

Computational Intelligence

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Three tasks:

1. Choice of an appropriate problem representation.
2. Choice / design of variation operators acting in problem representation.
3. Choice of strategy parameters (includes initialization).

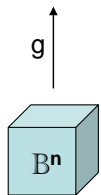
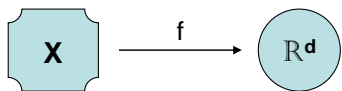
ad 1) different "schools":

- (a) operate on binary representation and define genotype/phenotype mapping
 - + can use standard algorithm
 - mapping may induce unintentional bias in search
- (b) no doctrine: use "most natural" representation
 - must design variation operators for specific representation
 - + if design done properly then no bias in search

ad 1a) genotype-phenotype mapping

original problem $f: X \rightarrow \mathbb{R}^d$

scenario: no standard algorithm for search space X available



- standard EA performs variation on binary strings $b \in B^n$
- fitness evaluation of individual b via $(f \circ g)(b) = f(g(b))$
where $g: B^n \rightarrow X$ is genotype-phenotype mapping
- selection operation independent from representation

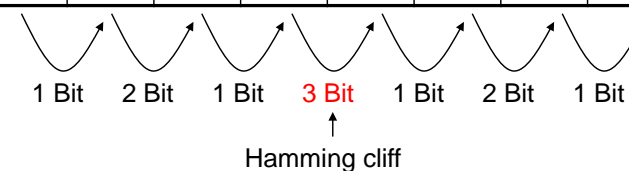
Genotype-Phenotype-Mapping $B^n \rightarrow [L, R] \subset \mathbb{R}$

- Standard encoding for $b \in B^n$

$$x = L + \frac{R - L}{2^n - 1} \sum_{i=0}^{n-1} b_{n-i} 2^i$$

→ Problem: *hamming cliffs*

000	001	010	011	100	101	110	111
0	1	2	3	4	5	6	7



← genotype
← phenotype

$L = 0, R = 7$
 $n = 3$

Genotype-Phenotype-Mapping $B^n \rightarrow [L, R] \subset \mathbb{R}$

- Gray encoding for $b \in B^n$

Let $a \in B^n$ standard encoded. Then $b_i = \begin{cases} a_i, & \text{if } i = 1 \\ a_{i-1} \oplus a_i, & \text{if } i > 1 \end{cases}$ $\oplus = \text{XOR}$

000	001	011	010	110	111	101	100	← genotype
0	1	2	3	4	5	6	7	← phenotype

OK, no hamming cliffs any longer ...

⇒ small changes in phenotype „lead to“ small changes in genotype

since we consider evolution in terms of Darwin (not Lamarck):

⇒ small changes in genotype lead to small changes in phenotype!

but: 1-Bit-change: $000 \rightarrow 100 \Rightarrow \ominus$

Genotype-Phenotype-Mapping $B^n \rightarrow \mathbb{P}^{\log(n)}$ (example only)

- e.g. standard encoding for $b \in B^n$

individual:

010	101	111	000	110	001	101	100	← genotype
0	1	2	3	4	5	6	7	← index

consider index and associated genotype entry as unit / record / struct;

sort units with respect to genotype value, old indices yield permutation:

000	001	010	100	101	101	110	111	← genotype
3	5	0	7	1	6	4	2	← old index

= permutation

ad 1a) genotype-phenotype mapping

typically required: strong causality

→ small changes in individual leads to small changes in fitness

→ small changes in genotype should lead to small changes in phenotype

but: how to find a genotype-phenotype mapping with that property?

necessary conditions:

- 1) $g: B^n \rightarrow X$ can be computed efficiently (otherwise it is senseless)
- 2) $g: B^n \rightarrow X$ is surjective (otherwise we might miss the optimal solution)
- 3) $g: B^n \rightarrow X$ *preserves closeness* (otherwise strong causality endangered)

Let $d(\cdot, \cdot)$ be a metric on B^n and $d_X(\cdot, \cdot)$ be a metric on X .

$\forall x, y, z \in B^n: d(x, y) \leq d(x, z) \Rightarrow d_X(g(x), g(y)) \leq d_X(g(x), g(z))$

ad 1b) use “most natural“ representation

typically required: strong causality

→ small changes in individual leads to small changes in fitness

→ need variation operators that obey that requirement

but: how to find variation operators with that property?

