Computational Intelligence

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- Fuzzy Sets

    - Basic Definitions and Results for Standard Operations

    - Algebraic Difference between Fuzzy and Crisp Sets

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Observation:

Communication between people is not precise but somehow <u>fuzzy</u> and <u>vague</u>.

"If the water is too hot then add a little bit of cold water."

Despite these shortcomings in human language we are able

● to process fuzzy / uncertain information and

● to accomplish complex tasks!

Goal:

Development of formal framework to process fuzzy statements in computer.

Consider the statement: "The water is hot."

Which temperature defines "hot"?

A single temperature T = 95° C?

No! Rather, an interval of temperatures: T $\in$ [70, 120] !

But who defines the limits of the intervals?

Some people regard temperatures > 60° C as hot, others already T > 50° C!

Idea: All people might agree that a temperature in the <u>set</u> [70, 120]
    defines a hot temperature!

If T = 65°C not all people regard this as hot. It does not belong to [70,120].

But it is hot to some <u>degree</u>.
Or: T = 65°C belongs to set of hot temperatures to some <u>degree</u>!

$\Rightarrow$ **Can be the concept for capturing fuzziness!** $\Rightarrow$ **Formalize this concept!**

Definition

A map F: X → [0,1] ⊂ ℝ that assigns its ***degree of membership*** F(x) to each x ∈ X is termed a **fuzzy set**.

Remark:

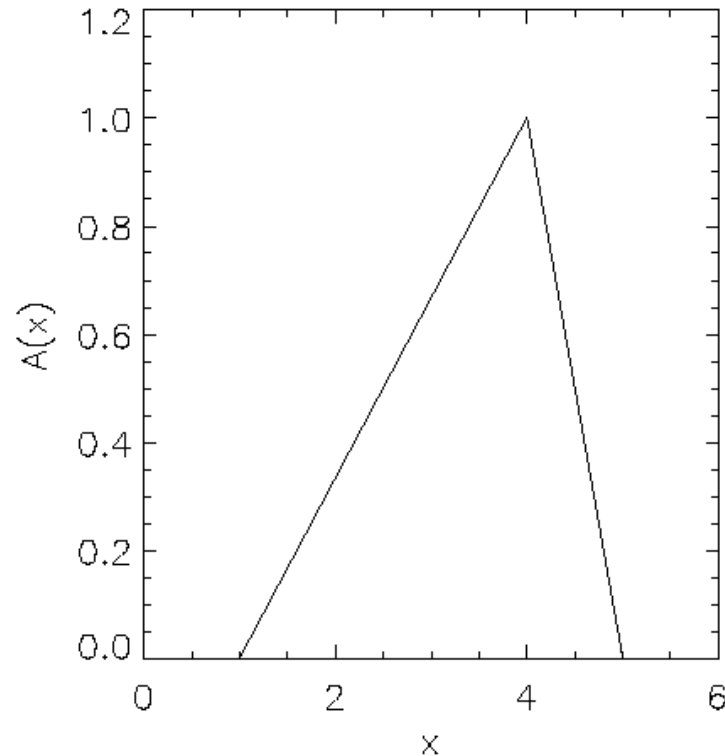A fuzzy set F is actually a map F(x). Shorthand notation is simply F.

Same point of view possible for traditional ("***crisp***") sets:

$$A(x) := 1_{[x \in A]} := 1_A(x) := \begin{cases} 1 & , \text{ if } x \in A \\ 0 & , \text{ if } x \notin A \end{cases}$$

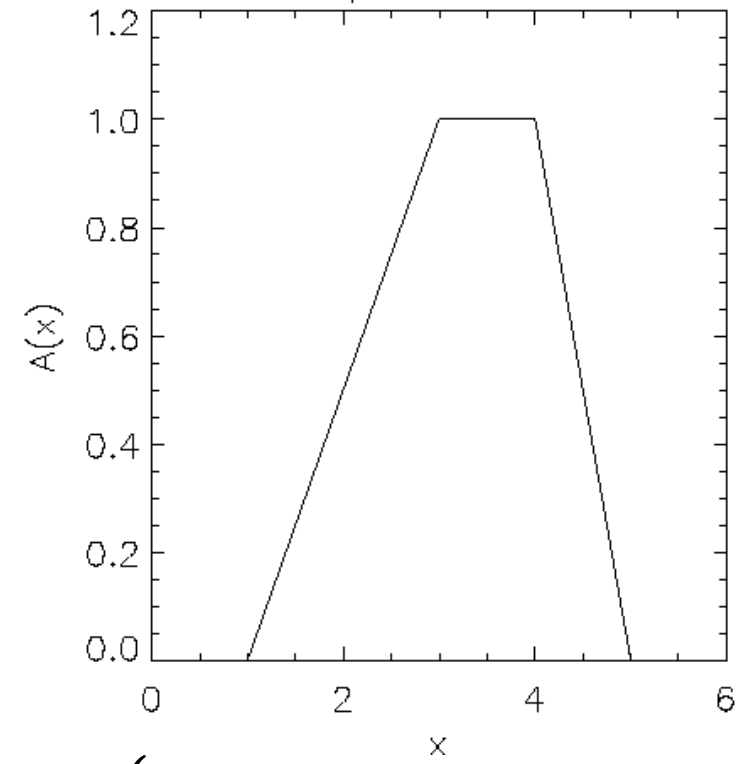characteristic / indicator function of (crisp) set A

