technische universität
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# Computational Intelligence

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- Fuzzy relations
- Fuzzy logic
  - Linguistic variables and terms
  - Inference from fuzzy statements

relations with <u>conventional sets</u> $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!

$\Rightarrow$ all set operations remain valid!

crisp membership function (of $x$ to relation $R$)

$$R(x_1, x_2, \ldots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \ldots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

**Definition**

***Fuzzy relation*** = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \times \mathcal{X}_n$ ■

→ each tuple $(x_1, ..., x_n)$ has a degree of membership to relation

→ degree of membership expresses
*strength of relationship* between elements of tuple

appropriate representation: n-dimensional membership matrix

**example**: Let X = { New York, Paris } and Y = { Bejing, New York, Dortmund }.

relation R = "very far away"

membership matrix →

| **relation R** | New York | Paris |
|---|---|---|
| Bejing | 1.0 | 0.9 |
| New York | 0.0 | 0.7 |
| Dortmund | 0.6 | 0.3 |

**Definition**

Let R(X, Y) be a fuzzy relation with membership matrix **R**. The ***inverse fuzzy relation*** to R(X,Y), denoted $R^{-1}$(Y, X), is a relation on Y x X with membership matrix **R**‘. ■

**Remark**: **R**‘ is the transpose of membership matrix **R**.

Evidently: $(R^{-1})^{-1} = R$ since (**R**‘)‘ = **R**

**Definition**

Let P(X, Y) and Q(Y, Z) be fuzzy relations. The operation ◦ on two relations, denoted P(X, Y) ◦ Q(Y, Z), is termed ***max-min-composition*** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$

■

**Theorem**

a) max-min composition on relations is associative.

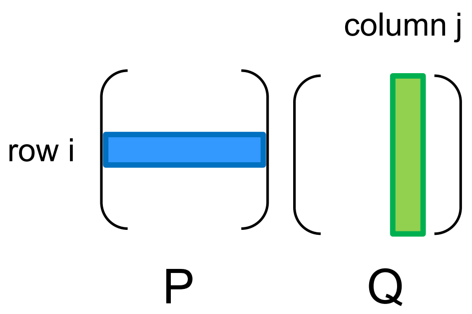
b) max-min composition on relations is not commutative.

c) $( P(X,Y) \circ Q(Y,Z) )^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y, X)$.

membership matrix of max-min composition
determinable via "fuzzy matrix multiplication": $R = P \circ Q$

fuzzy matrix multiplication $r_{ij} = \max_k \min\{p_{ik}, q_{kj}\}$

crisp matrix multiplication $r_{ij} = \sum_k p_{ik} \cdot q_{kj}$



further methods for realizing compositions of relations:

**max-prod composition**

$$(P \odot Q)(x,z) = \max_{y \in \mathcal{Y}} \{ P(x,y) \cdot Q(y,z) \}$$

**generalization: sup-t composition**

$$(P \circ Q)(x,z) = \sup_{y \in \mathcal{Y}} \{ t(P(x,y), Q(y,z)) \},$$ where t(.,.) is a t-norm

| | | |
|---|---|---|
| e.g.: | t(a,b) = min{a, b} | ⇒ max-min-composition |
| | t(a,b) = a · b | ⇒ max-prod-composition |

## Binary fuzzy relations on X x X : properties

- **reflexive** $\Leftrightarrow \forall\, x \in X : R(x,x) = 1$
- **irreflexive** $\Leftrightarrow \exists\, x \in X : R(x,x) < 1$
- **antireflexive** $\Leftrightarrow \forall\, x \in X : R(x,x) < 1$

- **symmetric** $\Leftrightarrow \forall\, (x,y) \in X \times X : R(x,y) = R(y,x)$
- **asymmetric** $\Leftrightarrow \exists\, (x,y) \in X \times X : R(x,y) \neq R(y,x)$
- **antisymmetric** $\Leftrightarrow \forall\, (x,y) \in X \times X : R(x,y) \neq R(y,x)$

- **transitive** $\Leftrightarrow \forall\, (x,z) \in X \times X : R(x,z) \geq \max_{y \in X} \min \{ R(x,y), R(y,z) \}$
- **intransitive** $\Leftrightarrow \exists\, (x,z) \in X \times X : R(x,z) < \max_{y \in X} \min \{ R(x,y), R(y,z) \}$
- **antitransitive** $\Leftrightarrow \forall\, (x,z) \in X \times X : R(x,z) < \max_{y \in X} \min \{ R(x,y), R(y,z) \}$

actually, here: max-min-transitivity (→ in general: sup-t-transitivity)

## binary fuzzy relation on X x X: <u>example</u>

Let **X** be a subset of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to“.

