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Computational Intelligence

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Plan for Today

- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Definition

A function c: $[0,1] \rightarrow [0,1]$ is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall\, a, b \in [0,1]: a \leq b \;\Rightarrow\; c(a) \geq c(b)$. monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall\, a \in [0,1]: c(c(a)) = a$ involutive

Examples:

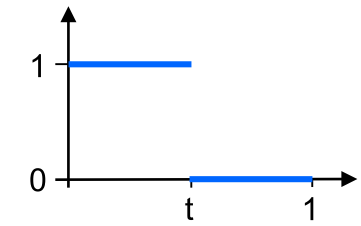
a) standard fuzzy complement $c(a) = 1 – a$

ad (A1): $c(0) = 1 – 0 = 1$ and $c(1) = 1 – 1 = 0$
ad (A2): $c'(a) = –1 < 0$ (monotone decreasing)

ad (A3): ☑
ad (A4): $1 – (1 – a) = a$

Fuzzy Complement: Examples

b) $c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$



ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t

☑

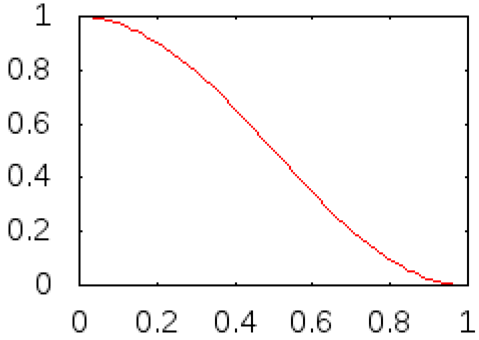
ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(\frac{1}{4})) = c(1) = 0 \neq \frac{1}{4}$ for $t = \frac{1}{2}$

Fuzzy Complement: Examples

c) $c(a) = \dfrac{1 + \cos(\pi\, a)}{2}$



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c'(a) = -\frac{1}{2}\,\pi \sin(\pi\, a) < 0$ since $\sin(\pi\, a) > 0$ for $a \in (0,1)$

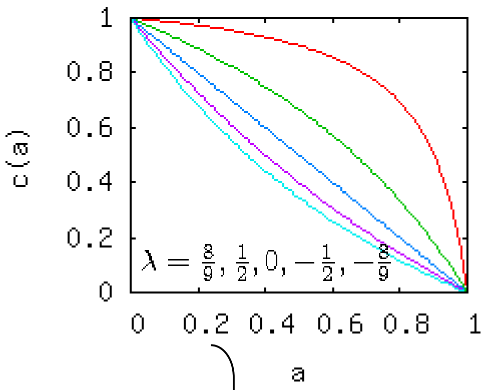
☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$$c\left(c\left(\tfrac{1}{3}\right)\right) = c\left(\tfrac{3}{4}\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) \neq \tfrac{1}{3}$$

d) c(a) = $\frac{1-a}{1+\lambda a}$ for $\lambda > -1$ **Sugeno class**



ad (A1): c(0) = 1 and c(1) = 0

ad (A2): $c(a) \geq c(b) \;\Leftrightarrow\; \frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b} \;\Leftrightarrow$

$$(1-a)(1+\lambda b) \geq (1-b)(1+\lambda a) \;\Leftrightarrow$$

$$b(\lambda+1) \geq a(\lambda+1) \;\Leftrightarrow\; b \geq a$$

☑

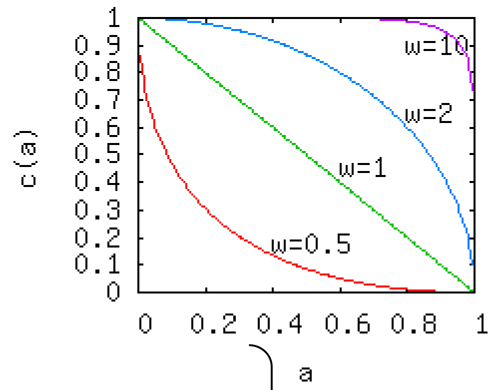
ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) \;=\; c\left(\frac{1-a}{1+\lambda a}\right) \;=\; \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda \frac{1-a}{1+\lambda a}} \;=\; \frac{a(\lambda+1)}{\lambda+1} \;=\; a$

