Computational Intelligence

Winter Term 2020/21

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- Fuzzy sets

  - Axioms of fuzzy complement, t- and s-norms

  - Generators

  - Dual tripels

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Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$

- $(A \cap B)(x) = \min \{ A(x), B(x) \}$

- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.

- Creation via generators.

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Definition

A function c: [0,1] → [0,1] is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall\, a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$.   monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall\, a \in [0,1]: c(c(a)) = a$ involutive

Examples:

a) standard fuzzy complement $c(a) = 1 - a$

    ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$ ad (A3): ☑
   ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)   ad (A4): $1 - (1 - a) = a$

b) $c(a) = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$
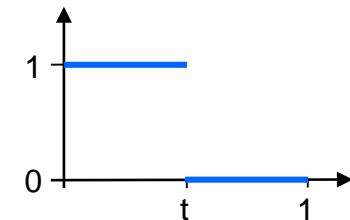
ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2):   monotone (actually: constant) from 0 to t and t to 1, decreasing at t

☑

ad (A3):   not valid → discontinuity at t

ad (A4):   not valid → counter example

$c(c(¼)) = c(1) = 0 \neq ¼$ for $t = ½$

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c) $c(a) = \dfrac{1 + \cos(\pi a)}{2}$
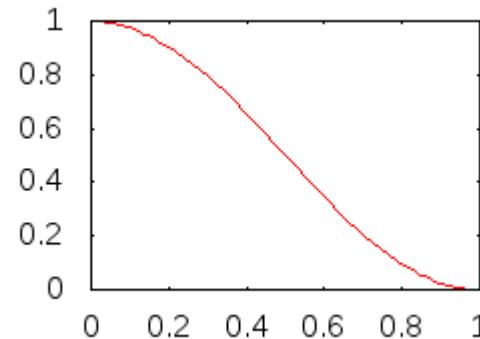
ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c'(a) = -\tfrac{1}{2} \pi \sin(\pi a) < 0$ since $\sin(\pi a) > 0$ for $a \in (0,1)$ ☑
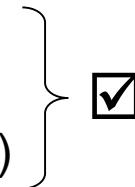
ad (A3):   is continuous as a composition of continuous functions

ad (A4):   not valid → counter example

$$c\left(c\left(\tfrac{1}{3}\right)\right) \;=\; c\left(\tfrac{3}{4}\right) \;=\; \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) \;\neq\; \tfrac{1}{3}$$

d) $c(a) = \dfrac{1-a}{1+\lambda\,a}$ for $\lambda > -1$ **Sugeno class**
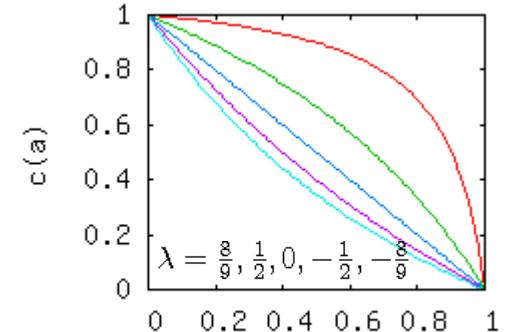
ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c(a) \geq c(b) \;\Leftrightarrow\; \dfrac{1-a}{1+\lambda\,a} \geq \dfrac{1-b}{1+\lambda\,b} \;\Leftrightarrow\;$

$\quad (1-a)\,(1+\lambda\,b) \geq (1-b)\,(1+\lambda\,a) \;\Leftrightarrow\;$

$\quad b\,(\lambda+1) \geq a\,(\lambda+1) \;\Leftrightarrow\; b \geq a$

☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) \;=\; c\left(\dfrac{1-a}{1+\lambda\,a}\right) \;=\; \dfrac{1-\frac{1-a}{1+\lambda\,a}}{1+\lambda\,\frac{1-a}{1+\lambda\,a}} \;=\; \dfrac{a\,(\lambda+1)}{\lambda+1} \;=\; a$