- R(x,x) = 1, since every city is certainly very close to itself.

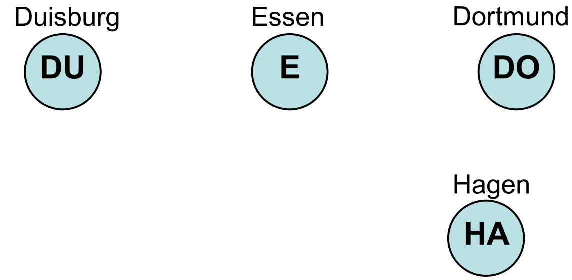
  ⇒ **reflexive**

- R(x,y) = R(y,x): if city x is very close to city y, then also vice versa.

  ⇒ **symmetric**

-

| R | DU | E | DO | HA |
|---|---|---|---|---|
| **DU** | 1 | 0.7 | 0.5 | 0.4 |
| **E** | 0.7 | 1 | 0.8 | 0.8 |
| **DO** | 0.5 | 0.8 | 1 | 0.9 |
| **HA** | 0.4 | 0.8 | 0.9 | 1 |




$R(DO,DU) = 0.5 < \max_y \min\{R(DO,y), R(y, DU)\} = 0.7$

$R(E,DO) = 0.8 \geq \max_y \min\{R(E,y), R(y, DO)\} = 0.8$

⇒ **intransitive**

**crisp:**
relation R is <u>equivalence relation</u> , R reflexive, symmetric, transitive

**fuzzy:**
relation R is <u>similarity relation</u> , R reflexive, symmetric, (max-min-) transitive

<u>examples:</u>

- *equivalence relation*: farm animals
  cattle, pigs, chicken, …
  R(cow, ox) = 1 but R(cow, hen) = 0

- *similarity relation*: farm animals
  cattle, pigs, chicken, horse, donkey, …
  R(mule, (male) donkey) = 0.5 and R(mule, (female) horse) = 0.5

What about **hybrids**?

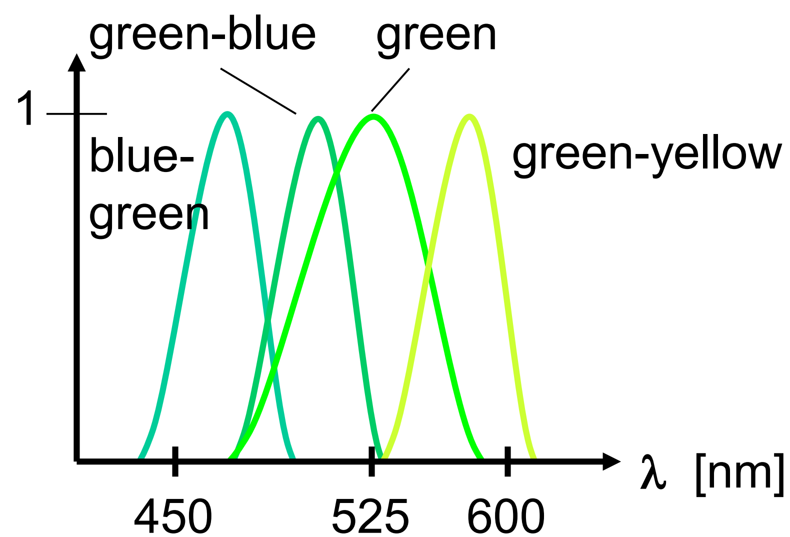
**linguistic variable:**

variable that can attain several values of lingustic / verbal nature

e.g.: **color** can attain values **red**, **green**, **blue**, **yellow**, …

values (red, green, …) of linguistic variable are called **linguistic terms**

linguistic terms are associated with fuzzy sets



## *fuzzy proposition*

p: *temperature* is *high*

linguistic variable (LV) — linguistic term (LT)

