# Computational Intelligence

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## Plan for Today



- Fuzzy Sets
  - Basic Definitions and Results for Standard Operations
  - Algebraic Difference between Fuzzy and Crisp Sets



## Fuzzy Systems: Introduction



**Observation:**

Communication between people is not precise but somehow fuzzy and vague.

"If the water is too hot then add a little bit of cold water."

Despite these shortcomings in human language we are able

- to process fuzzy / uncertain information and
- to accomplish complex tasks!

**Goal:**

Development of formal framework to process fuzzy statements in computer.



## Fuzzy Systems: Introduction



Consider the statement: "The water is hot."

Which temperature defines "hot"?

A single temperature T = 95° C?

No! Rather, an interval of temperatures: $T \in [70, 120]$ !

But who defines the limits of the intervals?

Some people regard temperatures > 60° C as hot, others already T > 50° C!

**Idea**: All people might agree that a temperature in the set [70, 120] defines a hot temperature!

If T = 65°C not all people regard this as hot. It does not belong to [70,120].

But it is hot to some degree.

Or: T = 65°C belongs to set of hot temperatures to some degree!

⇒ **Can be the concept for capturing fuzziness!** ⇒ **Formalize this concept!**

**Definition**

A map F: X → [0,1] $\subset \mathbb{R}$ that assigns its ***degree of membership*** F(x) to each x ∈ X is termed a **fuzzy set**.

**Remark:**

A fuzzy set F is actually a map F(x). Shorthand notation is simply F.

Same point of view possible for traditional ("***crisp***") sets:

$$A(x) := 1_{[x \in A]} := 1_A(x) := \begin{cases} 1 & \text{, if } x \in A \\ 0 & \text{, if } x \notin A \end{cases}$$

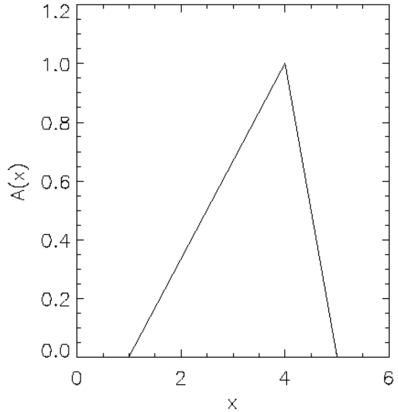
characteristic / indicator function of (crisp) set A

⇒ membership function interpreted as generalization of characteristic function



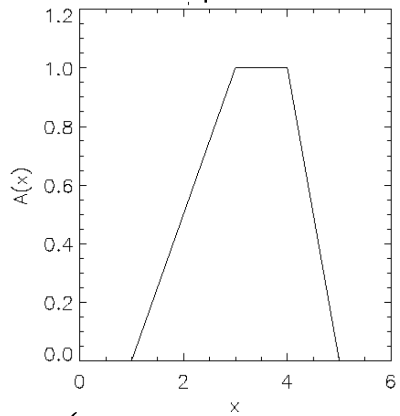


triangle function




$$A(x) = \begin{cases} \frac{1}{3}(x-1) & \text{if } 1 \le x < 4 \\ 5 - x & \text{if } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

trapezoidal

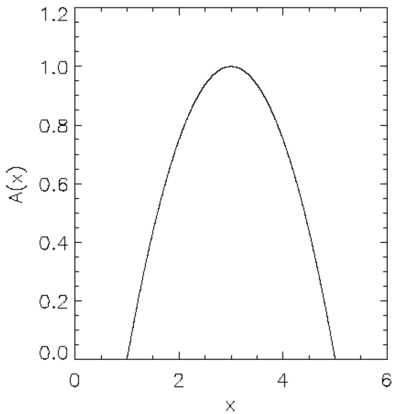



$$A(x) = \begin{cases} \frac{1}{2}(x-1) & \text{if } 1 \le x < 3 \\ 1 & \text{if } 3 \le x < 4 \\ 5 - x & \text{if } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$



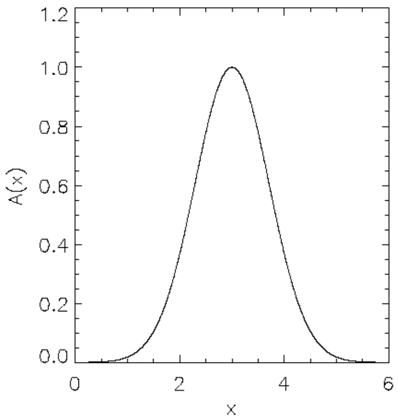


paraboloidal function




$$A(x) = \begin{cases} -\frac{(x-1)(x-5)}{4} & \text{if } 1 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

gaussoid function




$$A(x) = \exp\left(-\frac{(x-3)^2}{2}\right)$$





**Definition**

A fuzzy set F over the crisp set X is termed

a) ***empty*** if F(x) = 0 for all x ∈ X,

b) ***universal*** if F(x) = 1 for all x ∈ X.