⇒ need design guidelines ...

ad 2) design guidelines for variation operators

a) *reachability*

every $x \in X$ should be reachable from arbitrary $x_0 \in X$
after finite number of repeated variations with positive probability bounded from 0

b) *unbiasedness*

unless having gathered knowledge about problem
variation operator should not favor particular subsets of solutions
⇒ formally: maximum entropy principle

c) *control*

variation operator should have parameters affecting shape of distributions;
known from theory: weaken variation strength when approaching optimum

ad 2) design guidelines for variation operators **in practice**

binary search space $X = \mathbb{B}^n$

variation by k-point or uniform crossover and subsequent mutation

a) *reachability*:

regardless of the output of crossover
we can move from $x \in \mathbb{B}^n$ to $y \in \mathbb{B}^n$ in 1 step with probability

$$p(x, y) = p_m^{H(x,y)} (1 - p_m)^{n-H(x,y)} > 0$$

where $H(x,y)$ is Hamming distance between x and y .

Since $\min\{p(x,y) : x,y \in \mathbb{B}^n\} = \delta > 0$ we are done.

b) *unbiasedness*

don't prefer any direction or subset of points without reason

⇒ use maximum entropy distribution for sampling!

properties:

- distributes probability mass as uniform as possible
- additional knowledge can be included as constraints:
→ under given constraints sample as uniform as possible

Formally:**Definition:**

Let X be discrete random variable (r.v.) with $p_k = P\{X = x_k\}$ for some index set K .
The quantity

$$H(X) = - \sum_{k \in K} p_k \log p_k$$

is called the **entropy of the distribution** of X . If X is a continuous r.v. with p.d.f. $f_X(\cdot)$ then the entropy is given by

$$H(X) = - \int_{-\infty}^{\infty} f_X(x) \log f_X(x) dx$$

The distribution of a random variable X for which $H(X)$ is maximal is termed a **maximum entropy distribution**. ■

Knowledge available:

Discrete distribution with support $\{x_1, x_2, \dots, x_n\}$ with $x_1 < x_2 < \dots < x_n < \infty$

$$p_k = P\{X = x_k\}$$

⇒ leads to nonlinear constrained optimization problem:

$$\begin{aligned} - \sum_{k=1}^n p_k \log p_k &\rightarrow \max! \\ \text{s.t.} \quad \sum_{k=1}^n p_k &= 1 \end{aligned}$$

solution: via Lagrange (find stationary point of Lagrangian function)

$$L(p, a) = - \sum_{k=1}^n p_k \log p_k + a \left(\sum_{k=1}^n p_k - 1 \right)$$

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partial derivatives:

$$\begin{aligned} \frac{\partial L(p, a)}{\partial p_k} &= -1 - \log p_k + a \stackrel{!}{=} 0 && \Rightarrow p_k \stackrel{!}{=} e^{a-1} \\ \frac{\partial L(p, a)}{\partial a} &= \sum_{k=1}^n p_k - 1 \stackrel{!}{=} 0 \end{aligned}$$

$p_k = \frac{1}{n}$
uniform distribution

$$\Rightarrow \sum_{k=1}^n p_k = \sum_{k=1}^n e^{a-1} = n e^{a-1} \stackrel{!}{=} 1 \Leftrightarrow e^{a-1} = \frac{1}{n}$$

Knowledge available:

Discrete distribution with support $\{1, 2, \dots, n\}$ with $p_k = P\{X = k\}$ and $E[X] = \nu$

⇒ leads to nonlinear constrained optimization problem:

$$\begin{aligned} - \sum_{k=1}^n p_k \log p_k &\rightarrow \max! \\ \text{s.t.} \quad \sum_{k=1}^n p_k &= 1 \quad \text{and} \quad \sum_{k=1}^n k p_k = \nu \end{aligned}$$

solution: via Lagrange (find stationary point of Lagrangian function)

$$L(p, a, b) = - \sum_{k=1}^n p_k \log p_k + a \left(\sum_{k=1}^n p_k - 1 \right) + b \left(\sum_{k=1}^n k \cdot p_k - \nu \right)$$