☑

Fuzzy Complement: Examples

e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$

$a^w \leq b^w \Leftrightarrow a \leq b$

☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $$c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$$

$$= (1-(1-a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$$

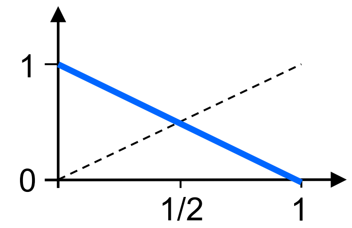
☑

Theorem

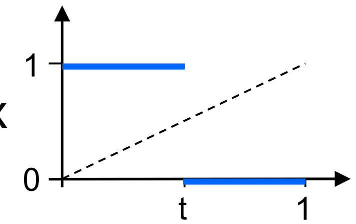
If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix



no fixed point → see example (b) → no intersection with bisectrix



assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b* (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2) ■

Theorem

If function $c:[0,1] \to [0,1]$ satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a^* with $c(a^*) = a^*$.

Proof:

Intermediate value theorem →

If $c(\cdot)$ continuous (A3) and $c(0) \geq c(1)$ (A1/A2)

then $\forall\ v \in [c(1), c(0)] = [0,1]: \exists\ a \in [0,1]: c(a) = v$.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points! ■

Examples:

(a) $c(a) = 1 - a$ ⇒ $a = 1 - a$ ⇒ $a^* = ½$

(b) $c(a) = (1 - a^w)^{1/w}$ ⇒ $a = (1 - a^w)^{1/w}$ ⇒ $a^* = (½)^{1/w}$

Theorem

$c: [0,1] \to [0,1]$ is involutive fuzzy complement iff
$\exists$ continuous function $g: [0,1] \to \mathbb{R}$ with

- $g(0) = 0$
- strictly monotone increasing
- $\forall\, a \in [0,1]: c(a) = g^{(-1)}(\, g(1) - g(a)\,).$ ■

defines an **increasing generator**

$g^{(-1)}(x)$ pseudo-inverse $= g^{-1}(\min\{\, g(1), x\,\})$

Examples

a) $g(x) = x \Rightarrow g^{(-1)}(x) = x \Rightarrow c(a) = 1 - a$ (Standard)

b) $g(x) = x^w \Rightarrow g^{(-1)}(x) = x^{1/w} \Rightarrow c(a) = (1 - a^w)^{1/w}$ (Yager class, $w > 0$)

c) $g(x) = \log(x+1) \Rightarrow g^{(-1)}(x) = e^x - 1 \Rightarrow c(a) = \exp(\, \log(2) - \log(a+1)\,) - 1$

$$= \frac{1-a}{1+a}$$ (Sugeno class. $\lambda = 1$)

Examples

d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda\, a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda\, a} > 0$ for $a \in [0, 1]$
- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda\, a) - 1}{\lambda}$, thus

$$
\begin{aligned}
c(a) &= g^{-1}\left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda\, a)}{\lambda}\right) \\
&= \frac{\exp(\log(1+\lambda) - \log(1+\lambda\, a)) - 1}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda\, a} - 1\right) = \frac{1-a}{1+\lambda\, a} \quad \text{(Sugeno Complement)}
\end{aligned}
$$

Theorem

c: $[0,1] \to [0,1]$ is involutive fuzzy complement iff

$\exists$continuous function f: $[0,1] \to \mathbb{R}$ with

- $f(1) = 0$
- strictly monotone decreasing
- $\forall\ a \in [0,1]$: $c(a) = f^{(-1)}(\ f(0) - f(a)\ )$. ■

$\Big\}$ defines a **decreasing generator**

$f^{(-1)}(x)$ pseudo-inverse

$= f^{-1}(\ \min\{\ f(0),\ x\ \}\ )$

Examples

a) $f(x) = k - k \cdot x \quad (k \geq 1)$ $\quad f^{(-1)}(x) = 1 - x/k$ $\quad c(a) = 1 - \frac{k-(k-ka)}{k} = 1 - a$

b) $f(x) = 1 - x^w$ $\quad f^{(-1)}(x) = (1 - x)^{1/w}$ $\quad c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w}$ (Yager)