Empty fuzzy set is denoted by $\mathbb{O}$. Universal set is denoted by $\mathbb{U}$. ■

**Definition**

Let A and B be fuzzy sets over the crisp set X.

a) A and B are termed ***equal***, denoted A = B, if A(x) = B(x) for all x ∈ X.

b) A is a ***subset*** of B, denoted A ⊆ B, if A(x) ≤ B(x) for all x ∈ X.

c) A is a ***strict subset*** of B, denoted A ⊂ B, if A ⊆ B and ∃ x ∈ X: A(x) < B(x). ■

**Remark:** A strict subset is also called a ***proper*** subset.

## Fuzzy Sets: Basic Relations



**Theorem**

Let A, B and C be fuzzy sets over the crisp set X. The following relations are valid:

a) reflexivity : $A \subseteq A$.

b) antisymmetry : $A \subseteq B$ and $B \subseteq A \;\Rightarrow\; A = B$.

c) transitivity : $A \subseteq B$ and $B \subseteq C \;\Rightarrow\; A \subseteq C$.

**Proof:** (via reduction to definitions and exploiting operations on crisp sets)

ad a) $\forall x \in X$: $A(x) \le A(x)$.

ad b) $\forall x \in X$: $A(x) \le B(x)$ and $B(x) \le A(x) \Rightarrow A(x) = B(x)$.

ad c) $\forall x \in X$: $A(x) \le B(x)$ and $B(x) \le C(x) \Rightarrow A(x) \le C(x)$. **q.e.d.**

**Remark**: Same relations valid for crisp sets. No Surprise! Why?



## Fuzzy Sets: Standard Operations



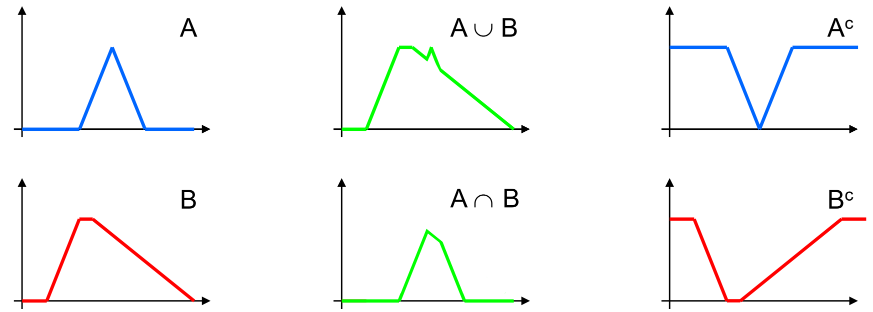
**Definition**

Let A and B be fuzzy sets over the crisp set X. The set C is the

a) ***union*** of A and B, denoted $C = A \cup B$, if $C(x) = \max\{ A(x), B(x) \}$ for all $x \in X$;

b) ***intersection*** of A and B, denoted $C = A \cap B$, if $C(x) = \min\{ A(x), B(x) \}$ for all $x \in X$;

c) ***complement*** of A, denoted $C = A^c$, if $C(x) = 1 - A(x)$ for all $x \in X$. ■