- LV may be associated with several LT : *high, medium, low, …*
- *high, medium, low* temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given **concrete crisp** temperature value v is interpreted as equal to the degree of membership *high*(v) of the fuzzy set *high*

## ***fuzzy proposition***

p: *V* is F

↙ linguistic variable (LV)

↘ linguistic term (LT)

actually:

p: *V* is F(v)

and

T(p) = F(v) for a concrete crisp value v

↘ trueness(p)

establishes connection between *degree of membership* of a fuzzy set and the *degree of trueness* of a fuzzy proposition

## ***fuzzy proposition***

p: IF *heating* is *hot,* THEN *energy consumption* is *high*

(*heating*: LV; *hot*: LT; *energy consumption*: LV; *high*: LT)

expresses relation between

a) temperature of heating and

b) quantity of energy consumption

p: (*heating, energy consumption*) ∈ R — relation

## ***fuzzy proposition***

p: IF *X* is A, THEN *Y* is B

| | | | |
|---|---|---|---|
| *X* | A | *Y* | B |
| LV | LT | LV | LT |

How can we determine / express degree of trueness T(p) ?

- For crisp, given values x, y we know A(x) and B(y)
- A(x) and B(y) must be processed to single value via relation R
- R( x, y ) = function( A(x), B(y) ) is fuzzy set over X x Y
- as before: interprete T(p) as degree of membership R(x,y)

## ***fuzzy proposition***

p: IF $X$ is A, THEN $Y$ is B

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over X x Y

$\forall\ (x,y) \in X \times Y:\quad R(x, y) = \text{Imp}(\ A(x), B(y)\ )$

What is Imp(·,·) ?

$\Rightarrow$ „appropriate" fuzzy implication $[0,1] \times [0,1] \to [0,1]$

**assumption:** we know an „appropriate“ Imp(a,b).

How can we determine the *degree of trueness* T(p) ?

**example:** (discrete case)

let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets

A:

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| 0.1 | 0.8 | 1.0 |

B:

| $y_1$ | $y_2$ |
|---|---|
| 0.5 | 1.0 |

⇒

| **R** | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $y_1$ | 1.0 | 0.7 | 0.5 |
| $y_2$ | 1.0 | 1.0 | 1.0 |

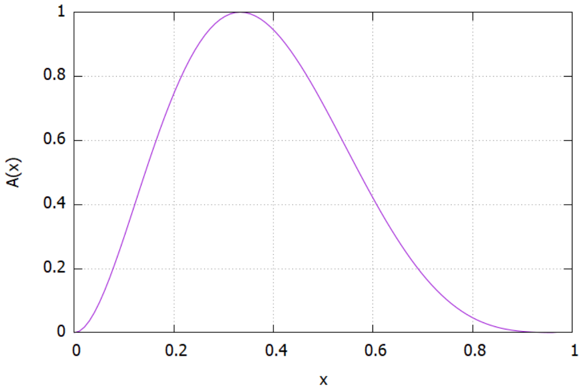
z.B.

$R(x_2, y_1) = Imp(A(x_2), B(y_1)) = Imp(0.8, 0.5) =$
$\min\{1.0, 0.7\} = 0.7$

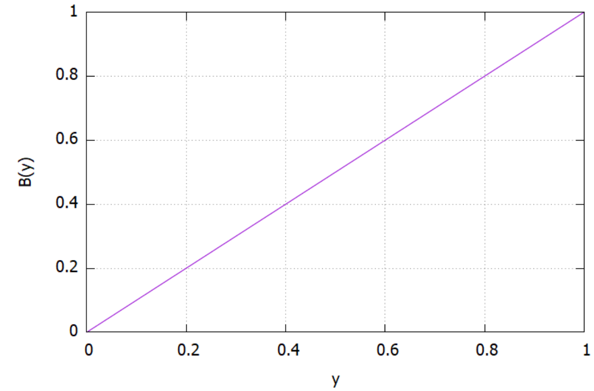
and T(p) for $(x_2,y_1)$ is $R(x_2, y_1) = 0.7$ ■

