Computational Intelligence

Winter Term 2020/21

Prof. Dr. Günter Rudolph

Lehrstuhl für Algorithm Engineering (LS 11)

Fakultät für Informatik

TU Dortmund

Plan for Today — Lecture 02

- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Fuzzy Sets — Lecture 02

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Fuzzy Complement: Axioms — Lecture 02

Definition

A function $c: [0,1] \to [0,1]$ is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$. — monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall a \in [0,1]: c(c(a)) = a$ — involutive

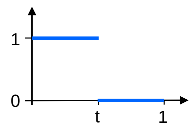
Examples:

a) standard fuzzy complement $c(a) = 1 - a$

ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$
ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)

ad (A3): ☑
ad (A4): $1 - (1 - a) = a$

b) $c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$



ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

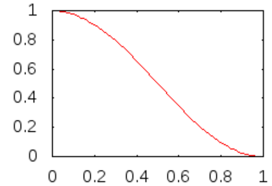
ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t ☑

ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(¼)) = c(1) = 0 \neq ¼$ for $t = ½$

c) $c(a) = \dfrac{1 + \cos(\pi\, a)}{2}$



ad (A1): $c(0) = 1$ and $c(1) = 0$

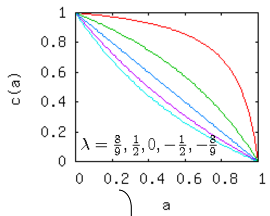
ad (A2): $c'(a) = -½\, \pi \sin(\pi\, a) < 0$ since $\sin(\pi\, a) > 0$ for $a \in (0,1)$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$$c\left(c\left(\frac{1}{3}\right)\right) = c\left(\frac{3}{4}\right) = \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right) \neq \frac{1}{3}$$

d) $c(a) = \dfrac{1-a}{1+\lambda\, a}$ for $\lambda > -1$ **Sugeno class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c(a) \ge c(b) \Leftrightarrow \dfrac{1-a}{1+\lambda\, a} \ge \dfrac{1-b}{1+\lambda\, b} \Leftrightarrow$

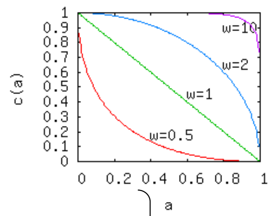
$(1-a)\,(1+\lambda\, b) \ge (1-b)\,(1+\lambda\, a) \Leftrightarrow$

$b\,(\lambda+1) \ge a\,(\lambda+1) \Leftrightarrow b \ge a$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left(\dfrac{1-a}{1+\lambda\, a}\right) = \dfrac{1 - \frac{1-a}{1+\lambda a}}{1+\lambda \frac{1-a}{1+\lambda a}} = \dfrac{a\,(\lambda+1)}{\lambda+1} = a$ ☑

e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \ge (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \ge 1 - b^w \Leftrightarrow$

$a^w \le b^w \Leftrightarrow a \le b$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$

$= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$ ☑

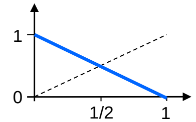
Fuzzy Complement: Fixed Points

Theorem

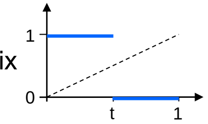
If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix



no fixed point → see example (b) → no intersection with bisectrix



assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b* (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2) ■

Fuzzy Complement: Fixed Points

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a* with c(a*) = a*.

Proof:

Intermediate value theorem →

If c(·) continuous (A3) and c(0) ≥ c(1) (A1/A2)

then ∀ v ∈ [c(1), c(0)] = [0,1]: ∃ a ∈ [0,1]: c(a) = v.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points! ■

Examples:

(a) c(a) = 1 – a ⇒ a = 1 – a ⇒ a* = ½

(b) c(a) = $(1 – a^w)^{1/w}$ ⇒ a = $(1 – a^w)^{1/w}$ ⇒ a* = $(½)^{1/w}$

Fuzzy Complement: 1st Characterization

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃continuous function g: [0,1] → ℝ with

- g(0) = 0
- strictly monotone increasing
- ∀ a ∈ [0,1]: c(a) = $g^{(-1)}$(g(1) – g(a)). ■

defines an **increasing generator**

$g^{(-1)}$(x) pseudo-inverse

Examples

a) g(x) = x ⇒ g^{-1}(x) = x ⇒ c(a) = 1 – a (Standard)

b) g(x) = x^w ⇒ g^{-1}(x) = $x^{1/w}$ ⇒ c(a) = $(1 – a^w)^{1/w}$ (Yager class, w > 0)

c) g(x) = log(x+1) ⇒ g^{-1}(x) = e^x – 1 ⇒ c(a) = exp(log(2) – log(a+1)) – 1

$= \frac{1-a}{1+a}$ (Sugeno class. λ = 1)