Definition

A function $t:[0,1] \times [0,1] \rightarrow [0,1]$ is a ***fuzzy intersection*** or ***t-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $t(a, 1) = a$ (boundary condition)

(A2) $b \leq d \Rightarrow t(a, b) \leq t(a, d)$ (monotonicity)

(A3) $t(a,b) = t(b, a)$ (commutative)

(A4) $t(a, t(b, d)) = t(t(a, b), d)$ (associative) ■

"nice to have"

(A5) $t(a, b)$ is continuous (continuity)

(A6) $t(a, a) < a$ for $0 < a < 1$ (subidempotent)

(A7) $a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow t(a_1, b_1) < t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with $t(a,a) = a$ for all $a \in [0,1]$.

- If $0 \le a \le b \le 1$ then

$$a \underset{\text{by assumption}}{=} t(a,a) \underset{\text{by monotonicity}}{\le} t(a,b) \underset{\text{by monotonicity}}{\le} t(a,1) \underset{\text{by boundary condition}}{=} a$$

  and hence $t(a,b) = a$.

- If $0 \le b \le a \le 1$ then

$$b \underset{\text{by assumption}}{=} t(b,b) \underset{\text{by monotonicity}}{\le} t(b,a) \underset{\text{by monotonicity}}{\le} t(b,1) \underset{\text{by boundary condition}}{=} b$$

  and hence $t(a,b) \underset{\text{by commutativity}}{=} t(b,a) = b$.

$t(a,b) = \min(a,b)$

is the only possible solution!

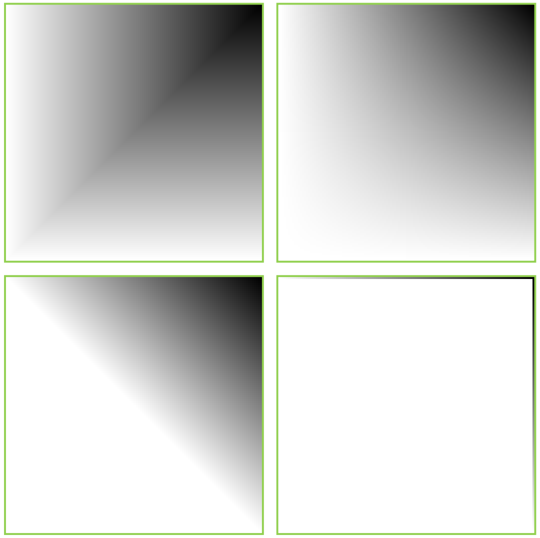
q.e.d.

Fuzzy Intersection: t-norm

Examples:

Name	Function
(a) Standard	$t(a, b) = \min \{ a, b \}$
(b) Algebraic Product	$t(a, b) = a \cdot b$
(c) Bounded Difference	$t(a, b) = \max \{ 0, a + b - 1 \}$
(d) Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$

(a) (b)



(c) (d)

Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): $t(a, 1) = a \cdot 1 = a$ ☑

ad (A2): $a \cdot b \leq a \cdot d \Leftrightarrow b \leq d$ ☑

ad (A3): $t(a, b) = a \cdot b = b \cdot a = t(b, a)$ ☑

ad (A4): $a \cdot (b \cdot d) = (a \cdot b) \cdot d$ ☑

Theorem

Function t: [0,1] x [0,1] → [0,1] is a t-norm ,

∃decreasing generator f:[0,1] → $\mathbb{R}$ with t(a, b) = f^{-1}(min{ f(0), f(a) + f(b) }). ■

Example:

f(x) = 1/x – 1 is decreasing generator since

- f(x) is continuous ☑
- f(1) = 1/1 – 1 = 0 ☑
- f'(x) = $-1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \frac{1}{x+1}$; f(0) = ∞ ⇒ min{ f(0), f(a) + f(b) } = f(a) + f(b)

$$\Rightarrow t(a, b) = f^{-1}\left(\frac{1}{a} + \frac{1}{b} - 2\right) = \frac{1}{\frac{1}{a}+\frac{1}{b}-1} = \frac{ab}{a+b-a\,b}$$