**example:** (continuous case)

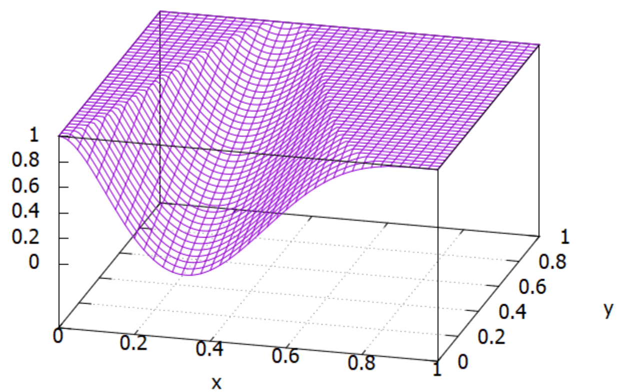
let Imp(a, b) = min{ 1, 1 – a + b } and consider fuzzy sets




$$A(x) = \frac{729}{16}\, x^2\, (1-x)^4 \text{ for } x \in [0,1]$$




$$B(y) = y \text{ for } y \in [0,1]$$




$$\Rightarrow \quad R(x,y) = \min\{1, 1 - A(x) + B(y)\} = \min\{1, 1 - \frac{789}{16}\, x^2\, (1-x)^4 + y\}$$

## toward inference from fuzzy statements:

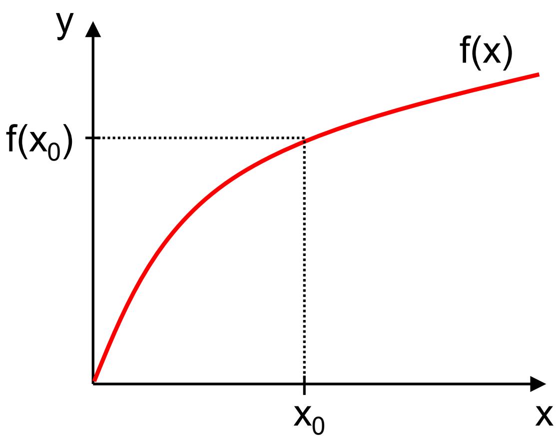
- let $R = \{ (x, y) : y = f(x) \}$ for a function $f: \mathbb{R} \to \mathbb{R}$

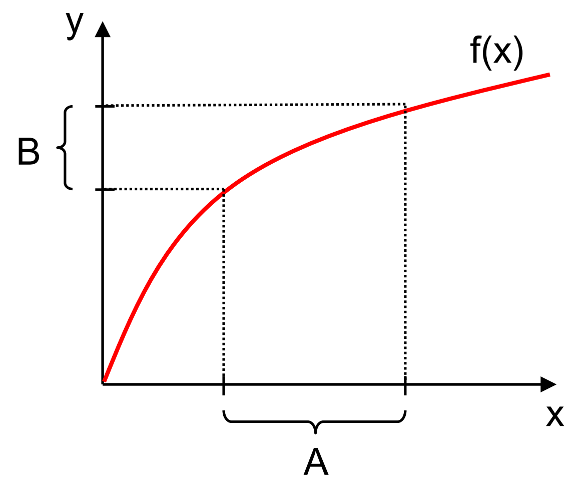
  IF $X = \{ x_0 \}$ THEN $Y = \{ f(x_0) \}$

- IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y} : y = f(x), x \in A \}$

crisp case:

functional relationship



## toward inference from fuzzy statements:

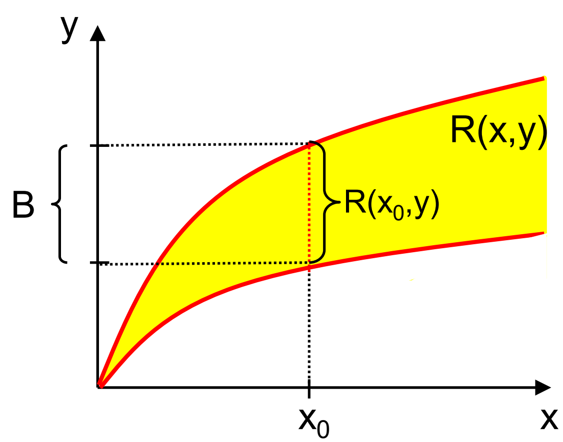
- let relationship between x and y be a relation R on $\mathcal{X} \times \mathcal{Y}$