⇒ membership function interpreted as generalization of characteristic function

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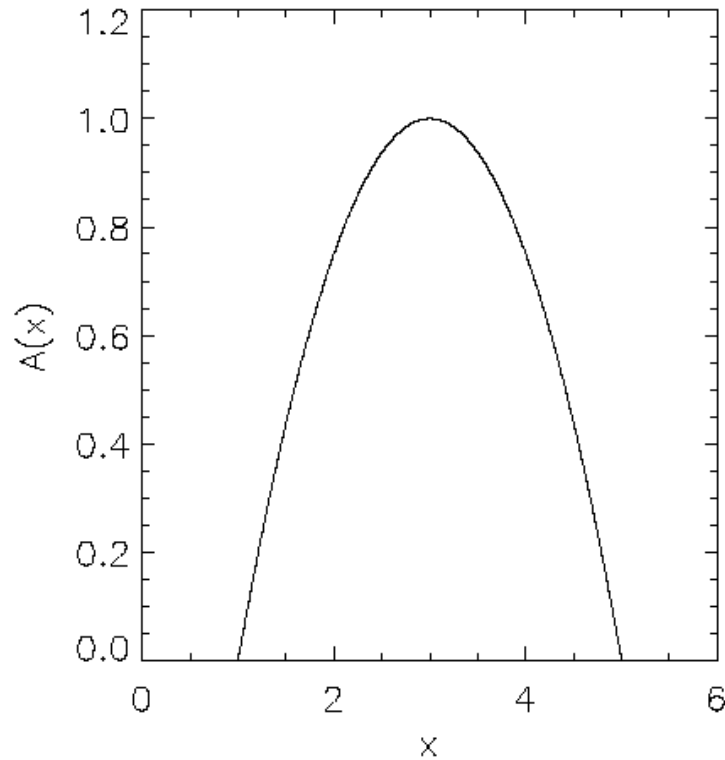dortmund

triangle function

$$A(x) = \begin{cases} \frac{1}{3}(x-1) & \text{if } 1 \le x < 4 \\ 5-x & \text{if } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$
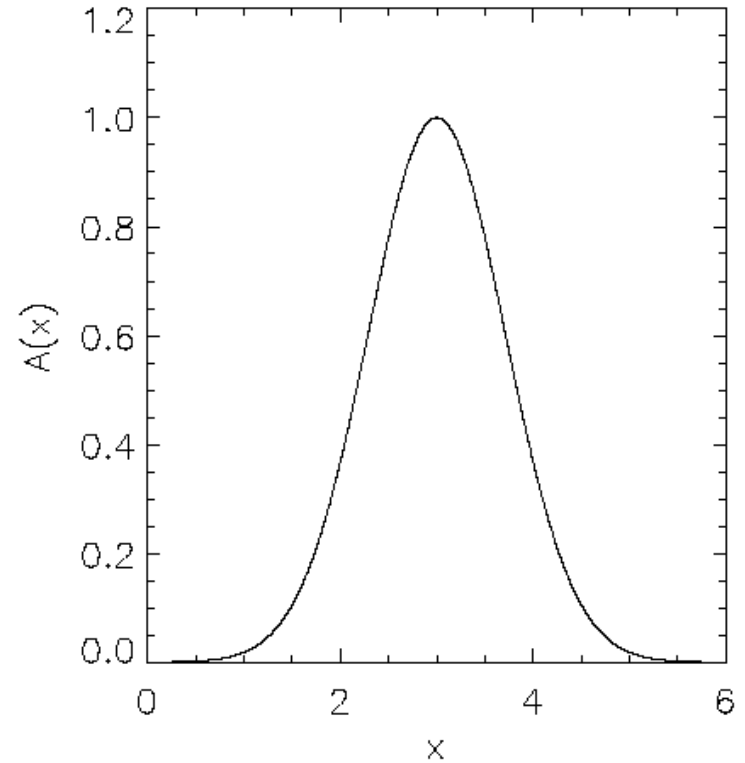
trapezoidal

$$A(x) = \begin{cases} \frac{1}{2}(x-1) & \text{if } 1 \le x < 3 \\ 1 & \text{if } 3 \le x < 4 \\ 5-x & \text{if } 4 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

paraboloidal function

gaussoid function

$$A(x) = \begin{cases} -\dfrac{(x-1)(x-5)}{4} & \text{if } 1 \le x < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$A(x) = \exp\left(-\frac{(x-3)^2}{2}\right)$$

Definition

A fuzzy set F over the crisp set X is termed

a) **empty** if F(x) = 0 for all x $\in$ X,

b) **universal** if F(x) = 1 for all x $\in$ X.

