

# Computational Intelligence

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### Three tasks:

1. Choice of an appropriate problem representation.
2. Choice / design of variation operators acting in problem representation.
3. Choice of strategy parameters (includes initialization).

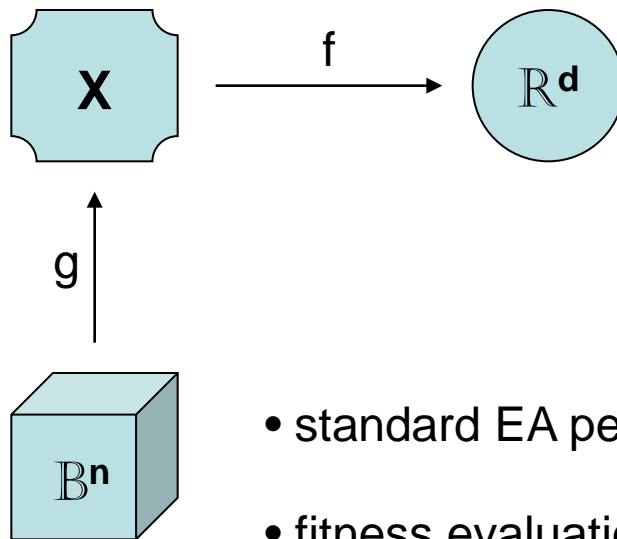
### ad 1) different “schools“:

- (a) operate on binary representation and define genotype/phenotype mapping
  - + can use standard algorithm
  - mapping may induce unintentional bias in search
- (b) no doctrine: use “most natural” representation
  - must design variation operators for specific representation
  - + if design done properly then no bias in search

### ad 1a) genotype-phenotype mapping

original problem  $f: X \rightarrow \mathbb{R}^d$

scenario: no standard algorithm for search space  $X$  available



- standard EA performs variation on binary strings  $b \in \mathbb{B}^n$
- fitness evaluation of individual  $b$  via  $(f \circ g)(b) = f(g(b))$   
where  $g: \mathbb{B}^n \rightarrow X$  is genotype-phenotype mapping
- selection operation independent from representation

## Genotype-Phenotype-Mapping $\mathbb{B}^n \rightarrow [L, R] \subset \mathbb{R}$

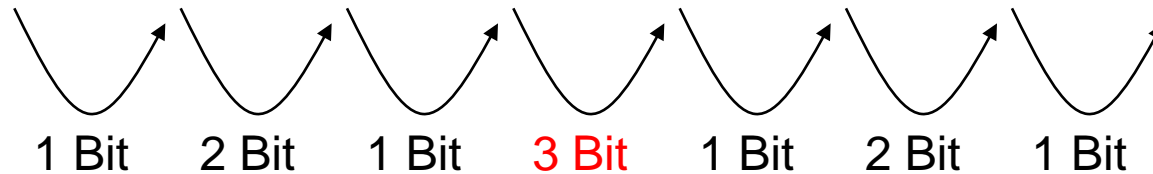
- Standard encoding for  $b \in \mathbb{B}^n$

$$x = L + \frac{R - L}{2^n - 1} \sum_{i=0}^{n-1} b_{n-i} 2^i$$

→ Problem: *hamming cliffs*

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7

← genotype  
← phenotype



↑  
Hamming cliff

$L = 0, R = 7$   
 $n = 3$

### Genotype-Phenotype-Mapping $\mathbb{B}^n \rightarrow [L, R] \subset \mathbb{R}$

- Gray encoding for  $b \in \mathbb{B}^n$

Let  $a \in \mathbb{B}^n$  standard encoded. Then  $b_i = \begin{cases} a_i, & \text{if } i = 1 \\ a_{i-1} \oplus a_i, & \text{if } i > 1 \end{cases}$   $\oplus = \text{XOR}$

000	001	011	010	110	111	101	100	← genotype
0	1	2	3	4	5	6	7	← phenotype

OK, no hamming cliffs any longer ...

⇒ small changes in phenotype „lead to“ small changes in genotype

since we consider evolution in terms of Darwin (not Lamarck):

⇒ small changes in genotype lead to small changes in phenotype!

**but:** 1-Bit-change:  $000 \rightarrow 100 \Rightarrow \text{☹}$

### Genotype-Phenotype-Mapping $\mathbb{B}^n \rightarrow \mathbb{P}^{\log(n)}$ (example only)

- e.g. standard encoding for  $b \in \mathbb{B}^n$

**individual:**

010	101	111	000	110	001	101	100	← genotype
0	1	2	3	4	5	6	7	← index

consider index and associated genotype entry as unit / record / struct;  
 sort units with respect to genotype value, old indices yield permutation:

000	001	010	100	101	101	110	111	← genotype
3	5	0	7	1	6	4	2	← old index

= permutation

ad 1a) genotype-phenotype mapping

typically required: strong causality

→ small changes in individual leads to small changes in fitness

→ small changes in genotype should lead to small changes in phenotype

**but:** how to find a genotype-phenotype mapping with that property?

**necessary conditions:**

- 1)  $g: \mathbb{B}^n \rightarrow X$  can be computed efficiently (otherwise it is senseless)
- 2)  $g: \mathbb{B}^n \rightarrow X$  is surjective (otherwise we might miss the optimal solution)
- 3)  $g: \mathbb{B}^n \rightarrow X$  *preserves closeness* (otherwise strong causality endangered)

Let  $d(\cdot, \cdot)$  be a metric on  $\mathbb{B}^n$  and  $d_X(\cdot, \cdot)$  be a metric on  $X$ .

$$\forall x, y, z \in \mathbb{B}^n : d(x, y) \leq d(x, z) \Rightarrow d_X(g(x), g(y)) \leq d_X(g(x), g(z))$$

ad 1b) use “most natural“ representation

typically required: strong causality

→ small changes in individual leads to small changes in fitness

→ need variation operators that obey that requirement

**but:** how to find variation operators with that property?

⇒ need design guidelines ...