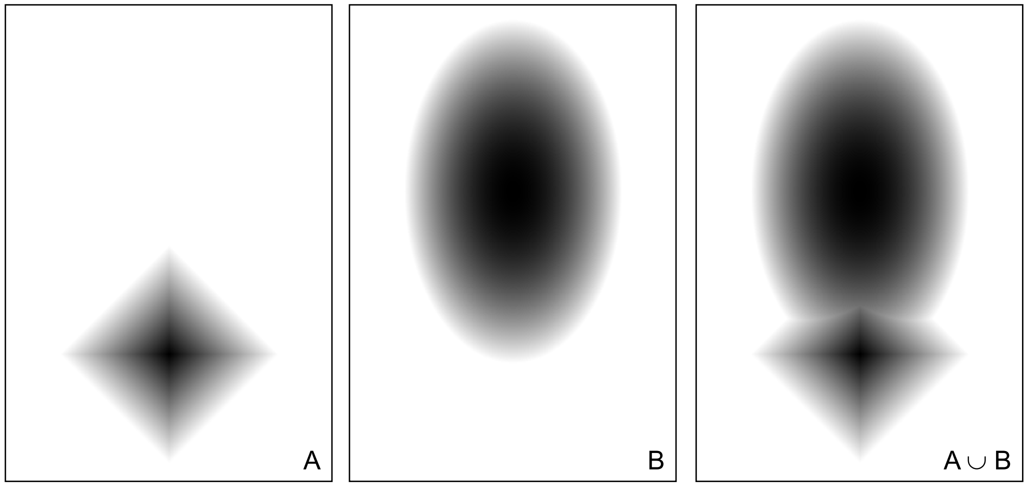





## Fuzzy Sets: Standard Operations in 2D



**standard fuzzy union**



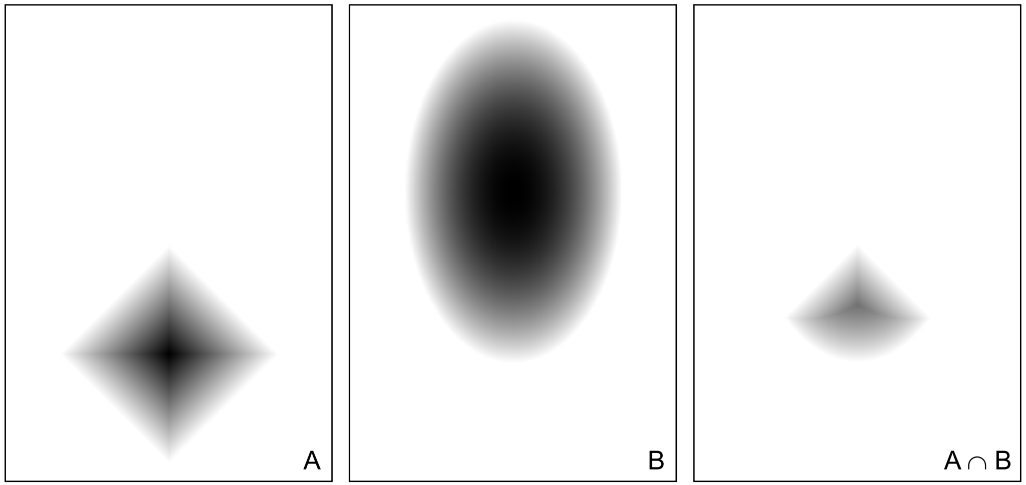

**interpretation:** membership = 0 is white, = 1 is black, in between is gray



## Fuzzy Sets: Standard Operations in 2D

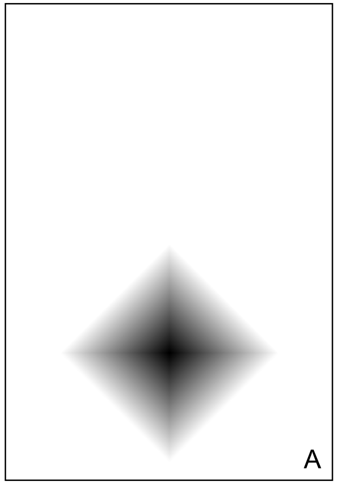


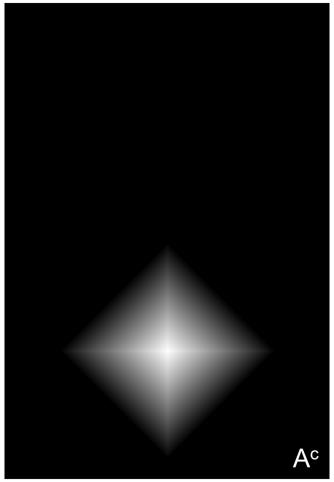
**standard fuzzy intersection**




**interpretation:** membership = 0 is white, = 1 is black, in between is gray

**standard fuzzy complement**




**interpretation:** membership = 0 is white, = 1 is black, in between is gray



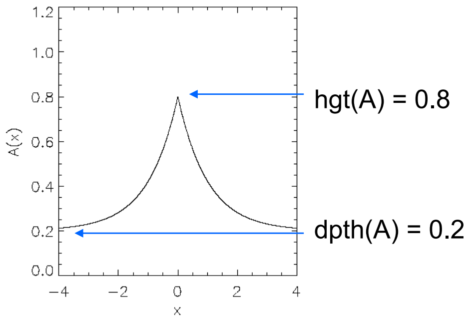


**Definition**

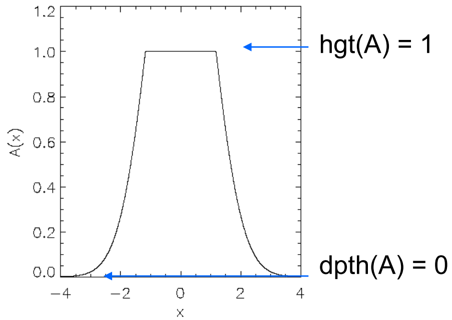
The fuzzy set A over the crisp set X has