Empty fuzzy set is denoted by $\mathbb{O}$. Universal set is denoted by $\mathbb{U}$. ∎

Definition

Let A and B be fuzzy sets over the crisp set X.

a) A and B are termed **equal**, denoted A = B, if A(x) = B(x) for all x $\in$ X.

b) A is a **subset** of B, denoted A $\subseteq$ B, if A(x) ≤ B(x) for all x $\in$ X.

c) A is a **strict subset** of B, denoted A $\subset$ B, if A $\subseteq$ B and $\exists$ x $\in$ X: A(x) < B(x). ∎

Remark: A strict subset is also called a **proper** subset.

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Theorem

Let A, B and C be fuzzy sets over the crisp set X. The following relations are valid:

a) reflexivity   : $A \subseteq A$.

b) antisymmetry : $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$.

c) transitivity   : $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

Proof: (via reduction to definitions and exploiting operations on crisp sets)

ad a) $\forall x \in X: A(x) \leq A(x)$.

ad b) $\forall x \in X: A(x) \leq B(x)$ and $B(x) \leq A(x) \Rightarrow A(x) = B(x)$.

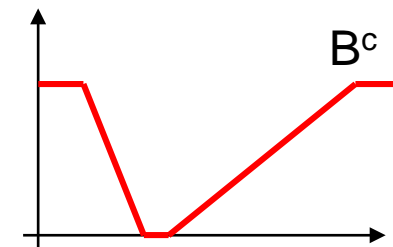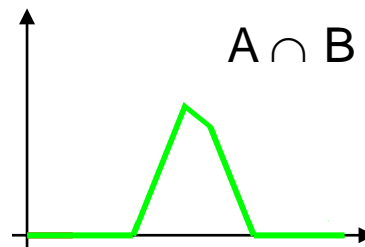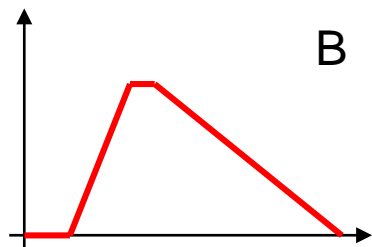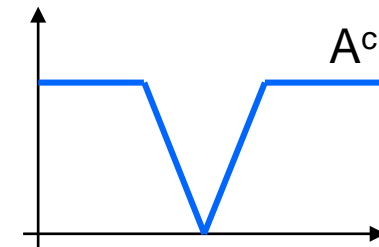ad c) $\forall x \in X: A(x) \leq B(x)$ and $B(x) \leq C(x) \Rightarrow A(x) \leq C(x)$.   **q.e.d.**

Remark: Same relations valid for crisp sets. No Surprise! Why?

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Definition

Let A and B be fuzzy sets over the crisp set X. The set C is the

a) **union** of A and B, denoted C = A $\cup$ B, if C(x) = max{ A(x), B(x) } for all x $\in$ X;

b) **intersection** of A and B, denoted C = A $\cap$ B, if C(x) = min{ A(x), B(x) } for all x $\in$ X;

c) **complement** of A, denoted C = A^c, if C(x) = 1 − A(x) for all x $\in$ X.   ∎
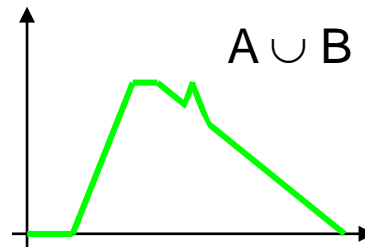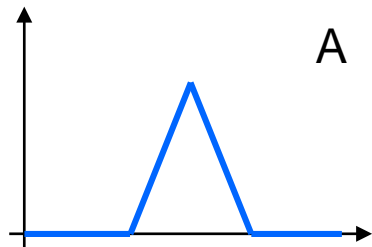
standard fuzzy union