### ad 2) design guidelines for variation operators

#### a) *reachability*

every  $x \in X$  should be reachable from arbitrary  $x_0 \in X$   
after finite number of repeated variations with positive probability bounded from 0

#### b) *unbiasedness*

unless having gathered knowledge about problem  
variation operator should not favor particular subsets of solutions  
 $\Rightarrow$  formally: maximum entropy principle

#### c) *control*

variation operator should have parameters affecting shape of distributions;  
known from theory: weaken variation strength when approaching optimum

ad 2) design guidelines for variation operators **in practice**

binary search space  $X = \mathbb{B}^n$

variation by k-point or uniform crossover and subsequent mutation

a) **reachability**:

regardless of the output of crossover

we can move from  $x \in \mathbb{B}^n$  to  $y \in \mathbb{B}^n$  in 1 step with probability

$$p(x, y) = p_m^{H(x,y)} (1 - p_m)^{n-H(x,y)} > 0$$

where  $H(x,y)$  is Hamming distance between  $x$  and  $y$ .

Since  $\min\{p(x,y): x,y \in \mathbb{B}^n\} = \delta > 0$  we are done.

### b) *unbiasedness*

don't prefer any direction or subset of points without reason

⇒ use maximum entropy distribution for sampling!

#### properties:

- distributes probability mass as uniform as possible
- additional knowledge can be included as constraints:
  - under given constraints sample as uniform as possible

Formally:

**Definition:**

Let  $X$  be discrete random variable (r.v.) with  $p_k = P\{ X = x_k \}$  for some index set  $K$ . The quantity

$$H(X) = - \sum_{k \in K} p_k \log p_k$$

is called the **entropy of the distribution** of  $X$ . If  $X$  is a continuous r.v. with p.d.f.  $f_X(\cdot)$  then the entropy is given by

$$H(X) = - \int_{-\infty}^{\infty} f_X(x) \log f_X(x) dx$$

The distribution of a random variable  $X$  for which  $H(X)$  is maximal is termed a **maximum entropy distribution**. ■

**Knowledge available:**

Discrete distribution with support  $\{x_1, x_2, \dots, x_n\}$  with  $x_1 < x_2 < \dots < x_n < \infty$

$$p_k = \mathbb{P}\{X = x_k\}$$

⇒ leads to nonlinear constrained optimization problem:

$$\begin{aligned} & - \sum_{k=1}^n p_k \log p_k \quad \rightarrow \max! \\ \text{s.t.} \quad & \sum_{k=1}^n p_k = 1 \end{aligned}$$

solution: via Lagrange (find stationary point of Lagrangian function)

$$L(p, a) = - \sum_{k=1}^n p_k \log p_k + a \left( \sum_{k=1}^n p_k - 1 \right)$$

$$L(p, a) = - \sum_{k=1}^n p_k \log p_k + a \left( \sum_{k=1}^n p_k - 1 \right)$$

partial derivatives:

$$\frac{\partial L(p, a)}{\partial p_k} = -1 - \log p_k + a \stackrel{!}{=} 0 \quad \Rightarrow \quad p_k \stackrel{!}{=} e^{a-1}$$

$$\frac{\partial L(p, a)}{\partial a} = \sum_{k=1}^n p_k - 1 \stackrel{!}{=} 0$$

$$\Rightarrow \sum_{k=1}^n p_k = \sum_{k=1}^n e^{a-1} = n e^{a-1} \stackrel{!}{=} 1 \quad \Leftrightarrow \quad e^{a-1} = \frac{1}{n}$$

$p_k = \frac{1}{n}$   
**uniform distribution**



**Knowledge available:**

Discrete distribution with support  $\{ 1, 2, \dots, n \}$  with  $p_k = P \{ X = k \}$  and  $E[ X ] = \nu$

$\Rightarrow$  leads to nonlinear constrained optimization problem:

$$\begin{aligned} & - \sum_{k=1}^n p_k \log p_k \quad \rightarrow \max! \\ \text{s.t.} \quad & \sum_{k=1}^n p_k = 1 \quad \text{and} \quad \sum_{k=1}^n k p_k = \nu \end{aligned}$$

solution: via Lagrange (find stationary point of Lagrangian function)

$$L(p, a, b) = - \sum_{k=1}^n p_k \log p_k + a \left( \sum_{k=1}^n p_k - 1 \right) + b \left( \sum_{k=1}^n k \cdot p_k - \nu \right)$$

$$L(p, a, b) = - \sum_{k=1}^n p_k \log p_k + a \left( \sum_{k=1}^n p_k - 1 \right) + b \left( \sum_{k=1}^n k \cdot p_k - \nu \right)$$

partial derivatives:

$$\frac{\partial L(p, a, b)}{\partial p_k} = -1 - \log p_k + a + b k \stackrel{!}{=} 0 \quad \Rightarrow \quad p_k = e^{a-1+bk}$$

$$\frac{\partial L(p, a, b)}{\partial a} = \sum_{k=1}^n p_k - 1 \stackrel{!}{=} 0$$

$$\frac{\partial L(p, a, b)}{\partial b} \stackrel{(*)}{=} \sum_{k=1}^n k p_k - \nu \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \sum_{k=1}^n p_k = e^{a-1} \sum_{k=1}^n (e^b)^k \stackrel{!}{=} 1$$

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$$\Rightarrow e^{a-1} = \frac{1}{\sum_{k=1}^n (e^b)^k} \quad \Rightarrow \quad p_k = e^{a-1+bk} = \frac{(e^b)^k}{\sum_{i=1}^n (e^b)^i}$$