$$L(p, a, b) = - \sum_{k=1}^n p_k \log p_k + a \left(\sum_{k=1}^n p_k - 1 \right) + b \left(\sum_{k=1}^n k \cdot p_k - \nu \right)$$

partial derivatives:

$$\begin{aligned} \frac{\partial L(p, a, b)}{\partial p_k} &= -1 - \log p_k + a + b k \stackrel{!}{=} 0 && \Rightarrow p_k = e^{a-1+bk} \\ \frac{\partial L(p, a, b)}{\partial a} &= \sum_{k=1}^n p_k - 1 \stackrel{!}{=} 0 \\ \frac{\partial L(p, a, b)}{\partial b} &\stackrel{(*)}{=} \sum_{k=1}^n k p_k - \nu \stackrel{!}{=} 0 \end{aligned}$$

\Downarrow
 $\sum_{k=1}^n p_k = e^{a-1} \sum_{k=1}^n (e^b)^k \stackrel{!}{=} 1$

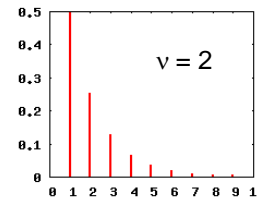
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$$\Rightarrow e^{a-1} = \frac{1}{\sum_{k=1}^n (e^b)^k} \Rightarrow p_k = e^{a-1+bk} = \frac{(e^b)^k}{\sum_{i=1}^n (e^b)^i}$$

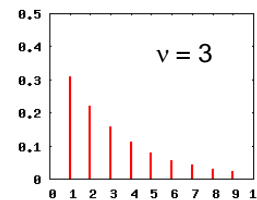
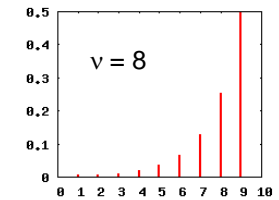
$$\Rightarrow \text{discrete Boltzmann distribution } p_k = \frac{q^k}{\sum_{i=1}^n q^i} \quad (q = e^b)$$

⇒ value of q depends on v via third condition: (*)

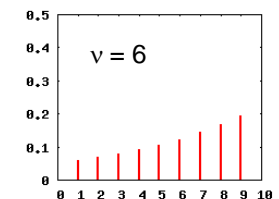
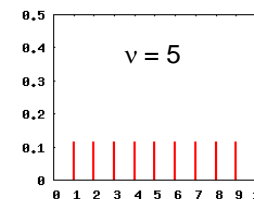
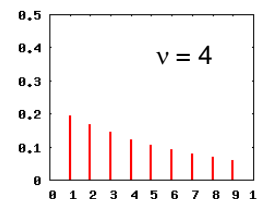
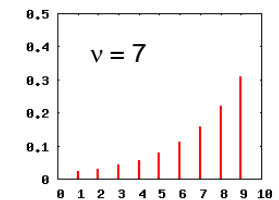
$$\sum_{k=1}^n k p_k = \frac{\sum_{k=1}^n k q^k}{\sum_{i=1}^n q^i} = \frac{1 - (n+1)q^n + nq^{n+1}}{(1-q)(1-q^n)} \stackrel{!}{=} \nu$$



Boltzmann distribution (n = 9)



specializes to uniform distribution if v = 5 (as expected)



Knowledge available:

Discrete distribution with support { 1, 2, ..., n } with $E[X] = \nu$ and $V[X] = \eta^2$

⇒ leads to nonlinear constrained optimization problem:

$$-\sum_{k=1}^n p_k \log p_k \rightarrow \max!$$

$$\text{s.t. } \sum_{k=1}^n p_k = 1 \quad \text{and} \quad \sum_{k=1}^n k p_k = \nu \quad \text{and} \quad \sum_{k=1}^n (k - \nu)^2 p_k = \eta^2$$

solution: in principle, via Lagrange (find stationary point of Lagrangian function)

but very complicated analytically, if possible at all

⇒ consider special cases only

note: constraints are linear equations in p_k

Special case: $n = 3$ and $E[X] = 2$ and $V[X] = \eta^2$

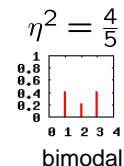
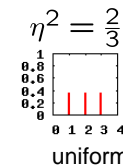
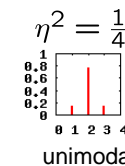
Linear constraints uniquely determine distribution:

$$\begin{aligned} \text{I. } & p_1 + p_2 + p_3 = 1 \\ \text{II. } & p_1 + 2p_2 + 3p_3 = 2 \\ \text{III. } & p_1 + 0 + p_3 = \eta^2 \end{aligned}$$

$$\begin{aligned} \text{II} - \text{I: } & p_2 + 2p_3 = 1 \\ \text{I} - \text{III: } & p_2 = 1 - \eta^2 \end{aligned}$$

$$\left. \begin{aligned} p_1 &= \frac{\eta^2}{2} \\ p_3 &= \frac{\eta^2}{2} \end{aligned} \right\} \begin{array}{l} \uparrow \\ \text{insertion in III.} \end{array}$$

$$\Rightarrow p = \left(\frac{\eta^2}{2}, 1 - \eta^2, \frac{\eta^2}{2} \right)$$



Knowledge available:

Discrete distribution with unbounded support $\{0, 1, 2, \dots\}$ and $E[X] = \nu$

\Rightarrow leads to infinite-dimensional nonlinear constrained optimization problem:

$$-\sum_{k=0}^{\infty} p_k \log p_k \rightarrow \max!$$

$$\text{s.t.} \quad \sum_{k=0}^{\infty} p_k = 1 \quad \text{and} \quad \sum_{k=0}^{\infty} k p_k = \nu$$

solution: via Lagrange (find stationary point of Lagrangian function)

$$L(p, a, b) = -\sum_{k=0}^{\infty} p_k \log p_k + a \left(\sum_{k=0}^{\infty} p_k - 1 \right) + b \left(\sum_{k=0}^{\infty} k \cdot p_k - \nu \right)$$

$$L(p, a, b) = -\sum_{k=0}^{\infty} p_k \log p_k + a \left(\sum_{k=0}^{\infty} p_k - 1 \right) + b \left(\sum_{k=0}^{\infty} k \cdot p_k - \nu \right)$$

partial derivatives:

$$\frac{\partial L(p, a, b)}{\partial p_k} = -1 - \log p_k + a + b k \stackrel{!}{=} 0 \quad \Rightarrow \quad p_k = e^{a-1+bk}$$

$$\frac{\partial L(p, a, b)}{\partial a} = \sum_{k=0}^{\infty} p_k - 1 \stackrel{!}{=} 0$$

$$\frac{\partial L(p, a, b)}{\partial b} \stackrel{(*)}{=} \sum_{k=0}^{\infty} k p_k - \nu \stackrel{!}{=} 0 \quad \Rightarrow \quad \sum_{k=0}^{\infty} p_k = e^{a-1} \sum_{k=0}^{\infty} (e^b)^k \stackrel{!}{=} 1$$

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$$\Rightarrow e^{a-1} = \frac{1}{\sum_{k=0}^{\infty} (e^b)^k} \quad \Rightarrow \quad p_k = e^{a-1+bk} = \frac{(e^b)^k}{\sum_{i=0}^{\infty} (e^b)^i}$$

set $q = e^b$ and insists that $q < 1 \Rightarrow \sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ ↑ insert

$$\Rightarrow p_k = (1-q)q^k \quad \text{for } k = 0, 1, 2, \dots \quad \text{geometrical distribution}$$

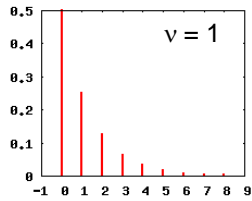
it remains to specify q ; to proceed recall that $\sum_{k=0}^{\infty} k q^k = \frac{q}{(1-q)^2}$

\Rightarrow value of q depends on ν via third condition: **(*)**

$$\sum_{k=0}^{\infty} k p_k = \frac{\sum_{k=0}^{\infty} k q^k}{\sum_{i=0}^{\infty} q^i} = \frac{q}{1-q} \stackrel{!}{=} \nu$$