A   B   A $\cup$ B
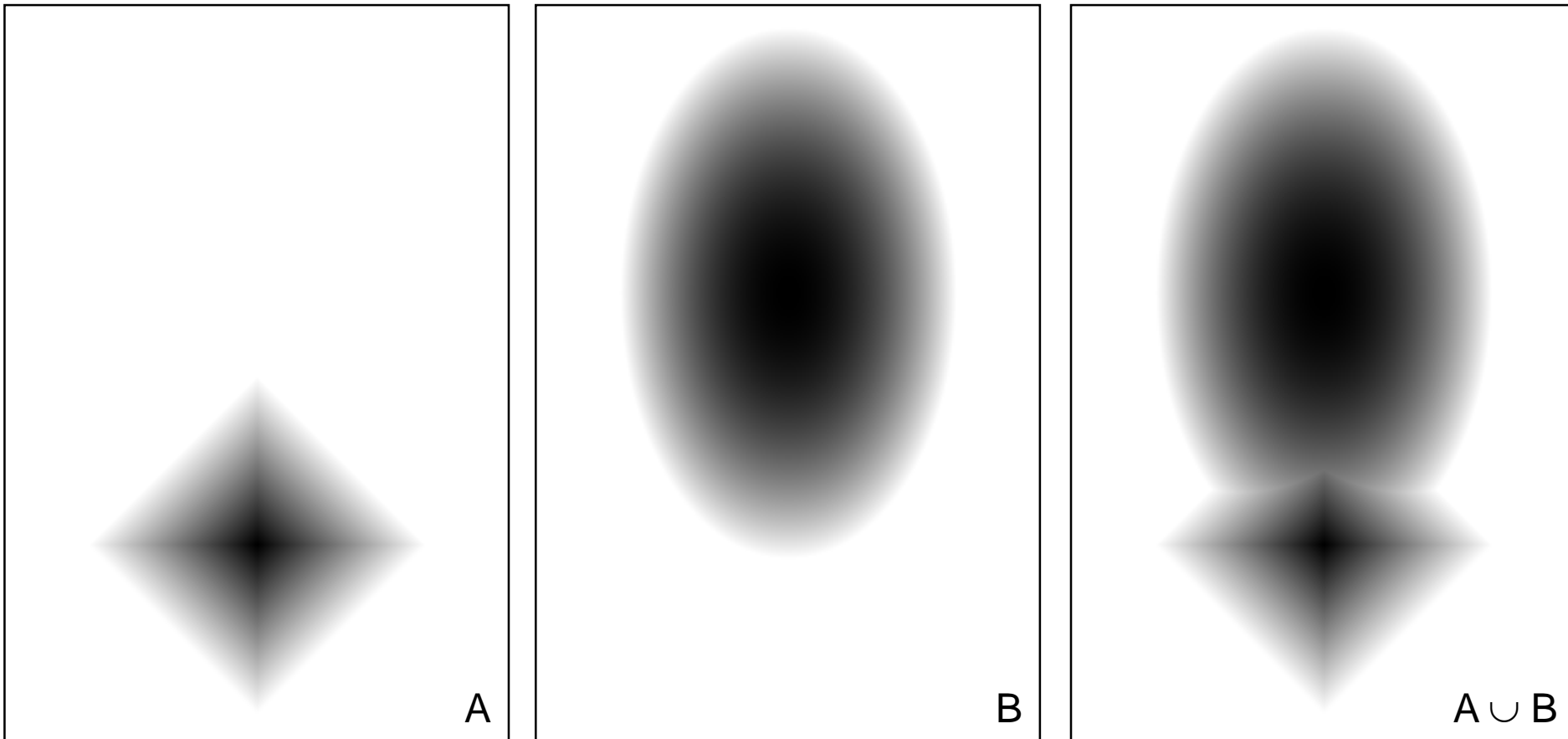
interpretation: membership = 0 is white, = 1 is black, in between is gray