☑

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e) $c(a) = (1 - a^w)^{1/w}$ for w > 0   **Yager class**
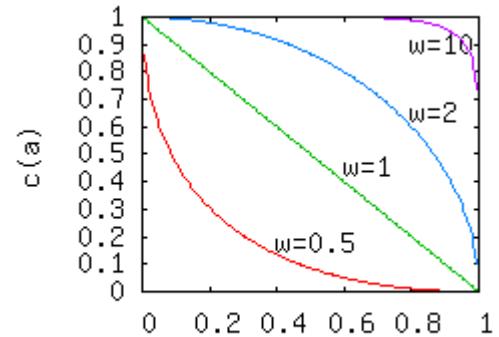
ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$

$a^w \leq b^w \Leftrightarrow a \leq b$

☑

ad (A3):   is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1 - a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1 - a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$

$= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$

☑

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix
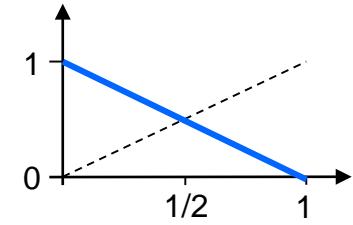
no fixed point → see example (b) → no intersection with bisectrix
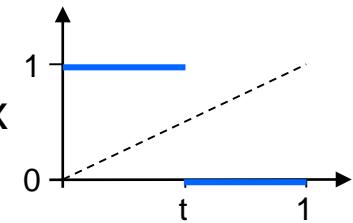
assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b*   (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2)   ∎

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Theorem

If function $c:[0,1] \rightarrow [0,1]$ satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a* with c(a*) = a*.

Proof:

Intermediate value theorem →

If c(·) continuous (A3) and c(0) ≥ c(1) (A1/A2)

then $\forall$ v $\in$ [c(1), c(0)] = [0,1]: $\exists$ a $\in$ [0,1]: c(a) = v.

$\Rightarrow$ there must be an intersection with bisectrix

$\Rightarrow$ a fixed point exists and by previous theorem there are no other fixed points! ∎

Examples:

(a) c(a) = 1 − a $\Rightarrow$ a = 1 − a $\Rightarrow$ a* = ½

(b) c(a) = $(1 - a^w)^{1/w}$ $\Rightarrow$ a = $(1 - a^w)^{1/w}$ $\Rightarrow$ a* = $(½)^{1/w}$

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Theorem

c: $[0,1] \to [0,1]$ is involutive fuzzy complement iff

$\exists$ continuous function g: $[0,1] \to \mathbb{R}$ with

• g(0) = 0

• strictly monotone increasing

• $\forall$ a $\in$ [0,1]: c(a) = $g^{(-1)}($ g(1) – g(a) $)$.   ∎

defines an
increasing generator

$g^{(-1)}(x)$ pseudo-inverse

Examples

a) g(x) = x $\Rightarrow g^{-1}(x) = x$ $\Rightarrow$ c(a) = 1 – a   (Standard)

b) $g(x) = x^w$ $\Rightarrow g^{-1}(x) = x^{1/w}$ $\Rightarrow c(a) = (1 - a^w)^{1/w}$ (Yager class, w > 0)

c) g(x) = log(x+1) $\Rightarrow g^{-1}(x) = e^x - 1$ $\Rightarrow$ c(a) = exp(log(2) – log(a+1)) – 1

$$= \frac{1 - a}{1 + a}$$ (Sugeno class. λ = 1)

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Examples

d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$

- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda a} > 0$ for $a \in [0, 1]$

- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda a) - 1}{\lambda}$, thus

$$
\begin{aligned}
c(a) &= g^{-1}\left(\frac{\log(1 + \lambda)}{\lambda} - \frac{\log(1 + \lambda a)}{\lambda}\right) \\
&= \frac{\exp(\log(1 + \lambda) - \log(1 + \lambda a)) - 1}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{1 + \lambda}{1 + \lambda a} - 1\right) = \frac{1 - a}{1 + \lambda a} \quad \text{(Sugeno Complement)}
\end{aligned}
$$

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃continuous function f: [0,1] → $\mathbb{R}$ with