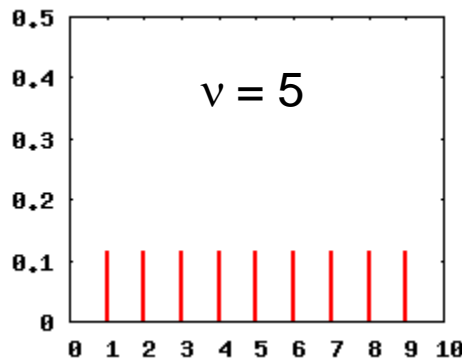
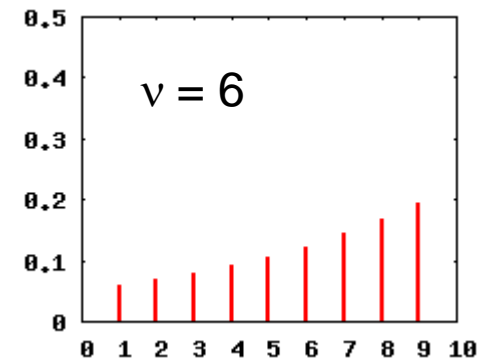
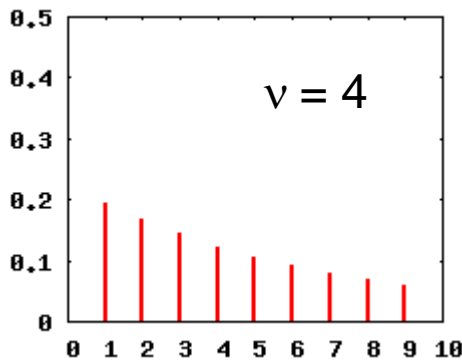
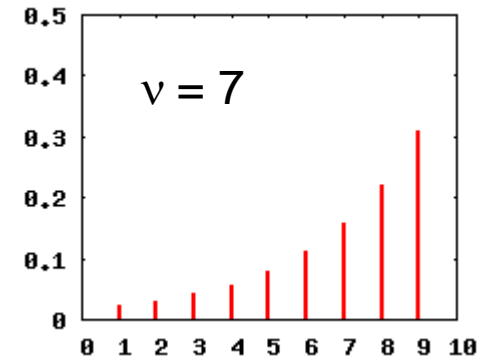
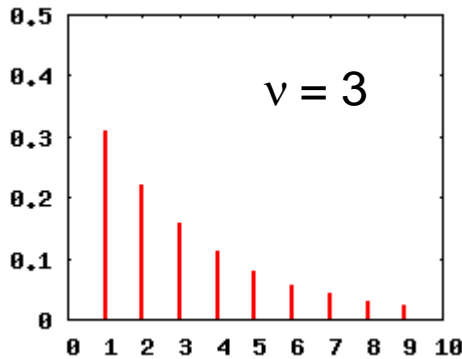
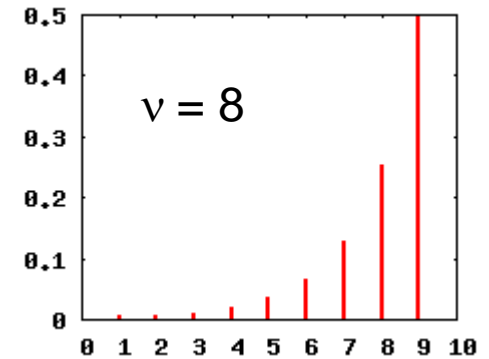
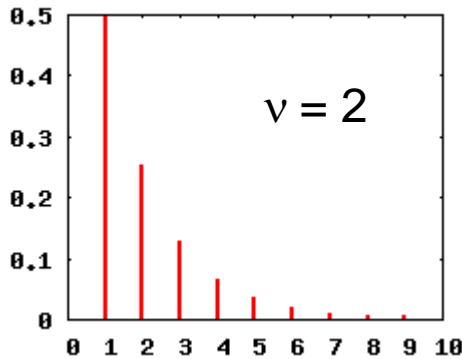
$$\Rightarrow \text{discrete Boltzmann distribution} \quad p_k = \frac{q^k}{\sum_{i=1}^n q^i} \quad (q = e^b)$$

$\Rightarrow$  value of  $q$  depends on  $\nu$  via third condition: (\*)

$$\sum_{k=1}^n k p_k = \frac{\sum_{k=1}^n k q^k}{\sum_{i=1}^n q^i} = \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)(1-q^n)} \stackrel{!}{=} \nu$$

## Boltzmann distribution ( $n = 9$ )

specializes to uniform  
distribution if  $v = 5$   
(as expected)



**Knowledge available:**

Discrete distribution with support  $\{ 1, 2, \dots, n \}$  with  $E[ X ] = \nu$  and  $V[ X ] = \eta^2$

⇒ leads to nonlinear constrained optimization problem:

$$\begin{aligned} & - \sum_{k=1}^n p_k \log p_k \quad \rightarrow \max! \\ \text{s.t.} \quad & \sum_{k=1}^n p_k = 1 \quad \text{and} \quad \sum_{k=1}^n k p_k = \nu \quad \text{and} \quad \sum_{k=1}^n (k - \nu)^2 p_k = \eta^2 \end{aligned}$$

solution: in principle, via Lagrange (find stationary point of Lagrangian function)

but very complicated analytically, if possible at all

⇒ consider special cases only

**note:** constraints  
are linear  
equations in  $p_k$

**Special case:**  $n = 3$  and  $E[X] = 2$  and  $V[X] = \eta^2$

Linear constraints uniquely determine distribution:

$$\text{I. } p_1 + p_2 + p_3 = 1$$

$$\text{II. } p_1 + 2p_2 + 3p_3 = 2$$

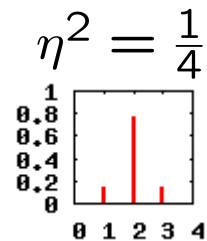
$$\text{III. } p_1 + 0 + p_3 = \eta^2$$

$$\text{II} - \text{I: } p_2 + 2p_3 = 1$$

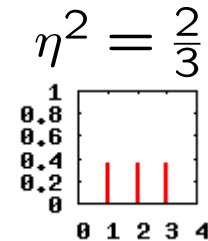
$$\text{I} - \text{III: } p_2 = 1 - \eta^2$$

$$\left. \begin{array}{l} p_1 = \frac{\eta^2}{2} \\ p_3 = \frac{\eta^2}{2} \end{array} \right\} \begin{array}{l} \uparrow \\ \text{insertion in III.} \end{array}$$