a) ***height*** hgt(A) = sup{ A(x) : x ∈ X },

b) ***depth*** dpth(A) = inf { A(x) : x ∈ X }. ■




$$A(x) = \frac{1}{5} + \frac{3}{5}\exp(-|x|)$$




$$A(x) = \min\left\{1, 2\exp\left(-\frac{x^2}{2}\right)\right\}$$





**Definition**

The fuzzy set A over the crisp set X is

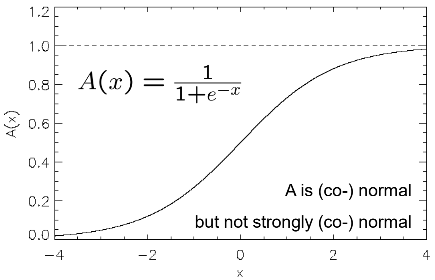
a) ***normal*** if hgt(A) = 1

b) ***strongly normal*** if ∃ x ∈ X: A(x) = 1

c) ***co-normal*** if dpth(A) = 0

d) ***strongly co-normal*** if ∃ x ∈ X: A(x) = 0

e) ***subnormal*** if 0 < A(x) < 1 for all x ∈ X. ■




**Remark:**

How to normalize a non-normal fuzzy set A?

$$A^*(x) = \frac{A(x)}{\text{hgt}(A)}$$





**Definition**

The ***cardinality*** card(A) of a fuzzy set A over the crisp set X is

$$\text{card}(A) := \begin{cases} \sum\limits_{x \in X} A(x) & \text{, if X countable} \\ \int\limits_X A(x)\,dx & \text{, if } X \subseteq \mathbb{R}^n \end{cases}$$ ■

**Examples**:

a) $A(x) = q^x$ with $q \in (0,1)$, $x \in \mathbb{N}_0$ $\Rightarrow \text{card}(A) = \sum\limits_{x \in X} A(x) = \sum\limits_{x=0}^{\infty} q^x = \frac{1}{1-q} < \infty$

b) $A(x) = 1/x$ with $x \in \mathbb{N}$ $\Rightarrow \text{card}(A) = \sum\limits_{x \in X} A(x) = \sum\limits_{x=1}^{\infty} \frac{1}{x} = \infty$

c) $A(x) = \exp(-|x|)$ with $x \in \mathbb{R}$ $\Rightarrow \text{card}(A) = \int\limits_{x \in X} A(x)dx = \int\limits_{x=-\infty}^{\infty} \exp(-|x|)dx = 2$

**Theorem**

For fuzzy sets A, B and C over a crisp set X the standard union operation is

a) ***commutative*** : $A \cup B = B \cup A$

b) ***associative*** : $A \cup (B \cup C) = (A \cup B) \cup C$

c) ***idempotent*** : $A \cup A = A$

d) ***monotone*** : $A \subseteq B \Rightarrow (A \cup C) \subseteq (B \cup C)$.

***Proof:*** (via reduction to definitions)

ad a) $A \cup B = \max \{ A(x), B(x) \} = \max \{ B(x), A(x) \} = B \cup A$.

ad b) $A \cup (B \cup C) = \max \{ A(x), \max\{ B(x), C(x) \} \} = \max \{ A(x), B(x), C(x) \}$
$= \max \{ \max \{ A(x), B(x) \}, C(x) \} = (A \cup B) \cup C$.

ad c) $A \cup A = \max \{ A(x), A(x) \} = A(x) = A$.

ad d) $A \cup C = \max \{ A(x), C(x) \} \le \max \{ B(x), C(x) \} = B \cup C$ since $A(x) \le B(x)$. **q.e.d.**





**Theorem**

For fuzzy sets A, B and C over a crisp set X the standard intersection operation is

a) ***commutative*** : $A \cap B = B \cap A$

b) ***associative*** : $A \cap (B \cap C) = (A \cap B) \cap C$

c) ***idempotent*** : $A \cap A = A$

d) ***monotone*** : $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$.

***Proof*:** (analogous to proof for standard union operation) ■





**Theorem**

For fuzzy sets A, B and C over a crisp set X there are the distributive laws

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

***Proof:***

ad a) $\max \{ A(x), \min \{ B(x), C(x) \} \} = \begin{cases} \max \{ A(x), B(x) \} & \text{if } B(x) \le C(x) \\ \max \{ A(x), C(x) \} & \text{otherwise} \end{cases}$

If $B(x) \le C(x)$ then $\max \{ A(x), B(x) \} \le \max \{ A(x), C(x) \}$.