standard fuzzy intersection

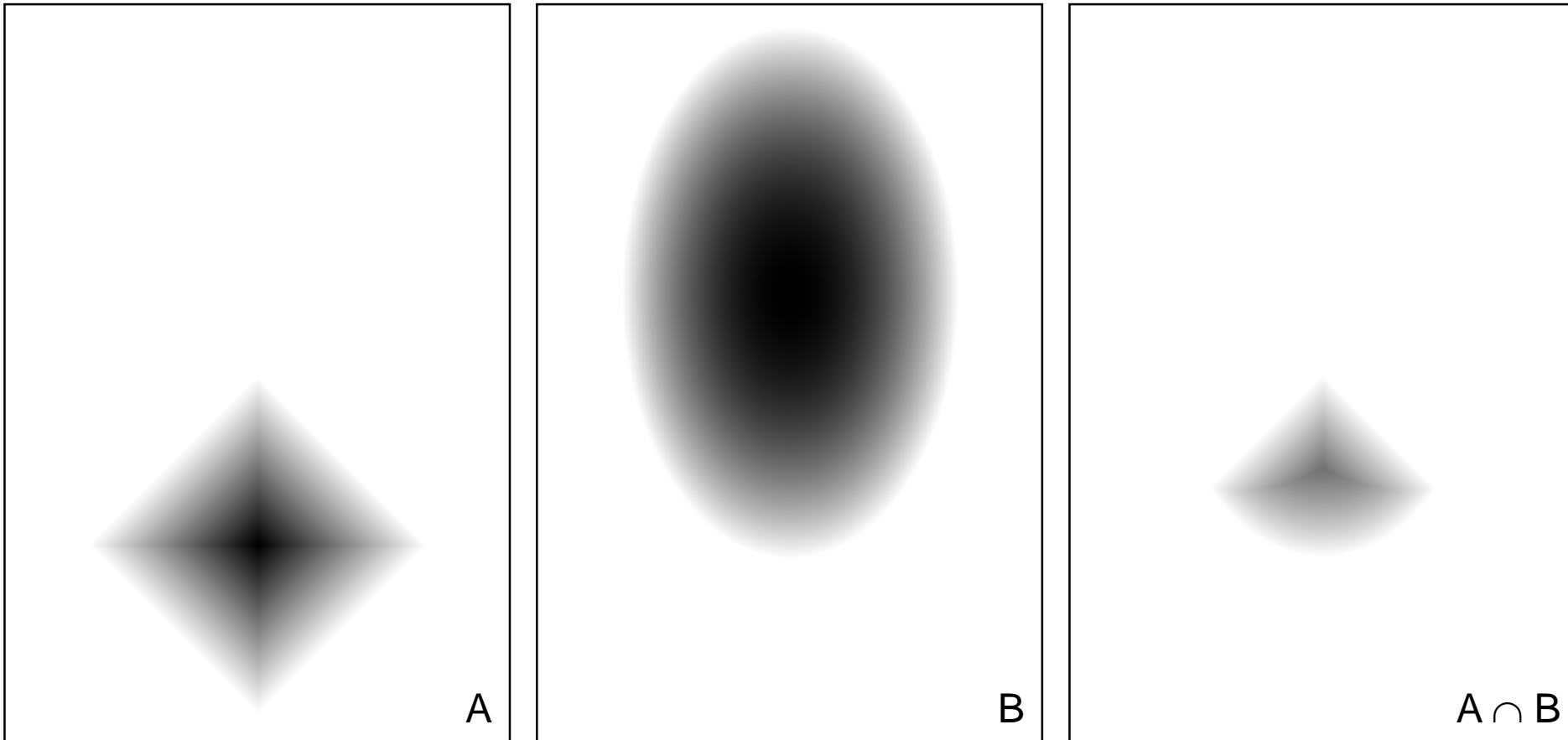**interpretation:** membership = 0 is white, = 1 is black, in between is gray

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standard fuzzy complement
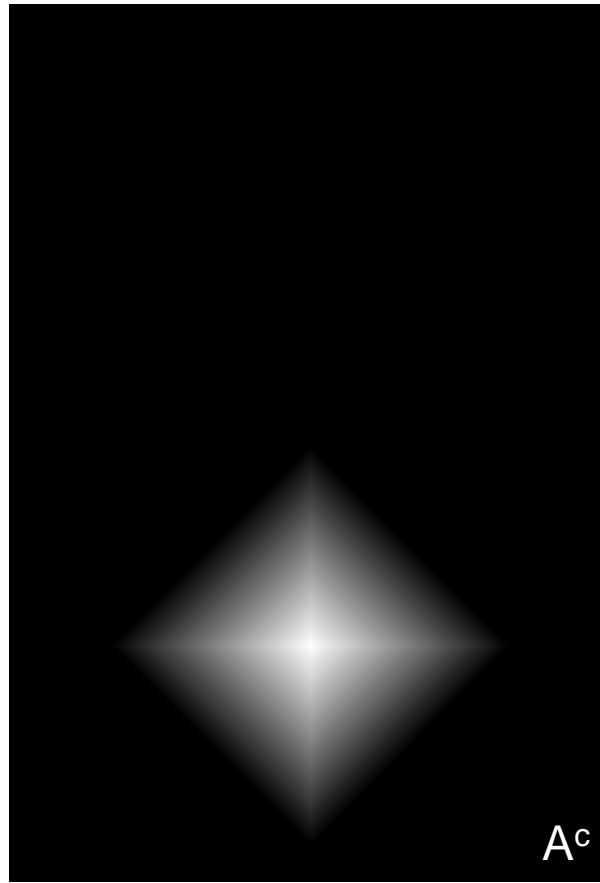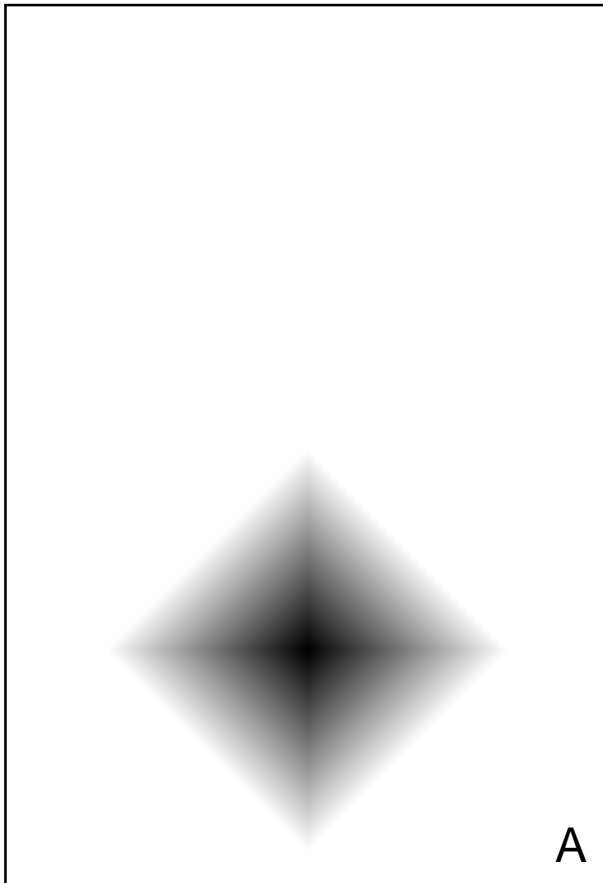
interpretation: membership = 0 is white, = 1 is black, in between is gray