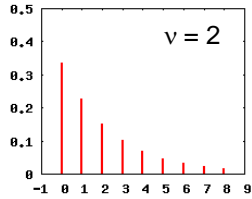
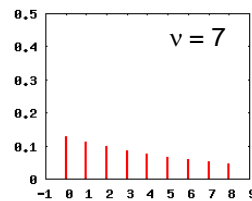
$$\Rightarrow q = \frac{\nu}{\nu+1} = 1 - \frac{1}{\nu+1}$$

$$\Rightarrow p_k = \frac{1}{\nu+1} \left(1 - \frac{1}{\nu+1} \right)^k$$

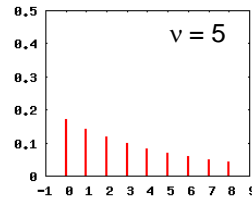
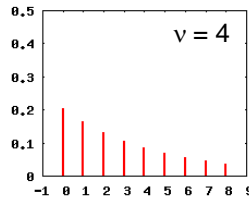
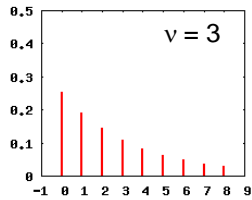
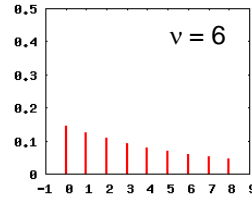


geometrical distribution

with $E[X] = v$



p_k only shown
for $k = 0, 1, \dots, 8$



Overview:

support $\{1, 2, \dots, n\}$ \Rightarrow *discrete uniform* distribution
and require $E[X] = \theta$ \Rightarrow *Boltzmann* distribution
and require $V[X] = \eta^2$ \Rightarrow N.N. (**not** Binomial distribution)

support \mathbb{N} \Rightarrow not defined!
and require $E[X] = \theta$ \Rightarrow *geometrical* distribution
and require $V[X] = \eta^2$ \Rightarrow ?

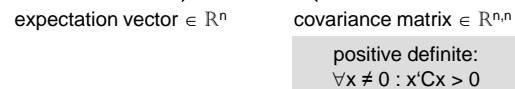
support \mathbb{Z} \Rightarrow not defined!
and require $E[|X|] = \theta$ \Rightarrow *bi-geometrical* distribution (*discrete Laplace* distr.)
and require $E[|X|^2] = \eta^2$ \Rightarrow N.N. (*discrete Gaussian* distr.)

support $[a, b] \subset \mathbb{R}$ \Rightarrow uniform distribution

support \mathbb{R}^+ with $E[X] = \theta$ \Rightarrow Exponential distribution

support \mathbb{R}
with $E[X] = \theta$, $V[X] = \eta^2$ \Rightarrow normal / Gaussian distribution $N(\theta, \eta^2)$

support \mathbb{R}^n
with $E[X] = \theta$
and $\text{Cov}[X] = C$ \Rightarrow multinormal distribution $N(\theta, C)$



for permutation distributions ?

\rightarrow uniform distribution on all possible permutations

```

set v[j] = j for j = 1, 2, ..., n
for i = n to 1 step -1
    draw k uniformly at random from { 1, 2, ..., i }
    swap v[i] and v[k]
endfor

```

generates
permutation
uniformly at
random in
 $\Theta(n)$ time

Guideline:

Only if you know something about the problem *a priori* or
if you have learnt something about the problem *during the search*
 \Rightarrow include that knowledge in search / mutation distribution (via constraints!)

ad 2) design guidelines for variation operators in practice

integer search space $X = Z^n$

- a) reachability
- b) unbiasedness
- c) control

- every recombination results in some $z \in Z^n$
 - mutation of z may then lead to any $z^* \in Z^n$ with positive probability in one step

ad a) support of mutation should be Z^n

ad b) need maximum entropy distribution over support Z^n

ad c) control variability by parameter

→ formulate as constraint of maximum entropy distribution

ad 2) design guidelines for variation operators in practice

$X = Z^n$

task: find (symmetric) maximum entropy distribution over Z with $E[|Z|] = \theta > 0$

⇒ need *analytic* solution of a ∞ -dimensional, nonlinear optimization problem with constraints!

$$H(p) = - \sum_{k=-\infty}^{\infty} p_k \log p_k \rightarrow \max!$$

s.t. $p_k = p_{-k} \quad \forall k \in Z, \quad$ (symmetry w.r.t. 0)

$$\sum_{k=-\infty}^{\infty} p_k = 1, \quad$$
 (normalization)

$$\sum_{k=-\infty}^{\infty} |k| p_k = \theta \quad$$
 (control "spread")

$$p_k \geq 0 \quad \forall k \in Z. \quad$$
 (nonnegativity)

result:

a random variable Z with support Z and probability distribution

$$p_k := P\{Z = k\} = \frac{q}{2-q} (1-q)^{|k|}, \quad k \in Z, \quad q \in (0, 1)$$

symmetric w.r.t. 0, unimodal, spread manageable by q and has max. entropy ■

generation of pseudo random numbers:

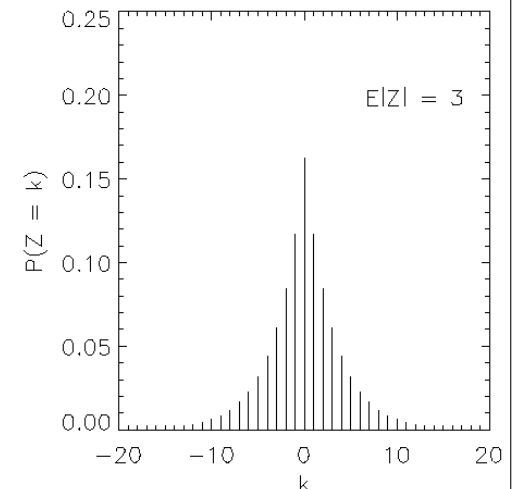
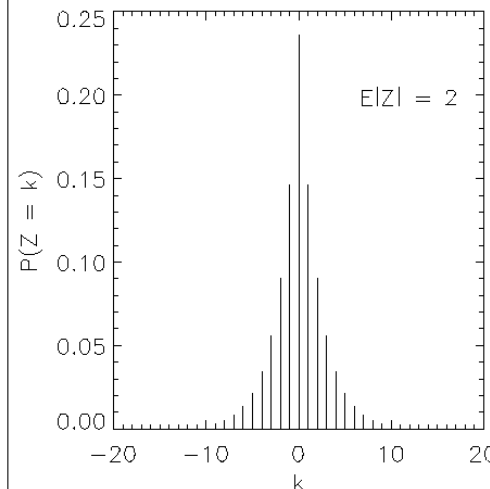
$$Z = G_1 - G_2$$

where

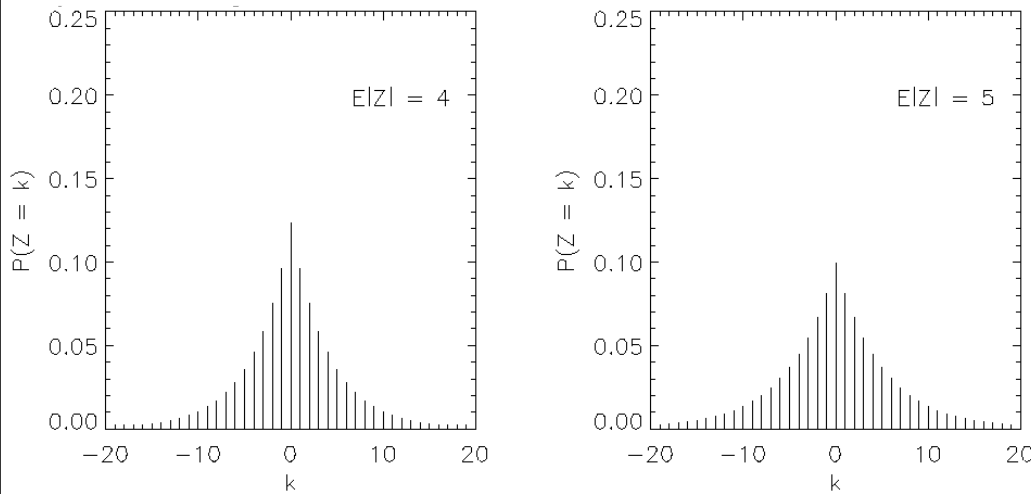
$$U_i \sim U(0, 1) \Rightarrow G_i = \left\lfloor \frac{\log(1 - U_i)}{\log(1 - q)} \right\rfloor, \quad i = 1, 2.$$

stochastic independent!

probability distributions for different mean step sizes $E|Z| = \theta$



probability distributions for different mean step sizes $E|Z| = \theta$



How to control the spread?

We must be able to adapt $q \in (0,1)$ for generating Z with variable $E|Z| = \theta$!

self-adaptation of q in open interval $(0,1)$?