Fuzzy Complement: 1st Characterization

Examples

d) $g(a) = \frac{1}{\lambda}\log_e(1+\lambda a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda a} > 0$ for $a \in [0,1]$
- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda a)-1}{\lambda}$, thus

$$\begin{aligned} c(a) &= g^{-1}\left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda a)}{\lambda}\right) \\ &= \frac{\exp(\log(1+\lambda) - \log(1+\lambda a)) - 1}{\lambda} \\ &= \frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda a} - 1\right) = \frac{1-a}{1+\lambda a} \quad \text{(Sugeno Complement)} \end{aligned}$$

Fuzzy Complement: 2nd Characterization

Theorem

c: [0,1] $\to$ [0,1] is involutive fuzzy complement iff

$\exists$continuous function f: [0,1] $\to \mathbb{R}$ with

- f(1) = 0
- strictly monotone decreasing
- $\forall$ a $\in$ [0,1]: c(a) = $f^{(-1)}$(f(0) – f(a)). ■

defines a **decreasing generator**

$f^{(-1)}(x)$ pseudo-inverse

Examples

a) $f(x) = k - k \cdot x$ $(k > 0)$ $f^{(-1)}(x) = 1 - x/k$ $c(a) = 1 - \frac{k-(k-ka)}{k} = 1 - a$

b) $f(x) = 1 - x^w$ $f^{(-1)}(x) = (1 - x)^{1/w}$ $c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w}$ (Yager)

Fuzzy Intersection: t-norm

Definition

A function t:[0,1] x [0,1] $\to$ [0,1] is a ***fuzzy intersection*** or ***t-norm*** iff $\forall$a,b,d $\in$ [0,1]

(A1)	t(a, 1) = a	(boundary condition)
(A2)	$b \le d \Rightarrow t(a, b) \le t(a, d)$	(monotonicity)
(A3)	t(a,b) = t(b, a)	(commutative)
(A4)	t(a, t(b, d)) = t(t(a, b), d)	(associative) ■

“nice to have”

(A5)	t(a, b) is continuous		(continuity)
(A6)	t(a, a) < a	for 0 < a < 1	(subidempotent)
(A7)	$a_1 < a_2$ and $b_1 \le b_2$ $\Rightarrow$ $t(a_1, b_1) < t(a_2, b_2)$		(strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Fuzzy Intersection: t-norm

Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with t(a,a) = a for all a $\in$ [0,1].

- If $0 \le a \le b \le 1$ then

$$a \underset{\text{by assumption}}{=} t(a,a) \underset{\text{by monotonicity}}{\le} t(a,b) \le t(a,1) \underset{\text{by boundary condition}}{=} a$$

  and hence t(a,b) = a.

- If $0 \le b \le a \le 1$ then

$$b \underset{\text{by assumption}}{=} t(b,b) \underset{\text{by monotonicity}}{\le} t(b,a) \le t(b,1) \underset{\text{by boundary condition}}{=} b$$

  and hence $t(a,b) \underset{\text{by commutativity}}{=} t(b,a) = b$.

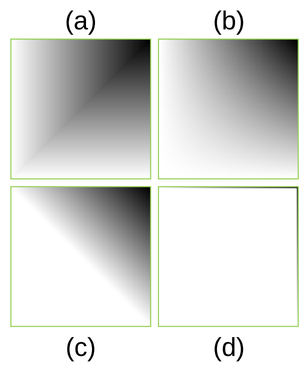
t(a,b) = min(a,b) is the only possible solution!

q.e.d.

Fuzzy Intersection: t-norm

Examples:

Name	Function
(a) Standard	t(a, b) = min { a, b }
(b) Algebraic Product	t(a, b) = a · b
(c) Bounded Difference	t(a, b) = max { 0, a + b – 1 }
(d) Drastic Product	$t(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$



Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): t(a, 1) = a · 1 = a ☑

ad (A2): a · b ≤ a · d ⇔ b ≤ d ☑

ad (A3): t(a, b) = a · b = b · a = t(b, a) ☑

ad (A4): a · (b · d) = (a · b) · d ☑

Fuzzy Intersection: Characterization

Theorem

Function t: [0,1] x [0,1] → [0,1] is a t-norm ,

∃decreasing generator f:[0,1] → $\mathbb{R}$ with t(a, b) = f^{-1}(min{ f(0), f(a) + f(b) }). ■

Example:

f(x) = 1/x – 1 is decreasing generator since

- f(x) is continuous ☑
- f(1) = 1/1 – 1 = 0 ☑
- f'(x) = $-1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \frac{1}{x+1}$; f(0) = ∞ ⇒ min{ f(0), f(a) + f(b) } = f(a) + f(b)

$$\Rightarrow t(a, b) = f^{-1}\left(\frac{1}{a} + \frac{1}{b} - 2\right) = \frac{1}{\frac{1}{a}+\frac{1}{b}-1} = \frac{ab}{a+b-a\,b}$$

Fuzzy Union: s-norm

Definition

A function s:[0,1] x [0,1] → [0,1] is a ***fuzzy union*** or ***s-norm*** iff ∀a,b,d ∈ [0,1]

(A1) s(a, 0) = a (boundary condition)

(A2) b ≤ d ⇒ s(a, b) ≤ s(a, d) (monotonicity)

(A3) s(a, b) = s(b, a) (commutative)

