# Computational Intelligence

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- Approximate Reasoning
- Fuzzy Control

**So far:**

- p: IF *X* is A THEN *Y* is B

  → R(x, y) = Imp( A(x), B(y) )

rule as relation; fuzzy implication

- rule: IF *X* is A THEN *Y* is B
  fact: *X* is A'
  conclusion: *Y* is B'

  → $B'(y) = \sup_{x \in X} t(A'(x), R(x, y))$

composition rule of inference

**Thus:**

- $B'(y) = \sup_{x \in X} t(A'(x), \mathrm{Imp}(A(x), B(y)))$

given : fuzzy rule

input : fuzzy set A'

output : fuzzy set B'

**here:**

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

$$B'(y) = \sup_{x \in X} t(\, A'(x), \mathrm{Imp}(\, A(x), B(y)\,)\,)$$

$$= \begin{cases} \sup\limits_{x \neq x_0} t(\, 0, \mathrm{Imp}(\, A(x), B(y)\,)\,) & \text{for } x \neq x_0 \\ t(\, 1, \mathrm{Imp}(\, A(x_0), B(y)\,)\,) & \text{for } x = x_0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x \neq x_0 & \text{since } t(0, a) = 0 \\ \mathrm{Imp}(\, A(x_0), B(y)\,) & \text{for } x = x_0 & \text{since } t(a, 1) = a \end{cases}$$

**Lemma:**

a) t(a, 1) = a

b) t(a, b) ≤ min { a, b }

c) t(0, a) = 0

***Proof:***

by a)

ad a) Identical to axiom 1 of t-norms.

ad b) From monotonicity (axiom 2) follows for b ≤ 1, that t(a, b) ≤ t(a, 1) = a.
Commutativity (axiom 3) and monotonicity lead in case of a ≤ 1 to
t(a, b) = t(b, a) ≤ t(b, 1) = b. Thus, t(a, b) is less than or
equal to a as well as b, which in turn implies t(a, b) ≤ min{ a, b }.

ad c) From b) follows 0 ≤ t(0, a) ≤ min { 0, a } = 0 and therefore t(0, a) = 0. ■

**Multiple rules:**

| | |
|---|---|
| IF X is $A_1$, THEN Y is $B_1$ | $\rightarrow R_1(x, y) = Imp_1(A_1(x), B_1(y))$ |
| IF X is $A_2$, THEN Y is $B_2$ | $\rightarrow R_2(x, y) = Imp_2(A_2(x), B_2(y))$ |
| IF X is $A_3$, THEN Y is $B_3$ | $\rightarrow R_3(x, y) = Imp_3(A_3(x), B_3(y))$ |
| … | … |
| IF X is $A_n$, THEN Y is $B_n$ | $\rightarrow R_n(x, y) = Imp_n(A_n(x), B_n(y))$ |
| X is A‘ | |

---

Y is B‘

**Multiple rules for <u>crisp input</u>:** $x_0$ is given

$B_1'(y) = Imp_1(A_1(x_0), B_1(y))$
…
$B_n'(y) = Imp_n(A_n(x_0), B_n(y))$

} aggregation of rules or local inferences necessary!

**aggregate!** $\Rightarrow$ $B'(y) = \mathbf{aggr}\{ B_1'(y), \ldots, B_n'(y) \}$, where $\mathbf{aggr} = \begin{cases} \min \\ \max \end{cases}$

**FITA**: "**F**irst **i**nference, **t**hen **a**ggregate!"

1. Each rule of the form IF *X* is $A_k$ THEN *Y* is $B_k$ must be transformed by an appropriate fuzzy implication $Imp_k(\cdot,\cdot)$ to a relation $R_k$ :
   $R_k(x, y) = Imp_k( A_k(x), B_k(y) )$.
2. Determine $B_k'(y) = R_k(x, y) \circ A'(x)$ for all k = 1, …, n (local inference).
3. Aggregate to $B'(y) = \beta( B_1'(y), \dots, B_n'(y) )$.

**FATI**: "**F**irst **a**ggregate, **t**hen **i**nference!"

1. Each rule of the form IF *X* ist $A_k$ THEN *Y* ist $B_k$ must be transformed by an appropriate fuzzy implication $Imp_k(\cdot, \cdot)$ to a relation $R_k$ :
   $R_k(x, y) = Imp_k( A_k(x), B_k(y) )$.
2. Aggregate $R_1, \dots, R_n$ to a **superrelation** with aggregating function $\boldsymbol{\alpha(\cdot)}$**:**
   $R(x, y) = \alpha( R_1(x, y), \dots, R_n(x, y) )$.
3. Determine $B'(y) = R(x, y) \circ A'(x)$ w.r.t. superrelation (inference).