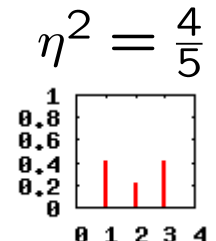
$$\Rightarrow p = \left( \frac{\eta^2}{2}, 1 - \eta^2, \frac{\eta^2}{2} \right)$$



unimodal



uniform



bimodal

**Knowledge available:**

Discrete distribution with unbounded support  $\{0, 1, 2, \dots\}$  and  $E[X] = \nu$

$\Rightarrow$  leads to infinite-dimensional nonlinear constrained optimization problem:

$$\begin{aligned} & - \sum_{k=0}^{\infty} p_k \log p_k \quad \rightarrow \max! \\ \text{s.t.} \quad & \sum_{k=0}^{\infty} p_k = 1 \quad \text{and} \quad \sum_{k=0}^{\infty} k p_k = \nu \end{aligned}$$

solution: via Lagrange (find stationary point of Lagrangian function)

$$L(p, a, b) = - \sum_{k=0}^{\infty} p_k \log p_k + a \left( \sum_{k=0}^{\infty} p_k - 1 \right) + b \left( \sum_{k=0}^{\infty} k \cdot p_k - \nu \right)$$

$$L(p, a, b) = - \sum_{k=0}^{\infty} p_k \log p_k + a \left( \sum_{k=0}^{\infty} p_k - 1 \right) + b \left( \sum_{k=0}^{\infty} k \cdot p_k - \nu \right)$$

partial derivatives:

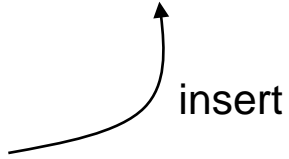
$$\frac{\partial L(p, a, b)}{\partial p_k} = -1 - \log p_k + a + b k \stackrel{!}{=} 0 \quad \Rightarrow \quad p_k = e^{a-1+bk}$$

$$\frac{\partial L(p, a, b)}{\partial a} = \sum_{k=0}^{\infty} p_k - 1 \stackrel{!}{=} 0$$

$$\frac{\partial L(p, a, b)}{\partial b} \stackrel{(*)}{=} \sum_{k=0}^{\infty} k p_k - \nu \stackrel{!}{=} 0 \quad \Rightarrow \quad \sum_{k=0}^{\infty} p_k = e^{a-1} \sum_{k=0}^{\infty} (e^b)^k \stackrel{!}{=} 1$$

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$$\Rightarrow e^{a-1} = \frac{1}{\sum_{k=0}^{\infty} (e^b)^k} \quad \Rightarrow \quad p_k = e^{a-1+bk} = \frac{(e^b)^k}{\sum_{i=0}^{\infty} (e^b)^i}$$

set  $q = e^b$  and insists that  $q < 1$   $\Rightarrow \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$  

$$\Rightarrow p_k = (1-q)q^k \quad \text{for } k = 0, 1, 2, \dots \quad \text{geometrical distribution}$$

it remains to specify  $q$ ; to proceed recall that  $\sum_{k=0}^{\infty} k q^k = \frac{q}{(1-q)^2}$

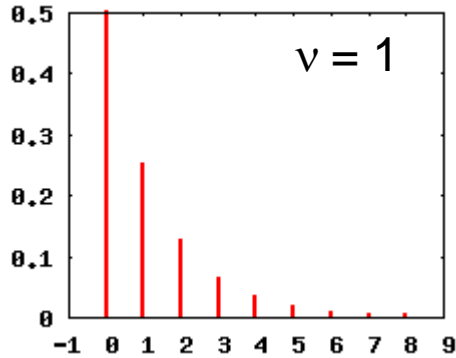
⇒ value of  $q$  depends on  $\nu$  via third condition: (\*)

$$\sum_{k=0}^{\infty} k p_k = \frac{\sum_{k=0}^{\infty} k q^k}{\sum_{i=0}^{\infty} q^i} = \frac{q}{1-q} \stackrel{!}{=} \nu$$

$$\Rightarrow q = \frac{\nu}{\nu + 1} = 1 - \frac{1}{\nu + 1}$$

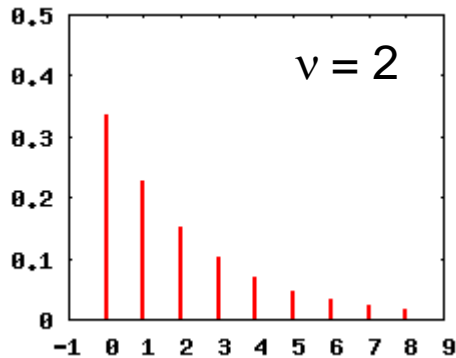
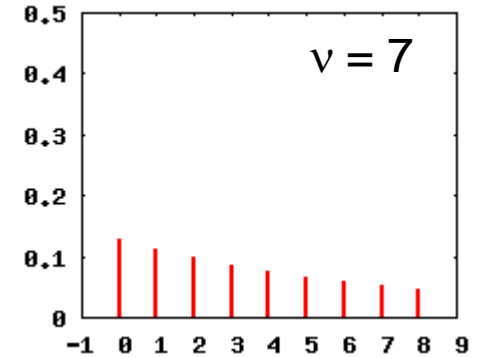
$$\Rightarrow p_k = \frac{1}{\nu + 1} \left( 1 - \frac{1}{\nu + 1} \right)^k$$