Definition

A function $s:[0,1] \times [0,1] \to [0,1]$ is a ***fuzzy union*** or ***s-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $s(a, 0) = a$ (boundary condition)

(A2) $b \leq d \Rightarrow s(a, b) \leq s(a, d)$ (monotonicity)

(A3) $s(a, b) = s(b, a)$ (commutative)

(A4) $s(a, s(b, d)) = s(s(a, b), d)$ (associative) ■

"nice to have"

(A5) $s(a, b)$ is continuous (continuity)

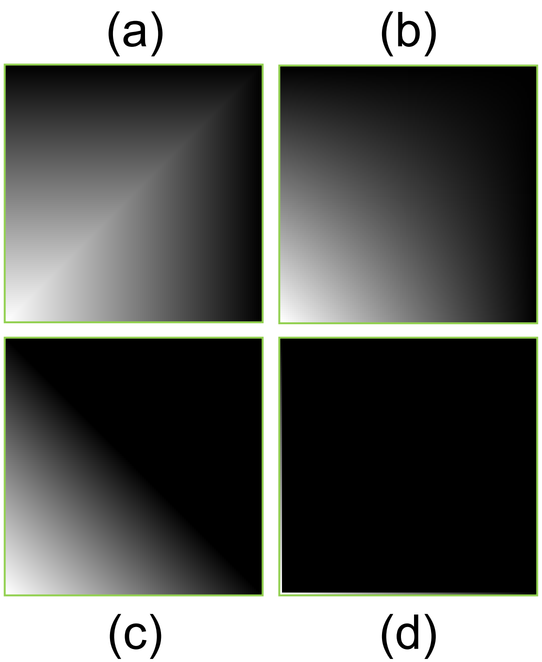
(A6) $s(a, a) > a$ for $0 < a < 1$ (superidempotent)

(A7) $a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow s(a_1, b_1) < s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	$s(a, b) = \max\{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min\{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$



Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$ ☑

ad (A2): $a + b - a \cdot b \le a + d - a \cdot d \Leftrightarrow b\,(1 - a) \le d\,(1 - a) \Leftrightarrow b \le d$ ☑

ad (A3): ☑

ad (A4): ☑

Theorem

Function s: [0,1] x [0,1] → [0,1] is a s-norm ⇔

∃ increasing generator $g:[0,1] \to \mathbb{R}$ with $s(a, b) = g^{-1}(\min\{ g(1), g(a) + g(b) \})$. ■

Example:

$g(x) = -\log(1 - x)$ is increasing generator since

- $g(x)$ is continuous ☑
- $g(0) = -\log(1 - 0) = 0$ ☑
- $g'(x) = 1/(1 - x) > 0$ (monotone increasing) ☑

inverse function is $g^{-1}(x) = 1 - \exp(-x)$; $g(1) = \infty \Rightarrow \min\{g(1), g(a) + g(b)\} = g(a) + g(b)$

$$\begin{aligned} \Rightarrow s(a, b) &= g^{-1}(-\log(1-a) - \log(1-b)) \\ &= 1 - \exp(\log(1-a) + \log(1-b)) \\ &= 1 - (1-a)(1-b) = a + b - a\,b \quad \text{(algebraic sum)} \end{aligned}$$

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition

A pair of t-norm $t(\cdot, \cdot)$ and s-norm $s(\cdot, \cdot)$ is said to be ***dual with regard to the fuzzy complement*** $c(\cdot)$ iff

