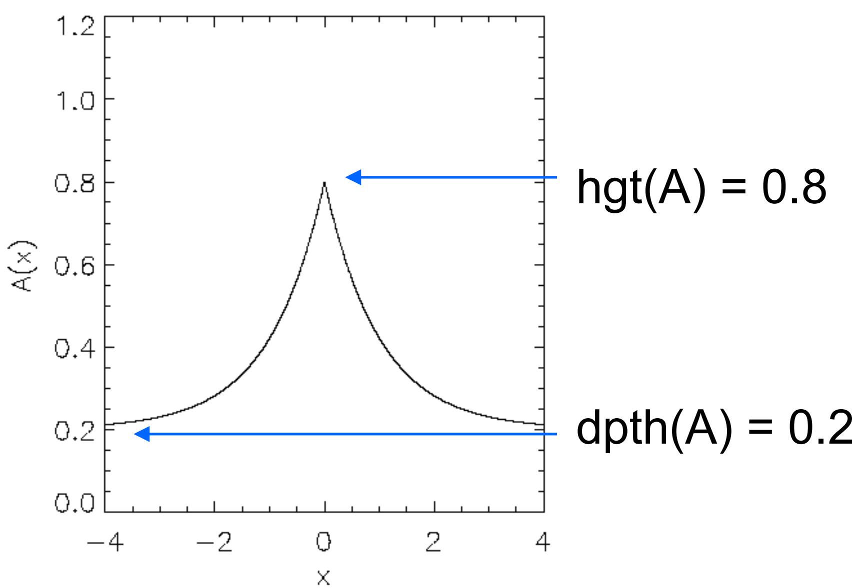
interpretation: membership = 0 is white, = 1 is black, in between is gray