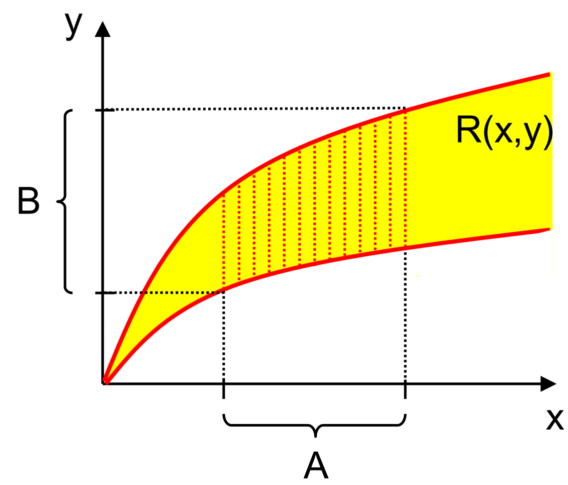
  IF $X = x_0$ THEN $Y \in B = \{ y \in \mathcal{Y} : (x_0, y) \in R \}$

- IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y} : (x, y) \in R,\ x \in A \}$

crisp case:

relational relationship



## toward inference from fuzzy statements:

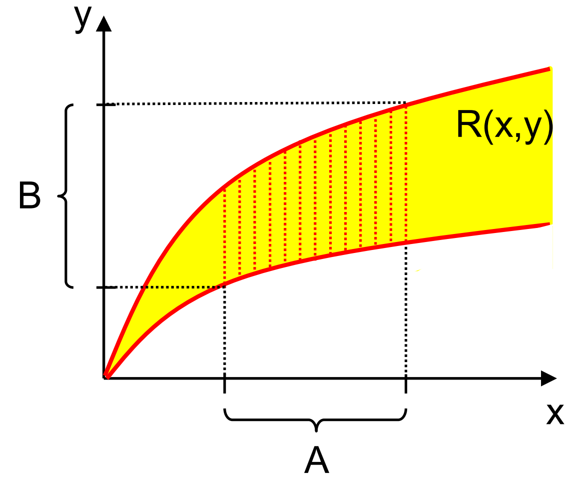
IF $X \in A$ THEN $Y \in B = \{ y \in \mathcal{Y} : (x, y) \in R, \; x \in A \}$

also expressible via characteristic functions of sets A, B, R:

$B(y) = 1$ iff $\exists\, x$: $A(x) = 1$ and $R(x, y) = 1$

$\Leftrightarrow \exists\, x$: $\min\{ A(x), R(x, y) \} = 1$

$\Leftrightarrow \max_{x \in \mathcal{X}} \min\{ A(x), R(x, y) \} = 1$




$\forall y \in \mathcal{Y} : B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$

## inference from fuzzy statements

**Now:** A‘, B‘ fuzzy sets over $\mathcal{X}$ resp. $\mathcal{Y}$

Assume: R(x,y) and A‘(x) are given.

Note:
A‘(x) is **not** the derivative of A(x)!
It is the membership function of fuzzy set A‘.

Idea: Generalize characteristic function of B(y) to membership function B‘(y)

$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$ characteristic functions

↓ ↓ ↓ ↓

$\forall y \in \mathcal{Y}: B'(y) = \sup_{x \in \mathcal{X}} \min \{ A'(x), R(x, y) \}$ membership functions

**composition rule of inference** (in matrix form): $\mathbf{B}^T = \mathbf{A} \circ \mathbf{R}$

## inference from fuzzy statements

- conventional:
  modus ponens

$$\frac{a \Rightarrow b \quad a}{b}$$

- fuzzy:
  generalized modus ponens (GMP)

IF *X* is A, THEN *Y* is B
*X* is A‘

---

*Y* is B‘

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
*heating* is warm

---

*energy consumption* is normal

**example: GMP**

consider

A:

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| 0.5 | 1.0 | 0.6 |

B:

| $y_1$ | $y_2$ |
|---|---|
| 1.0 | 0.4 |

with the rule: IF *X* is A THEN *Y* is B

given fact

A':

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| 0.6 | 0.9 | 0.7 |

$\Rightarrow$

| **R** | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $y_1$ | 1.0 | 1.0 | 1.0 |
| $y_2$ | 0.9 | 0.4 | 0.8 |

with Imp(a,b) = min{1, 1-a+b }

thus: A' ∘ R = B'

with max-min-composition

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

■

## inference from fuzzy statements

- conventional:
  modus tollens

$$\frac{\begin{array}{l} a \Rightarrow b \\ \overline{b} \end{array}}{\overline{a}}$$