- $c(\, t(a, b)\,) = s(\, c(a), c(b)\,)$
- $c(\, s(a, b)\,) = t(\, c(a), c(b)\,)$

for all $a, b \in [0,1]$. ■

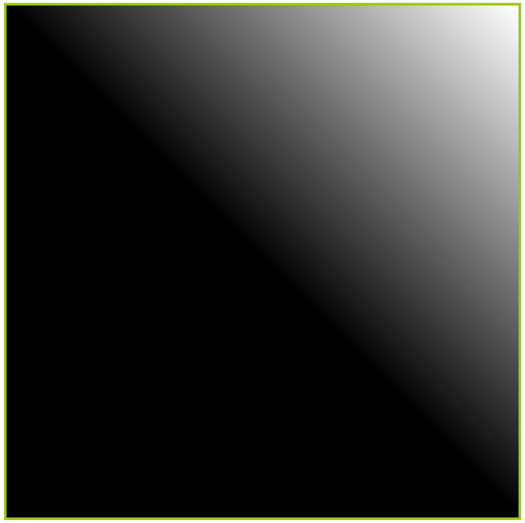
Definition

Let (c, s, t) be a tripel of fuzzy complement $c(\cdot)$, s- and t-norm.

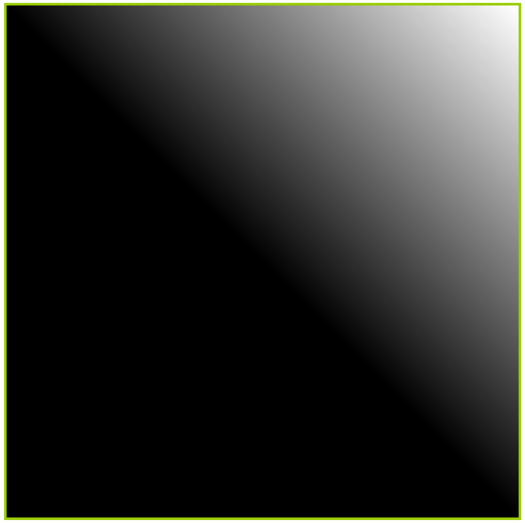
If t and s are dual to c then the tripel (c,s, t) is called a ***dual tripel***. ■

Examples of dual tripels

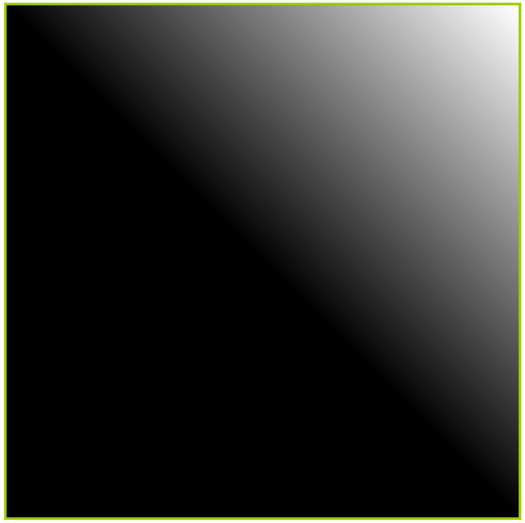
t-norm	s-norm	complement
$\min \{ a, b \}$	$\max \{ a, b \}$	$1 - a$
$a \cdot b$	$a + b - a \cdot b$	$1 - a$
$\max \{ 0, a + b - 1 \}$	$\min \{ 1, a + b \}$	$1 - a$



c(t(a, b))



s(c(a), c(b))



Dual Triple:

- bounded difference
- bounded sum
- standard complement

$\Rightarrow$ left image = right image

Non-Dual Triple:

- algebraic product
- bounded sum
- standard complement

$\Rightarrow$ left image $\neq$ right image

Why are dual triples so important?

⇒ allow equivalence transformations of fuzzy set expressions

⇒ required to transform into some equivalent normal form (standardized input)

⇒ e.g. two stages: intersection of unions $\bigcap_{i=1}^{n} (A_i \cup B_i)$

or union of intersections $\bigcup_{i=1}^{n} (A_i \cap B_i)$

<u>Example:</u>

$A \cup (B \cap (C \cap D)^c) =$	← not in normal form
$A \cup (B \cap (C^c \cup D^c)) =$	← equivalent if DeMorgan‘s law valid (dual triples!)
$A \cup (B \cap C^c) \cup (B \cap D^c)$	← equivalent (distributive lattice!)