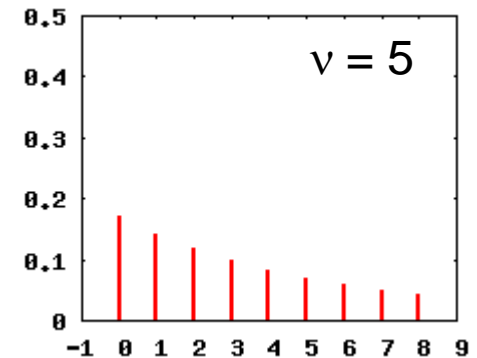
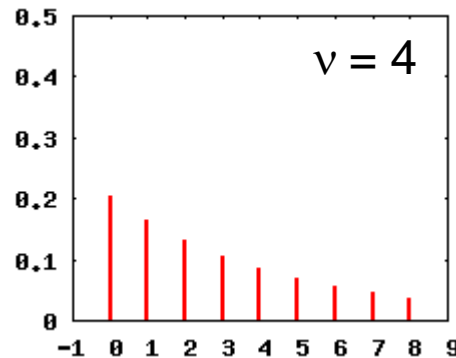
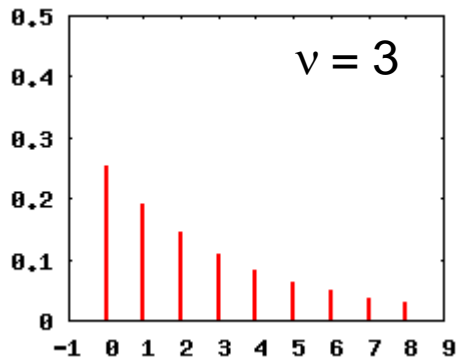
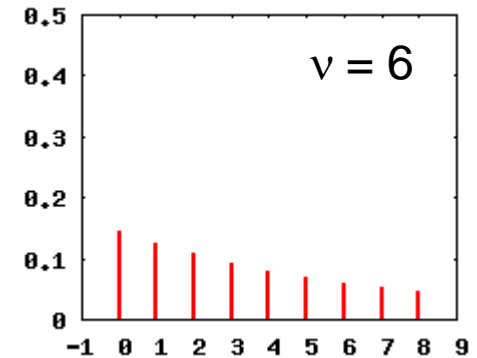


**geometrical distribution**

with  $E[x] = v$



$p_k$  only shown  
for  $k = 0, 1, \dots, 8$



**Overview:**

support  $\{ 1, 2, \dots, n \}$   $\Rightarrow$  *discrete uniform* distribution  
and require  $E[X] = \theta$   $\Rightarrow$  *Boltzmann* distribution  
and require  $V[X] = \eta^2$   $\Rightarrow$  N.N. (**not** Binomial distribution)

support  $\mathbb{N}$   $\Rightarrow$  not defined!  
and require  $E[X] = \theta$   $\Rightarrow$  *geometrical* distribution  
and require  $V[X] = \eta^2$   $\Rightarrow$  ?

support  $\mathbb{Z}$   $\Rightarrow$  not defined!  
and require  $E[|X|] = \theta$   $\Rightarrow$  *bi-geometrical* distribution (*discrete Laplace* distr.)  
and require  $E[|X|^2] = \eta^2$   $\Rightarrow$  N.N. (*discrete Gaussian* distr.)

support  $[a,b] \subset \mathbb{R}$   $\Rightarrow$  uniform distribution

support  $\mathbb{R}^+$  with  $E[X] = \theta$   $\Rightarrow$  Exponential distribution

support  $\mathbb{R}$   
with  $E[X] = \theta$ ,  $V[X] = \eta^2$   $\Rightarrow$  normal / Gaussian distribution  $N(\theta, \eta^2)$

support  $\mathbb{R}^n$   
with  $E[X] = \theta$   
and  $\text{Cov}[X] = C$   $\Rightarrow$  multinormal distribution  $N(\theta, C)$

expectation vector  $\in \mathbb{R}^n$

covariance matrix  $\in \mathbb{R}^{n,n}$

positive definite:  
 $\forall x \neq 0 : x'Cx > 0$

for permutation distributions ?

→ uniform distribution on all possible permutations

```
set v[j] = j for j = 1, 2, ..., n
for i = n to 1 step -1
  draw k uniformly at random from { 1, 2, ..., i }
  swap v[i] and v[k]
endfor
```

generates  
permutation  
uniformly at  
random in  
 $\Theta(n)$  time

### Guideline:

Only if you know something about the problem *a priori* or

if you have learnt something about the problem *during the search*

⇒ include that knowledge in search / mutation distribution (via constraints!)

### ad 2) design guidelines for variation operators **in practice**

integer search space  $X = \mathbb{Z}^n$

- a) reachability
- b) unbiasedness
- c) control

- every recombination results in some  $z \in \mathbb{Z}^n$   
- mutation of  $z$  may then lead to any  $z^* \in \mathbb{Z}^n$  with positive probability in one step

ad a) support of mutation should be  $\mathbb{Z}^n$

ad b) need maximum entropy distribution over support  $\mathbb{Z}^n$

ad c) control variability by parameter

→ formulate as constraint of maximum entropy distribution

ad 2) design guidelines for variation operators **in practice**

$$X = \mathbb{Z}^n$$

**task:** find (symmetric) maximum entropy distribution over  $\mathbb{Z}$  with  $E[|Z|] = \theta > 0$   
 $\Rightarrow$  need analytic solution of a  $\infty$ -dimensional, nonlinear optimization problem with constraints!

$$H(p) = - \sum_{k=-\infty}^{\infty} p_k \log p_k \quad \longrightarrow \quad \max!$$

$$\text{s.t.} \quad p_k = p_{-k} \quad \forall k \in \mathbb{Z}, \quad (\text{symmetry w.r.t. } 0)$$

$$\sum_{k=-\infty}^{\infty} p_k = 1, \quad (\text{normalization})$$

$$\sum_{k=-\infty}^{\infty} |k| p_k = \theta \quad (\text{control "spread"})$$

$$p_k \geq 0 \quad \forall k \in \mathbb{Z}. \quad (\text{nonnegativity})$$

**result:**

a random variable  $Z$  with support  $\mathbb{Z}$  and probability distribution

$$p_k := P\{Z = k\} = \frac{q}{2 - q} (1 - q)^{|k|}, \quad k \in \mathbb{Z}, \quad q \in (0, 1)$$