Definition

The fuzzy set A over the crisp set X has

a) **_height_** hgt(A) = sup{ A(x) : x ∈ X },

b) **_depth_** dpth(A) = inf { A(x) : x ∈ X }. ■

hgt(A) = 0.8

dpth(A) = 0.2

$$A(x) = \tfrac{1}{5} + \tfrac{3}{5}\exp(-|x|)$$
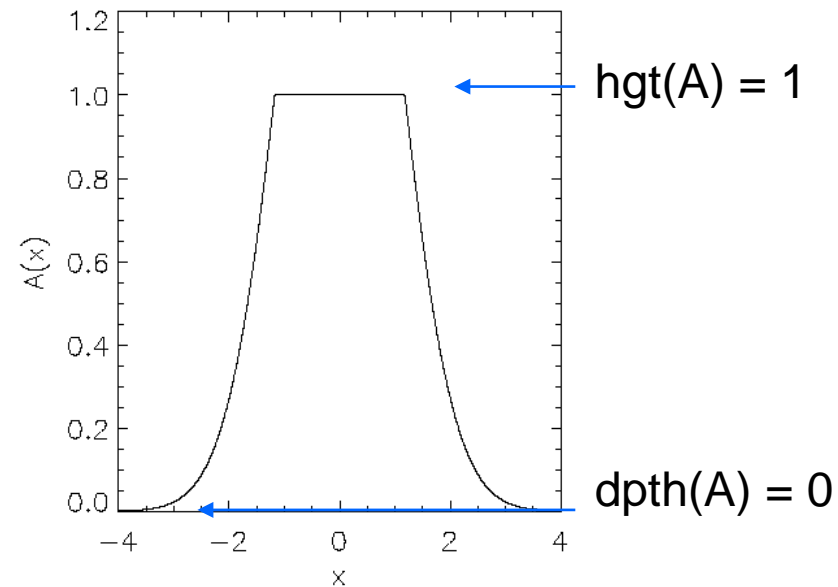
hgt(A) = 1

dpth(A) = 0

$$A(x) = \min\left\{ 1, 2\exp\left(-\tfrac{x^2}{2}\right)\right\}$$

Definition

The fuzzy set A over the crisp set X is

a) **normal**   if hgt(A) = 1

b) **strongly normal**   if $\exists$ x $\in$ X: A(x) = 1

c) **co-normal**   if dpth(A) = 0

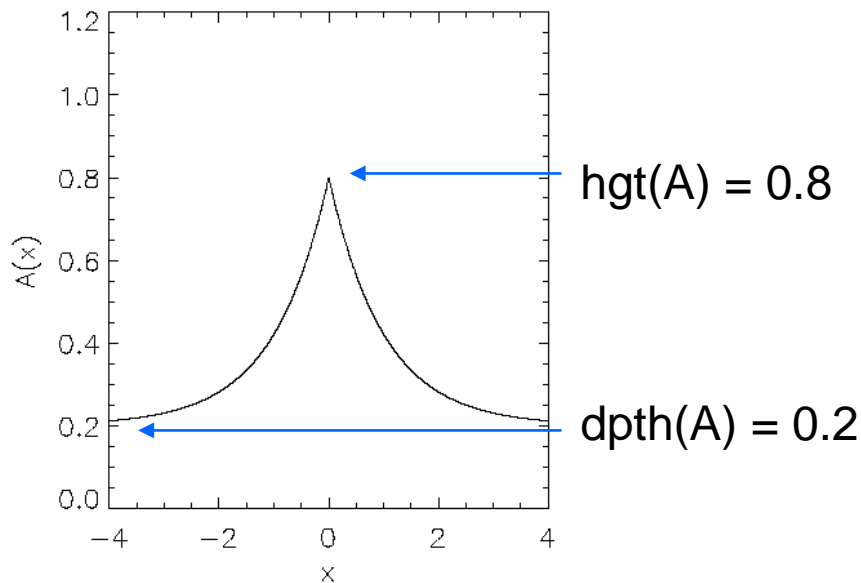d) **strongly co-normal** if $\exists$ x $\in$ X: A(x) = 0