Otherwise $\max \{ A(x), C(x) \} \le \max \{ A(x), B(x) \}$.

$\Rightarrow$ result is always the smaller max-expression

$\Rightarrow$ result is **min** $\{ \max \{ A(x), B(x) \}, \max \{ A(x), C(x) \} \} = (A \cup B) \cap (A \cup C)$.

ad b) analogous. ■





**Theorem**

If A is a fuzzy set over a crisp set X then

a) $A \cup \mathbb{O} = A$

b) $A \cup \mathbb{U} = \mathbb{U}$

c) $A \cap \mathbb{O} = \mathbb{O}$

d) $A \cap \mathbb{U} = A$.

***Proof:***

(via reduction to definitions)

ad a) $\max \{ A(x), 0 \} = A(x)$

ad b) $\max \{ A(x), 1 \} = \mathbb{U}(x) \equiv 1$

ad c) $\min \{ A(x), 0 \} = \mathbb{O}(x) \equiv 0$

ad d) $\min \{ A(x), 1 \} = A(x)$. ■

**Breakpoint:**

So far we know that fuzzy sets with operations $\cap$ and $\cup$ are a *distributive lattice*.

If we can show the validity of

- $(A^c)^c = A$
- $A \cup A^c = \mathbb{U}$
- $A \cap A^c = \mathbb{O}$

$\Rightarrow$ Fuzzy Sets would be Boolean Algebra! **Is it true ?**

**Theorem**

If A is a fuzzy set over a crisp set X then

a) $(A^c)^c = A$

b) $\frac{1}{2} \le (A \cup A^c)(x) < 1$ for $A(x) \in (0,1)$

c) $0 < (A \cap A^c)(x) \le \frac{1}{2}$ for $A(x) \in (0,1)$

**Remark:**

Recall the identities

$$\min\{a,b\} = \frac{a+b-|a-b|}{2}$$

$$\max\{a,b\} = \frac{a+b+|a-b|}{2}$$

***Proof*:**

ad a) $\forall\, x \in X$: $1 - (1 - A(x)) = A(x)$.

ad b) $\forall\, x \in X$: $\max\{ A(x), 1 - A(x) \} = \frac{1}{2} + | A(x) - \frac{1}{2} | \ge \frac{1}{2}$.

Value 1 only attainable for A(x) = 0 or A(x) = 1.

ad c) $\forall\, x \in X$: $\min\{ A(x), 1 - A(x) \} = \frac{1}{2} - | A(x) - \frac{1}{2} | \le \frac{1}{2}$.

Value 0 only attainable for A(x) = 0 or A(x) = 1.

**q.e.d.**





**Conclusion:**

Fuzzy sets with $\cup$ and $\cap$ are a distributive lattice.

But in general:

$$\left.\begin{array}{l} \text{a) } A \cup A^c \neq \mathbb{U} \\ \text{b) } A \cap A^c \neq \mathbb{O} \end{array}\right\} \Rightarrow$$ Fuzzy sets with $\cup$ and $\cap$ are **not** a Boolean algebra!

**Remarks:**

ad a) The **law of excluded middle** does not hold!

(„Everything must either be or not be!“)

ad b) The **law of noncontradiction** does not hold!

(„Nothing can both be and not be!“)

$\Rightarrow$ Nonvalidity of these laws generate the <u>desired</u> fuzziness!

**but**: Fuzzy sets still endowed with much algebraic structure (distributive lattice)!





**Theorem**

If A and B are fuzzy sets over a crisp set X with standard union, intersection, and complement operations then **DeMorgan**'s laws are valid:

a) $(A \cap B)^c = A^c \cup B^c$

b) $(A \cup B)^c = A^c \cap B^c$

***Proof:*** (via reduction to elementary identities)

ad a) $(A \cap B)^c(x) = 1 - \min\{ A(x), B(x) \} = \max\{ 1 - A(x), 1 - B(x) \} = A^c(x) \cup B^c(x)$

ad b) $(A \cup B)^c(x) = 1 - \max\{ A(x), B(x) \} = \min\{ 1 - A(x), 1 - B(x) \} = A^c(x) \cap B^c(x)$

**q.e.d.**

**Question** : Why restricting result above to “*<u>standard</u>*“ operations?

**Conjecture** : Most likely there also exist “*<u>nonstandard</u>*” operations!