symmetric w.r.t. 0, unimodal, spread manageable by  $q$  and has max. entropy ■

**generation of pseudo random numbers:**

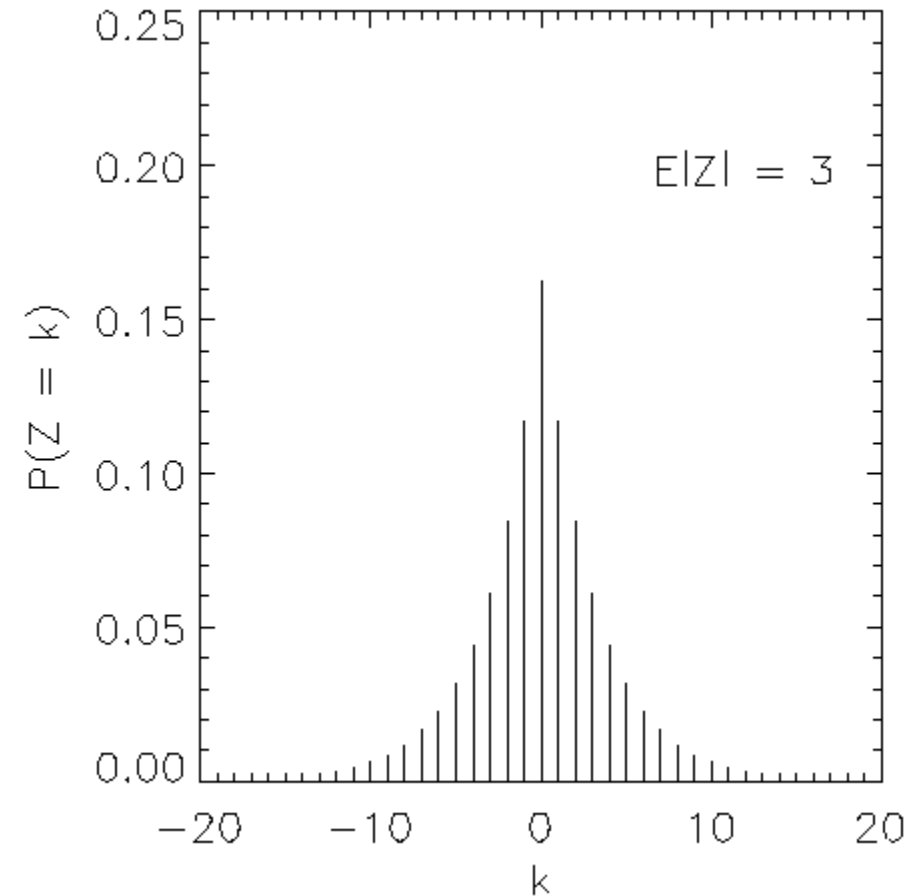
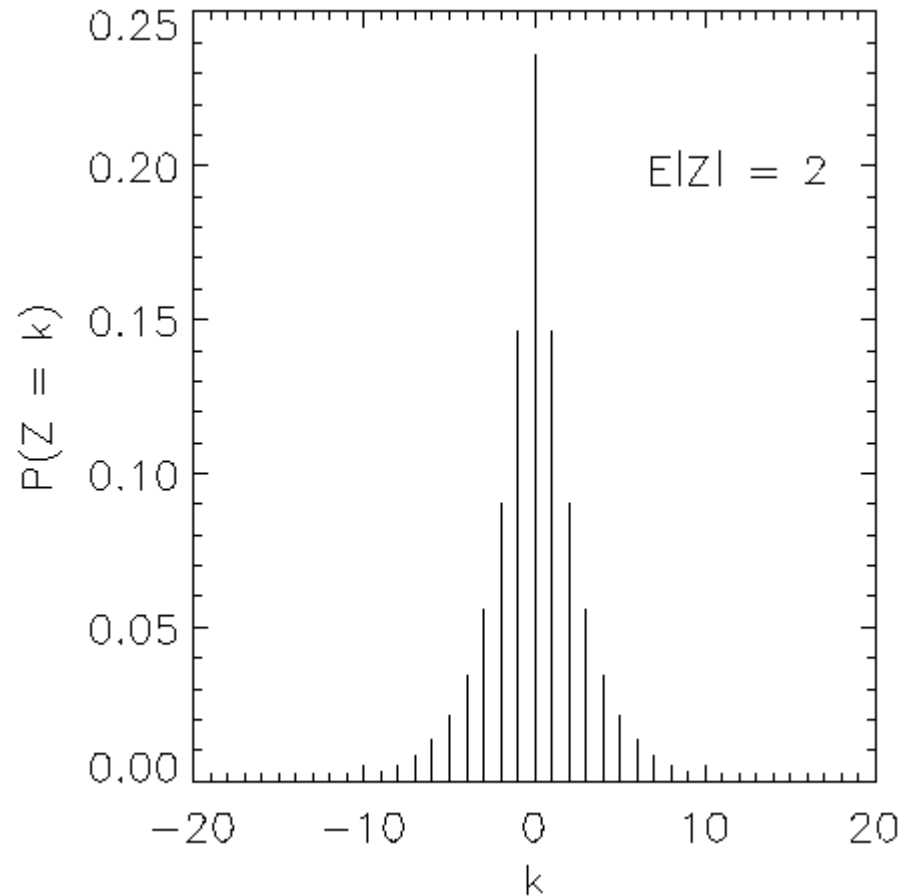
$$Z = G_1 - G_2$$

where

$$U_i \sim U(0, 1) \Rightarrow G_i = \left\lfloor \frac{\log(1 - U_i)}{\log(1 - q)} \right\rfloor, \quad i = 1, 2.$$