- fuzzy:
  generalized modus tollens (GMT)

IF *X* is A, THEN *Y* is B
*Y* is B‘

---

*X* is A‘

e.g.: IF *heating* is hot, THEN *energy consumption* is high
*energy consumption* is normal

---

*heating* is warm

**example: GMT**

consider

A:

| $x_1$ | $x_2$ | $x_3$ |
|---|---|---|
| 0.5 | 1.0 | 0.6 |

B:

| $y_1$ | $y_2$ |
|---|---|
| 1.0 | 0.4 |

with the rule: IF *X* is A THEN *Y* is B

given fact

B‘:

| $y_1$ | $y_2$ |
|---|---|
| 0.9 | 0.7 |

$\Rightarrow$

| **R** | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| $y_1$ | 1.0 | 1.0 | 1.0 |
| $y_2$ | 0.9 | 0.4 | 0.8 |

with Imp(a,b) = min{1, 1-a+b }

thus: $B' \circ R^{-1} = A'$

$$\begin{pmatrix} 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.9 & 0.9 \end{pmatrix}$$

with max-min-composition

■

## inference from fuzzy statements

- conventional:
  hypothetic syllogism

$$\frac{\begin{array}{l} a \Rightarrow b \\ b \Rightarrow c \end{array}}{a \Rightarrow c}$$

- fuzzy:
  generalized HS

$$\frac{\begin{array}{l} \text{IF } X \text{ is A, THEN } Y \text{ is B} \\ \text{IF } Y \text{ is B, THEN } Z \text{ is C} \end{array}}{\text{IF X is A, THEN Z is C}}$$

e.g.:

IF *heating* is hot, THEN *energy consumption* is high
IF *energy consumption* is high, THEN *living* is expensive

---

IF *heating* is hot, THEN *living* is expensive

**example: GHS**

let fuzzy sets A(x), B(x), C(x) be given

$\Rightarrow$ determine the three relations

$R_1(x,y) = \text{Imp}(A(x),B(y))$
$R_2(y,z) = \text{Imp}(B(y),C(z))$
$R_3(x,z) = \text{Imp}(A(x),C(z))$

and express them as matrices $R_1$, $R_2$, $R_3$

**We say:**

GHS is valid if $R_1 \circ R_2 = R_3$

**So, ...** what makes sense for Imp(·,·) ?

Imp(a,b) ought to express fuzzy version of implication $(a \Rightarrow b)$

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b$

But how can we calculate with fuzzy "boolean" expressions?

**request:** must be compatible to crisp version (and more) for $a,b \in \{ 0, 1 \}$

| a | b | $a \wedge b$ | t(a,b) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

| a | b | $a \vee b$ | s(a,b) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |

| a | $\bar{a}$ | c(a) |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |

**So, ...** what makes sense for $\mathrm{Imp}(\cdot,\cdot)$ ?

**1st approach: S implications**

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b$

fuzzy: $\mathrm{Imp}(a, b) = s(\, c(a), b)$

**2nd approach: R implications**

conventional: $a \Rightarrow b$ identical to $\max\{\, x \in \{0,1\} : a \wedge x \le b \,\}$

fuzzy: $\mathrm{Imp}(a, b) = \max\{\, x \in [0,1] : t(a, x) \le b \,\}$

**3rd approach: QL implications**

conventional: $a \Rightarrow b$ identical to $\bar{a} \vee b \equiv \bar{a} \vee (a \wedge b)$ law of absorption

fuzzy: $\mathrm{Imp}(a, b) = s(\, c(a), t(a, b)\,)$ (dual tripel ?)