e) **subnormal**   if 0 < A(x) < 1 for all x $\in$ X.   ∎

$$A(x) = \frac{1}{1+e^{-x}}$$

A is (co-) normal

but not strongly (co-) normal
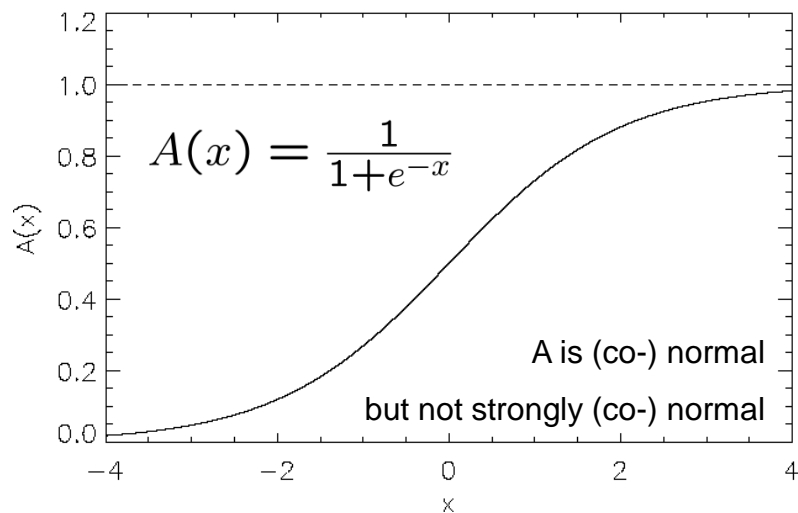
Remark:

How to normalize a non-normal fuzzy set A?

$$A^*(x) = \frac{A(x)}{\text{hgt}(A)}$$

Definition

The **cardinality** card(A) of a fuzzy set A over the crisp set X is

$$
\mathrm{card}(A) := \begin{cases} \displaystyle\sum_{x\in X} A(x) & \text{, if X countable} \\[2em] \displaystyle\int_X A(x)\,dx & \text{, if } X \subseteq \mathbb{R}^n \end{cases}
$$

∎

Examples:

a) $A(x) = q^x$ with $q \in (0,1)$, $x \in \mathbb{N}_0$ $\Rightarrow$ $\mathrm{card}(A) = \displaystyle\sum_{x\in X} A(x) = \sum_{x=0}^{\infty} q^x = \frac{1}{1-q} < \infty$

b) $A(x) = 1/x$ with $x \in \mathbb{N}$ $\Rightarrow$ $\mathrm{card}(A) = \displaystyle\sum_{x\in X} A(x) = \sum_{x=1}^{\infty} \frac{1}{x} = \infty$

c) $A(x) = \exp(-|x|)$ with $x \in \mathbb{R}$ $\Rightarrow$ $\mathrm{card}(A) = \displaystyle\int_{x\in X} A(x)\,dx = \int_{x=-\infty}^{\infty} \exp(-|x|)\,dx = 2$

Theorem

For fuzzy sets A, B and C over a crisp set X the <u>standard union operation</u> is

a)   *commutative*   : $A \cup B = B \cup A$

b)   *associative*   : $A \cup (B \cup C) = (A \cup B) \cup C$

c)   *idempotent*   : $A \cup A = A$

d)   *monotone*   : $A \subseteq B \Rightarrow (A \cup C) \subseteq (B \cup C)$.

Proof: (via reduction to definitions)

ad a) $A \cup B = \max \{ A(x), B(x) \} = \max \{ B(x), A(x) \} = B \cup A$.

ad b) $A \cup (B \cup C) = \max \{ A(x), \max\{ B(x), C(x) \} \} = \max \{ A(x), B(x), C(x) \}$
                    $= \max \{ \max \{ A(x), B(x) \}, C(x) \} = (A \cup B) \cup C$.

ad c) $A \cup A = \max \{ A(x), A(x) \} = A(x) = A$.

ad d) $A \cup C = \max \{ A(x), C(x) \} \leq \max \{ B(x), C(x) \} = B \cup C$ since $A(x) \leq B(x)$.   **q.e.d.**

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Theorem

For fuzzy sets A, B and C over a crisp set X the <u>standard intersection operation</u> is

a)   *commutative*   : $A \cap B = B \cap A$

b)   *associative*   : $A \cap (B \cap C) = (A \cap B) \cap C$

c)   *idempotent*   : $A \cap A = A$

d)   *monotone*   : $A \subseteq B \Rightarrow (A \cap C) \subseteq (B \cap C)$.

Proof: (analogous to proof for standard union operation)   ∎

Theorem

For fuzzy sets A, B and C over a crisp set X there are the <u>distributive laws</u>

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof:

ad a) max { A(x), min { B(x), C(x) } } = $\begin{cases} \max\{A(x), B(x)\} & \text{if } B(x) \leq C(x) \\ \max\{A(x), C(x)\} & \text{otherwise} \end{cases}$

If $B(x) \leq C(x)$ then max { A(x), B(x) } $\leq$ max { A(x), C(x) }.