- f(1) = 0

- strictly monotone decreasing

- ∀ a ∈ [0,1]: c(a) = f$^{(-1)}$(f(0) − f(a)).   ∎

defines a
decreasing generator

f$^{(-1)}$(x) pseudo-inverse

Examples

a) f(x) = $k - k \cdot$ x ($k > 0$)   f$^{(-1)}$(x) = 1 − x/k c(a) = $1 - \dfrac{k-(k-ka)}{k} = 1 - a$

b) f(x) = 1 − x^w f$^{(-1)}$(x) = (1 − x)$^{1/w}$ c(a) = f^{-1}(a^w) = (1 − a^w)$^{1/w}$ (Yager)

Definition

A function $t:[0,1] \times [0,1] \to [0,1]$ is a ***fuzzy intersection*** or ***t-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $t(a, 1) = a$ (boundary condition)

(A2) $b \leq d \Rightarrow t(a, b) \leq t(a, d)$ (monotonicity)

(A3) $t(a,b) = t(b, a)$ (commutative)

(A4) $t(a, t(b, d)) = t(t(a, b), d)$ (associative) ∎

"nice to have"

(A5) $t(a, b)$ is continuous (continuity)

(A6) $t(a, a) < a$ for $0 < a < 1$ (subidempotent)

(A7) $a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow t(a_1, b_1) < t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with $t(a,a) = a$ for all $a \in [0,1]$.

- If $0 \leq a \leq b \leq 1$ then

$$a = t(a,a) \leq t(a,b) \leq t(a, 1) = a$$

　　by assumption　　by monotonicity　　by boundary condition

and hence $t(a,b) = a$.

- If $0 \leq b \leq a \leq 1$ then

$$b = t(b,b) \leq t(b,a) \leq t(b, 1) = b$$

　　by assumption　　by monotonicity　　by boundary condition

and hence $t(a,b) = t(b,a) = b$.

　by commutativity

$t(a,b) = \min(a,b)$

is the only
possible solution!

q.e.d.

Examples:

Name	Function
(a) Standard	$t(a, b) = \min \{ a, b \}$
(b) Algebraic Product	$t(a, b) = a \cdot b$
(c) Bounded Difference	$t(a, b) = \max \{ 0, a + b - 1 \}$
(d) Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$
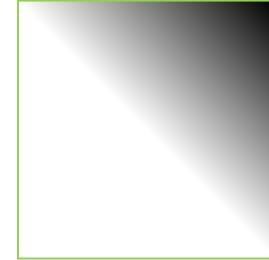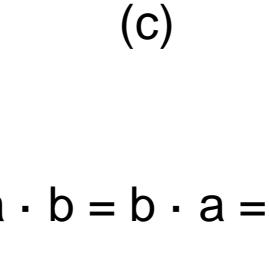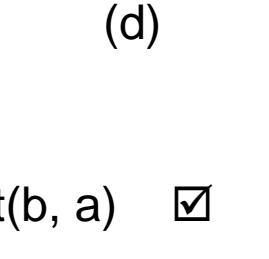
(a) (b)

(c) (d)
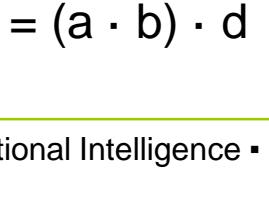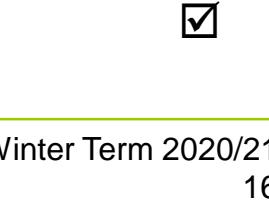
Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): $t(a, 1) = a \cdot 1 = a$ ☑   ad (A3): $t(a, b) = a \cdot b = b \cdot a = t(b, a)$ ☑

ad (A2): $a \cdot b \leq a \cdot d \Leftrightarrow b \leq d$ ☑   ad (A4): $a \cdot (b \cdot d) = (a \cdot b) \cdot d$ ☑

Theorem

Function t: [0,1] x [0,1] → [0,1] is a t-norm ,

∃decreasing generator f:[0,1] → $\mathbb{R}$ with t(a, b) = f^{-1}(min{ f(0), f(a) + f(b) }).   ■