**example: S implication** Imp(a, b) = s( $c_s$(a), b) ($c_s$ : std. complement)

1. Kleene-Dienes implication

   s(a, b) = max{ a, b } (standard) Imp(a,b) = max{ 1-a, b }

2. Reichenbach implication

   s(a, b) = a + b – ab (algebraic sum) Imp(a, b) = 1 – a + ab

3. Łukasiewicz implication

   s(a, b) = min{ 1, a + b } (bounded sum) Imp(a, b) = min{ 1, 1 – a + b }

**example: R implicationen** Imp(a, b) = max{ x ∈[0,1] : t(a, x) ≤ b }

1. Gödel implication

   t(a, b) = min{ a, b } (std.) $\text{Imp}(a, b) = \begin{cases} 1 & \text{, if } a \leq b \\ b & \text{, else} \end{cases}$

2. Goguen implication

   t(a, b) = ab (algeb. product) $\text{Imp}(a, b) = \begin{cases} 1 & \text{, if } a \leq b \\ \frac{b}{a} & \text{, else} \end{cases}$

3. Łukasiewicz implication

   t(a, b) = max{ 0, a + b – 1 } (bounded diff.) Imp(a, b) = min{ 1, 1 – a + b }

**example: QL implication** Imp(a, b) = s( c(a), t(a, b) )

1. Zadeh implication

   t(a, b) = min { a, b } (std.) Imp(a, b) = max{ 1 – a, min{a, b} }
   s(a,b) = max{ a, b } (std.)

2. „NN“ implication ☺ (Klir/Yuan 1994)

   t(a, b) = ab (algebr. prd.) Imp(a, b) = 1 – a + $a^2$b
   s(a,b) = a + b – ab (algebr. sum)

3. Kleene-Dienes implication

   t(a, b) = max{ 0, a + b – 1 } (bounded diff.) Imp(a, b) = max{ 1-a, b }
   s(a,b) = min { 1, a + b) (bounded sum)

## axioms for fuzzy implications

| | | |
|---|---|---|
| 1. | $a \le b$ implies $Imp(a, x) \ge Imp(b, x)$ | monotone in 1st argument |
| 2. | $a \le b$ implies $Imp(x, a) \le Imp(x, b)$ | monotone in 2nd argument |
| 3. | $Imp(0, a) = 1$ | dominance of falseness |
| 4. | $Imp(1, b) = b$ | neutrality of trueness |
| 5. | $Imp(a, a) = 1$ | identity |
| 6. | $Imp(a, Imp(b, x)) = Imp(b, Imp(a, x))$ | exchange property |
| 7. | $Imp(a, b) = 1$ iff $a \le b$ | boundary condition |
| 8. | $Imp(a, b) = Imp(c(b), c(a))$ | contraposition |
| 9. | $Imp(\cdot,\cdot)$ is continuous | continuity |

**Caution!**

Not all S-, R-, QL- implications obey <u>all</u> axioms for fuzzy implications!

| **Implication** | **Valid Axioms** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Kleene-Dienes | 1 | 2 | 3 | 4 | – | 6 | – | 8 | 9 | |
| Reichenbach | 1 | 2 | 3 | 4 | – | 6 | – | 8 | 9 | |
| Łukasiewicz | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ← |
| Gödel | 1 | 2 | 3 | 4 | 5 | 6 | 7 | – | – | |
| Goguen | 1 | 2 | 3 | 4 | 5 | 6 | 7 | – | 9 | |
| Zadeh | 1 | 2 | 3 | 4 | – | – | – | – | 9 | |
| Klir-Yuan | – | 2 | 3 | 4 | – | – | – | – | 9 | |

## characterization of fuzzy implication

**Theorem:**
Imp: $[0,1] \times [0,1] \rightarrow [0,1]$ satisfies axioms 1 - 9 for fuzzy implications for a certain fuzzy complement $c(\cdot)$ $\Leftrightarrow$

$\exists$ strictly monotone increasing, continuous function $f: [0,1] \rightarrow [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0,1]: \text{Imp}(a, b) = f^{-1}( \min\{ f(1) – f(a) + f(b), f(1)\} )$
- $\forall a \in [0,1]: c(a) = f^{-1}( f(1) – f(a) )$

**Proof:** Smets & Magrez (1987), p. 337f. ■

**examples:** (in tutorial)

**choosing an „appropriate" fuzzy implication ...**

**apt quotation:** (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem."

**guideline:**
GMP, GMT, GHS should be compatible with MP, MT, HS
for fuzzy implication in calculations with relations:
$B(y) = \sup \{ t( A(x), \text{Imp}( A(x), B(y) ) ) : x \in X \}$

**example:**
Gödel implication for t-norm = bounded difference