Otherwise   max { A(x), C(x) } $\leq$ max { A(x), B(x) }.

$\Rightarrow$ result is always the smaller max-expression

$\Rightarrow$ result is **min** { max { A(x), B(x) }, max { A(x), C(x) } } = $(A \cup B) \cap (A \cup C)$.

ad b) analogous.   ∎

Theorem

If A is a fuzzy set over a crisp set X then

a) $A \cup \mathbb{O} = A$

b) $A \cup \mathbb{U} = \mathbb{U}$

c) $A \cap \mathbb{O} = \mathbb{O}$

d) $A \cap \mathbb{U} = A.$

Proof:

(via reduction to definitions)

ad a) $\max \{ A(x), 0 \} = A(x)$

ad b) $\max \{ A(x), 1 \} = \mathbb{U}(x) \equiv 1$

ad c) $\min \{ A(x), 0 \} = \mathbb{O}(x) \equiv 0$

ad d) $\min \{ A(x), 1 \} = A(x).$ ■

Breakpoint:

So far we know that fuzzy sets with operations $\cap$ and $\cup$ are a *distributive lattice*.

If we can show the validity of

• $(A^c)^c = A$

• $A \cup A^c = \mathbb{U}$

• $A \cap A^c = \mathbb{O}$ $\Rightarrow$ Fuzzy Sets would be Boolean Algebra! **Is it true ?**

Theorem

If A is a fuzzy set over a crisp set X then

a) $(A^c)^c = A$

b) $\frac{1}{2} \leq (A \cup A^c)(x) < 1$ for $A(x) \in (0,1)$

c) $0 < (A \cap A^c)(x) \leq \frac{1}{2}$ for $A(x) \in (0,1)$

Remark:

Recall the identities

$$\min\{a, b\} = \frac{a+b-|a-b|}{2}$$

$$\max\{a, b\} = \frac{a+b+|a-b|}{2}$$

Proof:

ad a) $\forall\, x \in X:\ 1 - (1 - A(x)) = A(x).$

ad b) $\forall\, x \in X:\ \max\{\, A(x), 1 - A(x)\,\} = \frac{1}{2} + |\,A(x) - \frac{1}{2}\,| \geq \frac{1}{2}.$

   Value 1 only attainable for $A(x) = 0$ or $A(x) = 1$.

ad c) $\forall\, x \in X:\ \min\{\, A(x), 1 - A(x)\,\} = \frac{1}{2} - |\,A(x) - \frac{1}{2}\,| \leq \frac{1}{2}.$
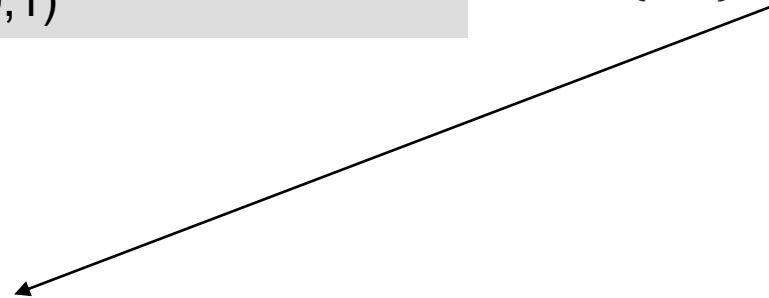
   Value 0 only attainable for $A(x) = 0$ or $A(x) = 1$.

q.e.d.

Conclusion:

Fuzzy sets with $\cup$ and $\cap$ are a distributive lattice.

But in general:

a) $A \cup A^c \neq \mathbb{U}$

b) $A \cap A^c \neq \mathbb{O}$

$\Rightarrow$ Fuzzy sets with $\cup$ and $\cap$ are **not** a Boolean algebra!

Remarks:

ad a)   The **law of excluded middle** does not hold!

("Everything must either be or not be!")

ad b)   The **law of noncontradiction** does not hold!

("Nothing can both be and not be!")

$\Rightarrow$ Nonvalidity of these laws generate the <u>desired</u> fuzziness!

but: Fuzzy sets still endowed with much algebraic structure (distributive lattice)!

Theorem

If A and B are fuzzy sets over a crisp set X with standard union, intersection, and complement operations then **DeMorgan**'s laws are valid:

a) $(A \cap B)^c = A^c \cup B^c$

b) $(A \cup B)^c = A^c \cap B^c$

Proof: (via reduction to elementary identities)

ad a) $(A \cap B)^c(x) = 1 - \min \{ A(x), B(x) \} = \max \{ 1 - A(x), 1 - B(x) \} = A^c(x) \cup B^c(x)$

ad b) $(A \cup B)^c(x) = 1 - \max \{ A(x), B(x) \} = \min \{ 1 - A(x), 1 - B(x) \} = A^c(x) \cap B^c(x)$

q.e.d.

Question : Why restricting result above to "*standard*" operations?

Conjecture : Most likely there also exist "*nonstandard*" operations!