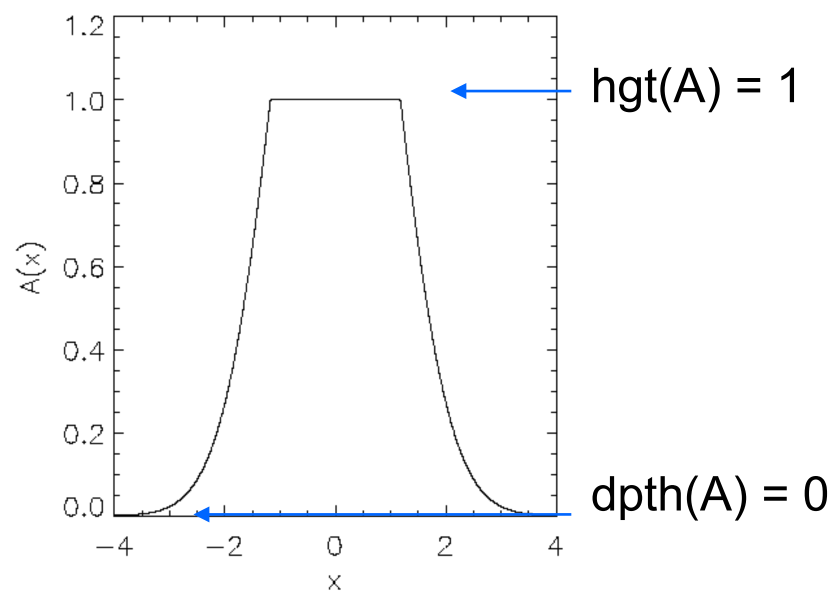
Definition

The fuzzy set A over the crisp set X has

a) ***height*** hgt(A) = sup{ A(x) : x ∈ X },

b) ***depth*** dpth(A) = inf { A(x) : x ∈ X }. ■





$$A(x) = \frac{1}{5} + \frac{3}{5}\exp(-|x|)$$

$$A(x) = \min\left\{1, 2\exp\left(-\frac{x^2}{2}\right)\right\}$$

Definition

The fuzzy set A over the crisp set X is

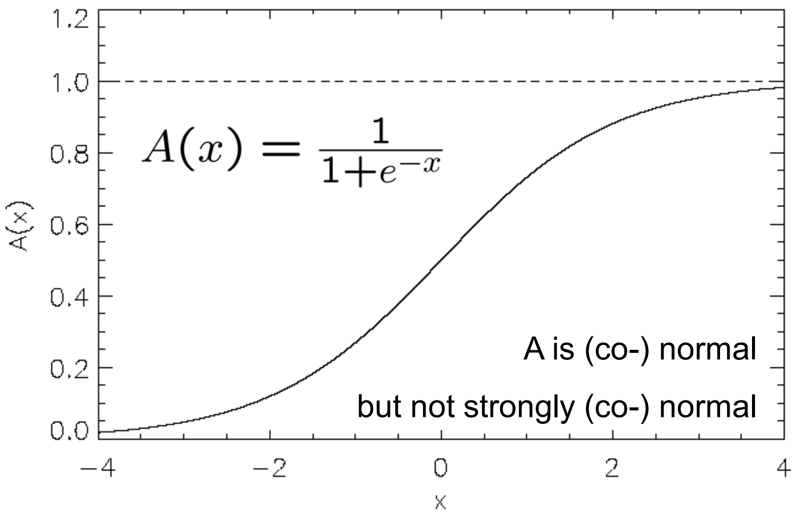
a) ***normal*** if hgt(A) = 1

b) ***strongly normal*** if $\exists$ x $\in$ X: A(x) = 1

c) ***co-normal*** if dpth(A) = 0

d) ***strongly co-normal*** if $\exists$ x $\in$ X: A(x) = 0

e) ***subnormal*** if 0 < A(x) < 1 for all x $\in$ X. ■



Remark:

How to normalize a non-normal fuzzy set A?

$$A^*(x) = \frac{A(x)}{\text{hgt}(A)}$$

Definition

The ***cardinality*** card(A) of a fuzzy set A over the crisp set X is

$$\text{card}(A) := \begin{cases} \sum\limits_{x \in X} A(x) & \text{, if X countable} \\ \int\limits_X A(x)\,dx & \text{, if } X \subseteq \mathbb{R}^n \end{cases}$$

■

Examples:

a) A(x) = q^x with q ∈ (0,1), x ∈ N_0 $\Rightarrow$ card(A) = $\sum\limits_{x \in X} A(x) = \sum\limits_{x=0}^{\infty} q^x = \frac{1}{1-q} < \infty$

b) A(x) = 1/x with x ∈ N $\Rightarrow$ card(A) = $\sum\limits_{x \in X} A(x) = \sum\limits_{x=1}^{\infty} \frac{1}{x} = \infty$

c) A(x) = exp(-|x|) with x ∈ $\mathbb{R}$ $\Rightarrow$ card(A) = $\int\limits_{x \in X} A(x)dx = \int\limits_{x=-\infty}^{\infty} \exp(-|x|)dx = 2$

Theorem

For fuzzy sets A, B and C over a crisp set X the standard union operation is

a) ***commutative*** : $A \cup B = B \cup A$

b) ***associative*** : $A \cup (B \cup C) = (A \cup B) \cup C$

c) ***idempotent*** : $A \cup A = A$

d) ***monotone*** : $A \subseteq B \Rightarrow (A \cup C) \subseteq (B \cup C)$.

Proof: (via reduction to definitions)

ad a) $A \cup B = \max \{ A(x), B(x) \} = \max \{ B(x), A(x) \} = B \cup A$.

ad b) $A \cup (B \cup C) = \max \{ A(x), \max\{ B(x), C(x) \} \} = \max \{ A(x), B(x) , C(x) \}$
$= \max \{ \max \{ A(x), B(x) \} , C(x) \} = (A \cup B) \cup C$.

ad c) $A \cup A = \max \{ A(x), A(x) \} = A(x) = A$.

ad d) $A \cup C = \max \{ A(x), C(x) \} \leq \max \{ B(x), C(x) \} = B \cup C$ since $A(x) \leq B(x)$. **q.e.d.**

Theorem

For fuzzy sets A, B and C over a crisp set X the standard intersection operation is

a) ***commutative*** : $A \cap B = B \cap A$

b) ***associative*** : $A \cap (B \cap C) = (A \cap B) \cap C$

c) ***idempotent*** : $A \cap A = A$

d) ***monotone*** : $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$.

Proof: (analogous to proof for standard union operation) ■

Theorem

For fuzzy sets A, B and C over a crisp set X there are the distributive laws

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof:

ad a) $\max\{ A(x), \min\{ B(x), C(x) \} \} = \begin{cases} \max\{ A(x), B(x) \} & \text{if } B(x) \le C(x) \\ \max\{ A(x), C(x) \} & \text{otherwise} \end{cases}$

If $B(x) \le C(x)$ then $\max\{ A(x), B(x) \} \le \max\{ A(x), C(x) \}$.