(A4) s(a, s(b, d)) = s(s(a, b), d) (associative) ■

"nice to have"

(A5) s(a, b) is continuous (continuity)

(A6) s(a, a) > a for 0 < a < 1 (superidempotent)

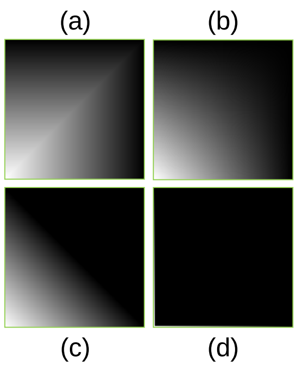
(A7) $a_1 < a_2$ and $b_1 \le b_2$ ⇒ $s(a_1, b_1) < s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Fuzzy Union: s-norm

Examples:

Name	Function
Standard	s(a, b) = max { a, b }
Algebraic Sum	s(a, b) = a + b – a · b
Bounded Sum	s(a, b) = min { 1, a + b }
Drastic Union	s(a, b) = { a if b = 0; b if a = 0; 1 otherwise



Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): s(a, 0) = a + 0 – a · 0 = a ☑

ad (A2): a + b – a · b ≤ a + d – a · d ⇔ b (1 – a) ≤ d (1 – a) ⇔ b ≤ d ☑

ad (A3): ☑

ad (A4): ☑

Fuzzy Union: Characterization

Theorem

Function s: [0,1] x [0,1] → [0,1] is a s-norm ⇔

∃ increasing generator g:[0,1] → $\mathbb{R}$ with s(a, b) = g^{-1}(min{ g(1), g(a) + g(b) }). ■

Example:

g(x) = –log(1 – x) is increasing generator since

- g(x) is continuous ☑
- g(0) = –log(1 – 0) = 0 ☑
- g'(x) = 1/(1 – x) > 0 (monotone increasing) ☑

inverse function is $g^{-1}(x)$ = 1 – exp(–x); g(1) = ∞ ⇒ min{g(1), g(a) + g(b)} = g(a) + g(b)

$$\begin{aligned}\Rightarrow s(a, b) &= g^{-1}(-\log(1-a) - \log(1-b)) \\ &= 1 - \exp(\log(1-a) + \log(1-b)) \\ &= 1 - (1-a)\,(1-b) = a + b - a\,b \quad \text{(algebraic sum)}\end{aligned}$$

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition

A pair of t-norm t(·, ·) and s-norm s(·, ·) is said to be ***dual with regard to the fuzzy complement*** c(·) iff

- c(t(a, b)) = s(c(a), c(b))
- c(s(a, b)) = t(c(a), c(b))

for all a, b $\in$ [0,1]. ■

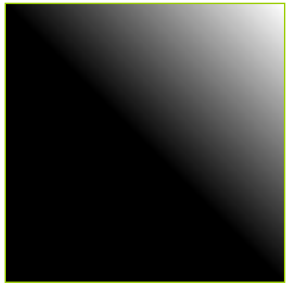
Definition

Let (c, s, t) be a tripel of fuzzy complement c(·), s- and t-norm.

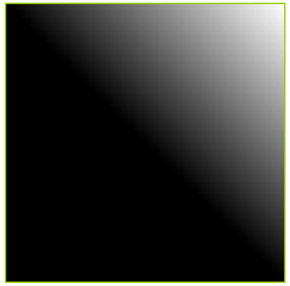
If t and s are dual to c then the tripel (c,s, t) is called a ***dual tripel***. ■

Examples of dual tripels

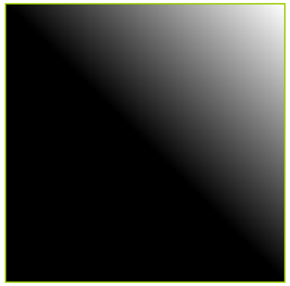
t-norm	s-norm	complement
min { a, b }	max { a, b }	1 – a
a · b	a + b – a · b	1 – a
max { 0, a + b – 1 }	min { 1, a + b }	1 – a



c(t(a, b))



s(c(a), c(b))



Dual Triple:

- bounded difference
- bounded sum
- standard complement

$\Rightarrow$ left image = right image

Non-Dual Triple:

- algebraic product
- bounded sum
- standard complement

$\Rightarrow$ left image ≠ right image

Why are dual triples so important?

$\Rightarrow$ allow equivalence transformations of fuzzy set expressions

$\Rightarrow$ required to transform into some equivalent normal form (standardized input)

$\Rightarrow$ e.g. two stages: intersection of unions $\displaystyle\bigcap_{i=1}^{n} (A_i \cup B_i)$

or union of intersections $\displaystyle\bigcup_{i=1}^{n} (A_i \cap B_i)$

Example:

$A \cup (B \cap (C \cap D)^c) \;=$	$\leftarrow$ not in normal form
$A \cup (B \cap (C^c \cup D^c)) \;=$	$\leftarrow$ equivalent if DeMorgan's law valid (dual triples!)
$A \cup (B \cap C^c) \cup (B \cap D^c)$	$\leftarrow$ equivalent (distributive lattice!)