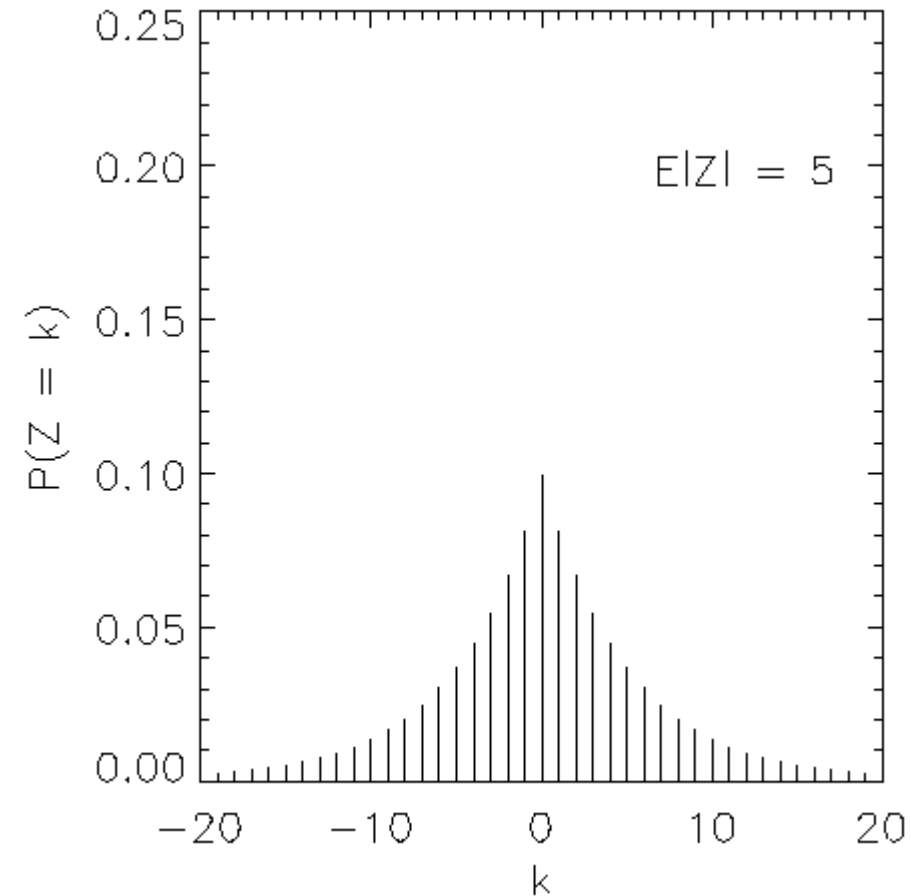
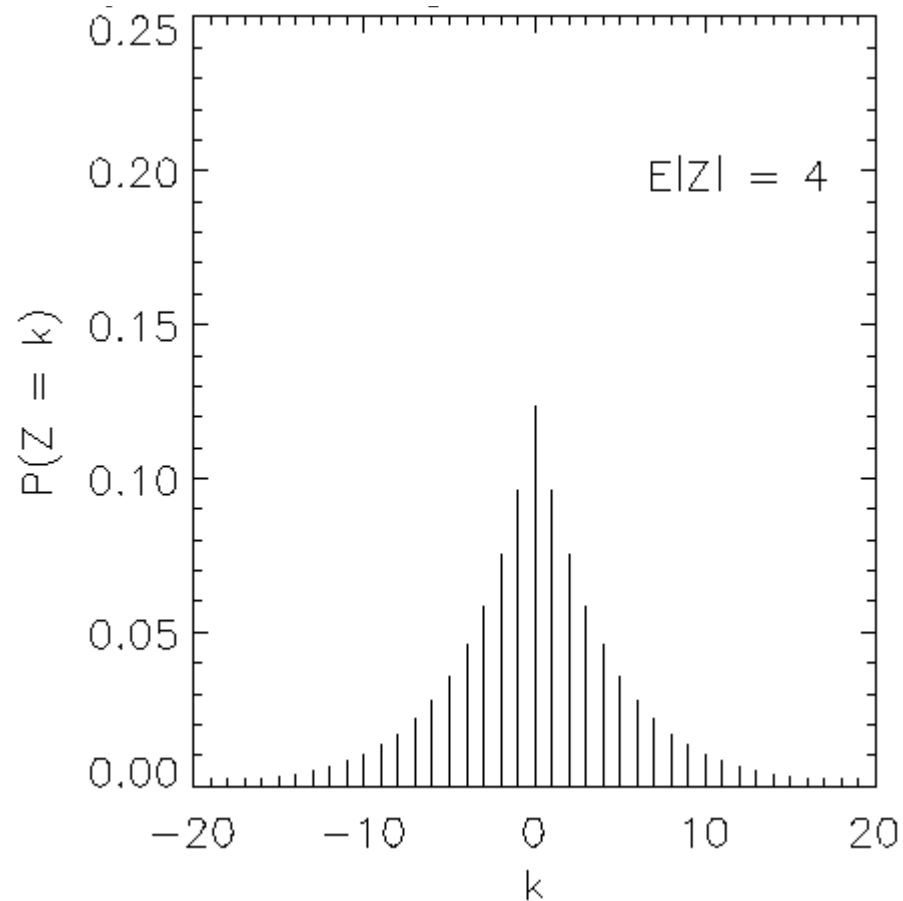
stochastic  
independent!

probability distributions for different mean step sizes  $E|Z| = \theta$





probability distributions for different mean step sizes  $E|Z| = \theta$



### How to control the spread?

We must be able to adapt  $q \in (0,1)$  for generating  $Z$  with variable  $E|Z| = \theta$  !

self-adaptation of  $q$  in open interval  $(0,1)$  ?