Otherwise $\max\{ A(x), C(x) \} \le \max\{ A(x), B(x) \}$.

$\Rightarrow$ result is always the smaller max-expression

$\Rightarrow$ result is **min** $\{ \max\{ A(x), B(x) \}, \max\{ A(x), C(x) \} \} = (A \cup B) \cap (A \cup C)$.

ad b) analogous. ■

Theorem

If A is a fuzzy set over a crisp set X then

a) $A \cup \mathbb{O} = A$

b) $A \cup \mathbb{U} = \mathbb{U}$

c) $A \cap \mathbb{O} = \mathbb{O}$

d) $A \cap \mathbb{U} = A$.

Proof:

(via reduction to definitions)

ad a) $\max \{ A(x), 0 \} = A(x)$

ad b) $\max \{ A(x), 1 \} = \mathbb{U}(x) \equiv 1$

ad c) $\min \{ A(x), 0 \} = \mathbb{O}(x) \equiv 0$

ad d) $\min \{ A(x), 1 \} = A(x)$. ■

Breakpoint:

So far we know that fuzzy sets with operations $\cap$ and $\cup$ are a *distributive lattice*.

If we can show the validity of

- $(A^c)^c = A$
- $A \cup A^c = \mathbb{U}$
- $A \cap A^c = \mathbb{O}$

$\Rightarrow$ Fuzzy Sets would be Boolean Algebra! **Is it true ?**

Theorem

If A is a fuzzy set over a crisp set X then

a) $(A^c)^c = A$

b) $\frac{1}{2} \le (A \cup A^c)(x) < 1$ for $A(x) \in (0,1)$

c) $0 < (A \cap A^c)(x) \le \frac{1}{2}$ for $A(x) \in (0,1)$

Remark:

Recall the identities

$$\min\{a, b\} = \frac{a+b-|a-b|}{2}$$

$$\max\{a, b\} = \frac{a+b+|a-b|}{2}$$

Proof:

ad a) $\forall x \in X: 1 - (1 - A(x)) = A(x)$.

ad b) $\forall x \in X: \max\{ A(x), 1 - A(x) \} = \frac{1}{2} + | A(x) - \frac{1}{2} | \ge \frac{1}{2}$.

Value 1 only attainable for A(x) = 0 or A(x) = 1.

ad c) $\forall x \in X: \min\{ A(x), 1 - A(x) \} = \frac{1}{2} - | A(x) - \frac{1}{2} | \le \frac{1}{2}$.

Value 0 only attainable for A(x) = 0 or A(x) = 1.

q.e.d.

Conclusion:

Fuzzy sets with $\cup$ and $\cap$ are a distributive lattice.

But in general:

a) $A \cup A^c \neq \mathbb{U}$

b) $A \cap A^c \neq \mathbb{O}$

$\Rightarrow$ Fuzzy sets with $\cup$ and $\cap$ are **not** a Boolean algebra!

Remarks:

ad a) The **law of excluded middle** does not hold!

(„Everything must either be or not be!“)

ad b) The **law of noncontradiction** does not hold!

(„Nothing can both be and not be!“)

$\Rightarrow$ Nonvalidity of these laws generate the <u>desired</u> fuzziness!

but: Fuzzy sets still endowed with much algebraic structure (distributive lattice)!

Theorem

If A and B are fuzzy sets over a crisp set X with standard union, intersection, and complement operations then **DeMorgan**‘s laws are valid:

a) $(A \cap B)^c = A^c \cup B^c$

b) $(A \cup B)^c = A^c \cap B^c$

Proof: (via reduction to elementary identities)

ad a) $(A \cap B)^c(x) = 1 - \min \{ A(x), B(x) \} = \max \{ 1 - A(x), 1 - B(x) \} = A^c(x) \cup B^c(x)$

ad b) $(A \cup B)^c(x) = 1 - \max \{ A(x), B(x) \} = \min \{ 1 - A(x), 1 - B(x) \} = A^c(x) \cap B^c(x)$

q.e.d.

Question : Why restricting result above to “*standard*“ operations?

Conjecture : Most likely there also exist “*nonstandard*” operations!