Example:

f(x) = 1/x − 1 is decreasing generator since

- f(x) is continuous   ☑

- f(1) = 1/1 − 1 = 0   ☑

- f'(x) = −1/x^2 < 0 (monotone decreasing)   ☑

inverse function is f^{-1}(x) = $\dfrac{1}{x+1}$;   f(0) = ∞ ⇒ min{ f(0), f(a) + f(b) } = f(a) + f(b)

$$\Rightarrow t(a, b) \;=\; f^{-1}\left(\tfrac{1}{a} + \tfrac{1}{b} - 2 \right) \;=\; \frac{1}{\frac{1}{a}+\frac{1}{b}-1} \;=\; \frac{ab}{a+b-a\,b}$$

Definition

A function $s:[0,1] \times [0,1] \rightarrow [0,1]$ is a ***fuzzy union*** or ***s-norm*** iff　$\forall a,b,d \in [0,1]$

(A1)　$s(a, 0) = a$　　　　　　　　　　　　　　　　　(boundary condition)

(A2)　$b \leq d \Rightarrow s(a, b) \leq s(a, d)$　　　　　　　　(monotonicity)

(A3)　$s(a, b) = s(b, a)$　　　　　　　　　　　　　　(commutative)

(A4)　$s(a, s(b, d)) = s(s(a, b), d)$　　　　　　　　(associative)　　　　■

"nice to have"

(A5)　$s(a, b)$ is continuous　　　　　　　　　　　(continuity)

(A6)　$s(a, a) > a$　　　　　　　for $0 < a < 1$　　　(superidempotent)

(A7)　$a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow s(a_1, b_1) < s(a_2, b_2)$　　(strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	$s(a, b) = \max \{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min \{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$
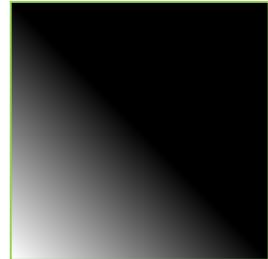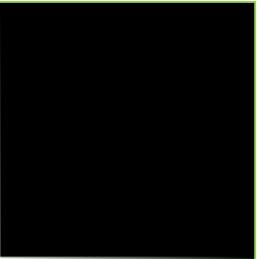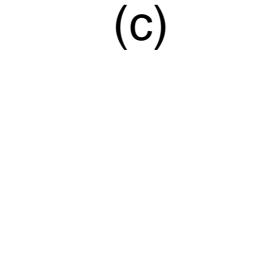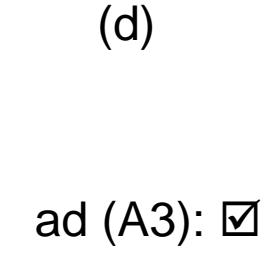
(a)　(b)

(c)　(d)
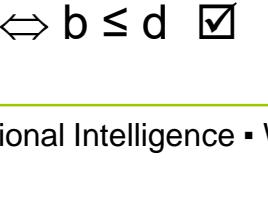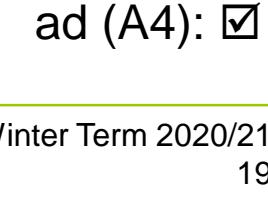
Is algebraic sum a t-norm?　Check the 4 axioms!

ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$　☑

ad (A2): $a + b - a \cdot b \leq a + d - a \cdot d \Leftrightarrow b (1 - a) \leq d (1 - a) \Leftrightarrow b \leq d$　☑

ad (A3): ☑

ad (A4): ☑

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Theorem

Function s: $[0,1] \times [0,1] \to [0,1]$ is a s-norm $\Leftrightarrow$

$\exists$ increasing generator $g:[0,1] \to \mathbb{R}$ with $s(a, b) = g^{-1}(\min\{ g(1), g(a) + g(b) \})$. ■

Example:

$g(x) = -\log(1 - x)$ is increasing generator since

- $g(x)$ is continuous　　　　　　　　　　　　☑

- $g(0) = -\log(1 - 0) = 0$　　　　　　　　　☑

- $g'(x) = 1/(1 - x) > 0$ (monotone increasing)　☑

inverse function is $g^{-1}(x) = 1 - \exp(-x)$; $g(1) = \infty \Rightarrow \min\{g(1), g(a) + g(b)\} = g(a) + g(b)$

$$\Rightarrow s(a, b) = g^{-1}(-\log(1 - a) - \log(1 - b))$$
$$= 1 - \exp(\log(1 - a) + \log(1 - b))$$
$$= 1 - (1 - a)(1 - b) = a + b - a\,b \quad \text{(algebraic sum)}$$