→ make mean step size $E[|Z|]$ adjustable!

$$E[|Z|] = \sum_{k=-\infty}^{\infty} |k| p_k = \theta = \frac{2(1-q)}{q(2-q)} \Leftrightarrow q = 1 - \frac{\theta}{(1+\theta^2)^{1/2} + 1}$$

\downarrow
 $\in \mathbb{R}_+$

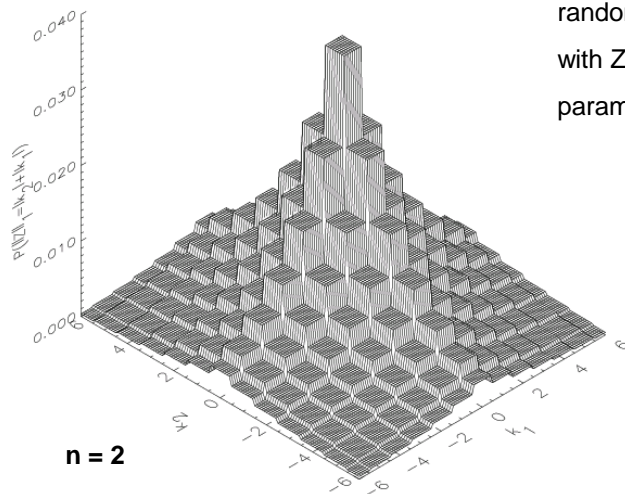
\downarrow
 $\in (0,1)$

→ θ adjustable by mutative self adaptation

→ get q from θ

like mutative step size control of σ in EA with search space \mathbb{R}^n !

n - dimensional generalization



random vector $Z = (Z_1, Z_2, \dots, Z_n)$
 with $Z_i = G_{1,i} - G_{2,i}$ (stoch. indep.);
 parameter q for all G_{1i}, G_{2i} equal

n - dimensional generalization

$$P\{Z_i = k\} = \frac{q}{2-q} (1-q)^{|k|}$$

$$P\{Z_1 = k_1, Z_2 = k_2, \dots, Z_n = k_n\} = \prod_{i=1}^n P\{Z_i = k_i\} =$$

$$\left(\frac{q}{2-q}\right)^n \prod_{i=1}^n (1-q)^{|k_i|} = \left(\frac{q}{2-q}\right)^n (1-q)^{\sum_{i=1}^n |k_i|}$$

$$= \left(\frac{q}{2-q}\right)^n (1-q)^{\|k\|_1}$$

⇒ n-dimensional distribution is symmetric w.r.t. ℓ_1 norm!

⇒ all random vectors with same step length have same probability!

How to control $E[\|Z\|_1]$?

$$E[\|Z\|_1] = E\left[\sum_{i=1}^n |Z_i|\right] \stackrel{\text{linearity of } E[\cdot]}{=} \sum_{i=1}^n E[|Z_i|] \stackrel{\text{identical distributions for } Z_i}{=} n \cdot E[|Z_1|]$$

by def.
linearity of $E[\cdot]$
identical distributions for Z_i

$$\underbrace{n \cdot E[|Z_1|]}_{= \theta} = n \cdot \frac{2(1-q)}{q(2-q)} \Leftrightarrow q = 1 - \frac{\theta/n}{(1 + (\theta/n)^2)^{1/2} + 1}$$

self-adaptation
calculate from θ

Algorithm:

- individual : $(x, \theta) \in \mathbb{Z}^n \times \mathbb{R}_+$
- mutation : $\theta^{(t+1)} = \theta^{(t)} \cdot \exp(N)$, $N \sim N(0, 1/n)$.
if $\theta^{(t+1)} < 1$ then $\theta_{t+1} = 1$
calculate new q for G_i from θ_{t+1}
 $\forall j = 1, \dots, n : X_j^{(t+1)} = X_j^{(t)} + (G_{1,j} - G_{2,j})$
- recombination : discrete (uniform crossover)
- selection : (μ, λ) -selection

(Rudolph, PPSN 1994)

ad 2) design guidelines for variation operators **in practice**

continuous search space $X = \mathbb{R}^n$

- a) reachability
- b) unbiasedness
- c) control

⇒ leads to CMA-ES !