→ make mean step size  $E[|Z|]$  adjustable!

$$E[|Z|] = \sum_{k=-\infty}^{\infty} |k| p_k = \theta = \frac{2(1-q)}{q(2-q)} \Leftrightarrow q = 1 - \frac{\theta}{(1+\theta^2)^{1/2} + 1}$$

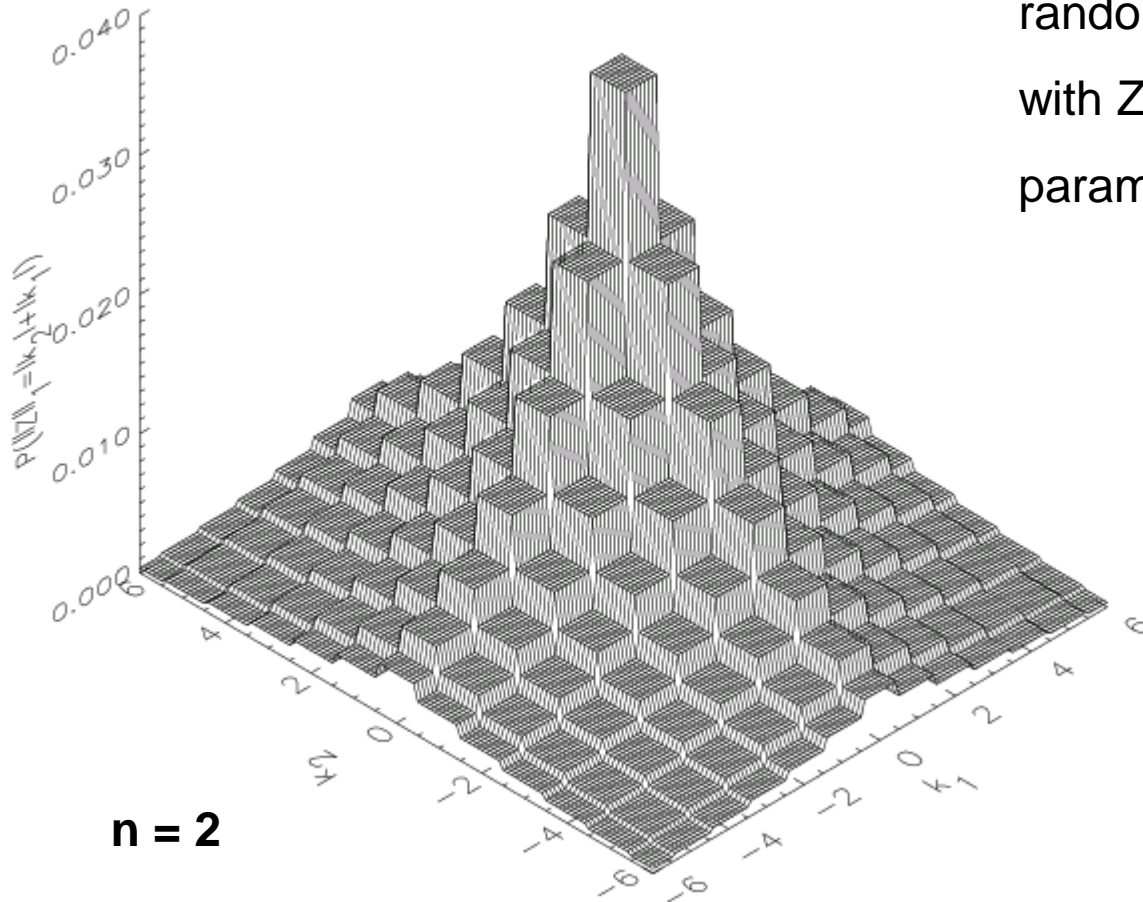
$\downarrow$   
 $\in \mathbb{R}_+$

$\downarrow$   
 $\in (0,1)$

→  $\theta$  adjustable by mutative self adaptation

→ get  $q$  from  $\theta$

like mutative step size control  
of  $\sigma$  in EA with search space  $\mathbb{R}^n$  !

**n - dimensional generalization****n = 2**

random vector  $Z = (Z_1, Z_2, \dots, Z_n)$   
with  $Z_i = G_{1,i} - G_{2,i}$  (stoch. indep.);  
parameter  $q$  for all  $G_{1,i}, G_{2,i}$  equal

n - dimensional generalization

$$P\{Z_i = k\} = \frac{q}{2 - q} (1 - q)^{|k|}$$

$$P\{Z_1 = k_1, Z_2 = k_2, \dots, Z_n = k_n\} = \prod_{i=1}^n P\{Z_i = k_i\} =$$

$$\begin{aligned} \left(\frac{q}{2 - q}\right)^n \prod_{i=1}^n (1 - q)^{|k_i|} &= \left(\frac{q}{2 - q}\right)^n (1 - q)^{\sum_{i=1}^n |k_i|} \\ &= \left(\frac{q}{2 - q}\right)^n (1 - q)^{\|k\|_1}. \end{aligned}$$

⇒ n-dimensional distribution is symmetric w.r.t.  $\ell_1$  norm!

⇒ all random vectors with same step length have same probability!

How to control  $E[ \| Z \|_1 ]$  ?

$$E[ \| Z \|_1 ] \underset{\substack{\uparrow \\ \text{by def.}}}{=} E \left[ \sum_{i=1}^n |Z_i| \right] \underset{\substack{\uparrow \\ \text{linearity of } E[\cdot]}}{=} \sum_{i=1}^n E[|Z_i|] \underset{\substack{\uparrow \\ \text{identical distributions for } Z_i}}{=} n \cdot E[|Z_1|]$$

$$\underbrace{n \cdot E[|Z_1|]}_{= \theta} = n \cdot \frac{2(1-q)}{q(2-q)} \Leftrightarrow q = 1 - \frac{\theta/n}{(1 + (\theta/n)^2)^{1/2} + 1}$$

self-adaptation
calculate from  $\theta$

**Algorithm:**

individual :  $(x, \theta) \in \mathbb{Z}^n \times \mathbb{R}_+$

mutation :  $\theta^{(t+1)} = \theta^{(t)} \cdot \exp(N), \quad N \sim N(0, 1/n).$

if  $\theta^{(t+1)} < 1$  then  $\theta_{t+1} = 1$

calculate new  $q$  for  $G_i$  from  $\theta_{t+1}$

$\forall j = 1, \dots, n : X_j^{(t+1)} = X_j^{(t)} + (G_{1,j} - G_{2,j})$

recombination : discrete (uniform crossover)

selection :  $(\mu, \lambda)$ -selection

(Rudolph, PPSN 1994)

ad 2) design guidelines for variation operators **in practice**

continuous search space  $X = \mathbb{R}^n$

- a) reachability
- b) unbiasedness
- c) control

⇒ leads to CMA-ES !