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan's laws

| **Definition**	**Definition**
A pair of t-norm $t(\cdot, \cdot)$ and s-norm $s(\cdot, \cdot)$ is said to be ***dual with regard to the fuzzy complement*** $c(\cdot)$ iff • $c(t(a, b)) = s(c(a), c(b))$ • $c(s(a, b)) = t(c(a), c(b))$ for all $a, b \in [0,1]$. ∎	Let (c, s, t) be a tripel of fuzzy complement $c(\cdot)$, s- and t-norm. If t and s are dual to c then the tripel (c, s, t) is called a ***dual tripel***. ∎

Examples of dual tripels

t-norm	s-norm	complement
min { a, b }	max { a, b }	1 – a
a · b	a + b – a · b	1 – a
max { 0, a + b – 1 }	min { 1, a + b }	1 – a

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c(t(a, b))
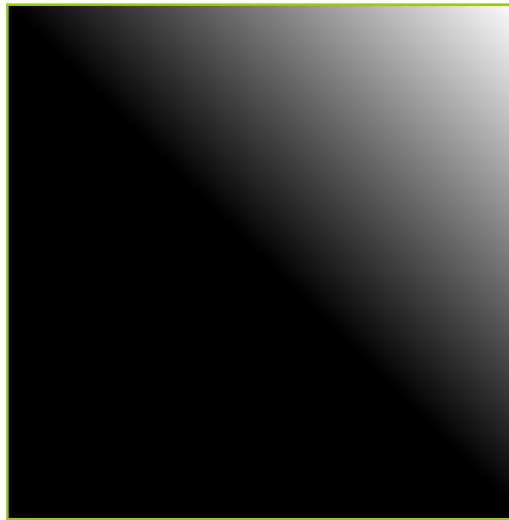
s(c(a), c(b))
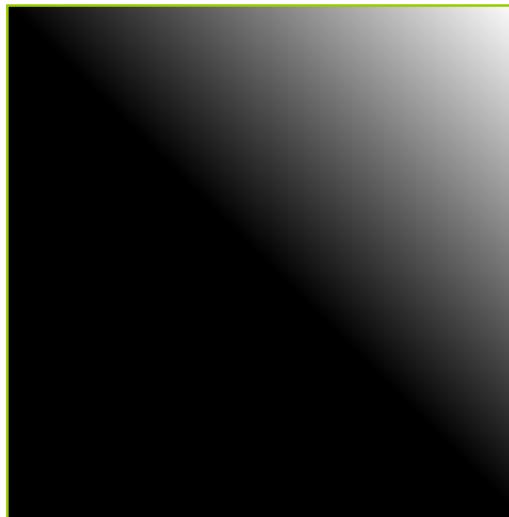
Dual Triple:

- bounded difference

- bounded sum

- standard complement

⇒ left image = right image

Non-Dual Triple:

- algebraic product

- bounded sum

- standard complement

⇒ left image ≠ right image

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Why are dual triples so important?

$\Rightarrow$ allow equivalence transformations of fuzzy set expressions

$\Rightarrow$ required to transform into some equivalent normal form (standardized input)

$\Rightarrow$ e.g. two stages: intersection of unions $\quad\bigcap\limits_{i=1}^{n}(A_i \cup B_i)$

$\qquad\qquad\qquad$ or union of intersections $\quad\bigcup\limits_{i=1}^{n}(A_i \cap B_i)$

Example:

$A \cup (B \cap (C \cap D)^c) =$ $\qquad$ $\leftarrow$ not in normal form

$A \cup (B \cap (C^c \cup D^c)) =$ $\qquad$ $\leftarrow$ equivalent if DeMorgan's law valid (dual triples!)

$A \cup (B \cap C^c) \cup (B \cap D^c)$ $\qquad$ $\leftarrow$ equivalent (distributive lattice!)