**1. Which principle is better? FITA or FATI?**

**2. Equivalence of FITA and FATI ?**

**FITA:** $B'(y) = \beta(\, B_1'(y), \ldots, B_n'(y)\,)$

$= \beta(\, R_1(x, y) \circ A'(x), \ldots, R_n(x, y) \circ A'(x)\,)$

**FATI:** $B'(y) = R(x, y) \circ A'(x)$

$= \alpha(\, R_1(x, y), \ldots, R_n(x, y)\,) \circ A'(x)$

**special case:**

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

**On the equivalence of FITA and FATI:**

**FITA:**
$$B'(y) = \beta(\, B_1'(y), \ldots, B_n'(y) \,)$$
$$= \beta(\, Imp_1(A_1(x_0), B_1(y)\,), \ldots, Imp_n(A_n(x_0), B_n(y)\,)\,)$$

**FATI:**
$$B'(y) = R(x, y) \circ A'(x)$$
$$= \sup_{x \in X} t(\, A'(x), R(x, y)\,)$$
(from now: special case)
$$= R(x_0, y)$$
$$= \alpha(\, Imp_1(\, A_1(x_0), B_1(y)\,), \ldots, Imp_n(\, A_n(x_0), B_n(y)\,)\,)$$

**evidently:** sup-t-composition with arbitrary t-norm and $\boldsymbol{\alpha(\cdot) = \beta(\cdot)}$

- **AND-connected premises**

IF $X_1 = A_{11}$ AND $X_2 = A_{12}$ AND … AND $X_m = A_{1m}$ THEN $Y = B_1$
…
IF $X_n = A_{n1}$ AND $X_2 = A_{n2}$ AND … AND $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$A_k(x_1,\ldots, x_m) = \min \{ A_{k1}(x_1), A_{k2}(x_2), \ldots, A_{km}(x_m) \}$ or in general: t-norm

- **OR-connected premises**

IF $X_1 = A_{11}$ OR $X_2 = A_{12}$ OR … OR $X_m = A_{1m}$ THEN $Y = B_1$
…
IF $X_n = A_{n1}$ OR $X_2 = A_{n2}$ OR … OR $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$A_k(x_1,\ldots, x_m) = \max \{ A_{k1}(x_1), A_{k2}(x_2), \ldots, A_{km}(x_m) \}$ or in general: s-norm

**important:**

- if rules of the form **IF *X* is A THEN *Y* is B** interpreted as <u>logical</u> implication

  ⇒ R(x, y) = Imp( A(x), B(y) ) makes sense

- we obtain: B'(y) = $\sup_{x \in X}$ t( A'(x), R(x, y) )

⇒ the worse the match of premise A'(x), the larger is the fuzzy set B'(y)

⇒ follows immediately from axiom 1: a ≤ b implies Imp(a, z) ≥ Imp(b, z)

**interpretation of output set B'(y):**

- B'(y) is the set of values that are still possible
- each rule leads to an additional restriction of the values that are still possible

⇒ resulting fuzzy sets $B'_k(y)$ obtained from single rules must be mutually <u>intersected</u>!

⇒ aggregation via B'(y) = **min** { $B_1'(y)$, …, $B_n'(y)$ }

**important:**

- if rules of the form **IF *X* is A THEN *Y* is B** are not interpreted as logical implications, then the function Fct(·) in

$$R(x, y) = Fct( A(x), B(y) )$$

can be chosen as required for desired interpretation.

- frequent choice (especially in fuzzy control):

  - $R(x, y) = \min \{ A(x), B(x) \}$ Mamdani – “implication“
  - $R(x, y) = A(x) \cdot B(x)$ Larsen – “implication“

⇒ of course, they are no implications but specific t-norms!

⇒ thus, if relation R(x, y) is given,
then the *composition rule of inference*

$$B'(y) = A'(x) \circ R(x, y) = \sup_{x \in X} \min \{ A'(x), R(x, y) \}$$

still can lead to a conclusion via fuzzy logic.

**example:** [JM96, S. 244ff.]

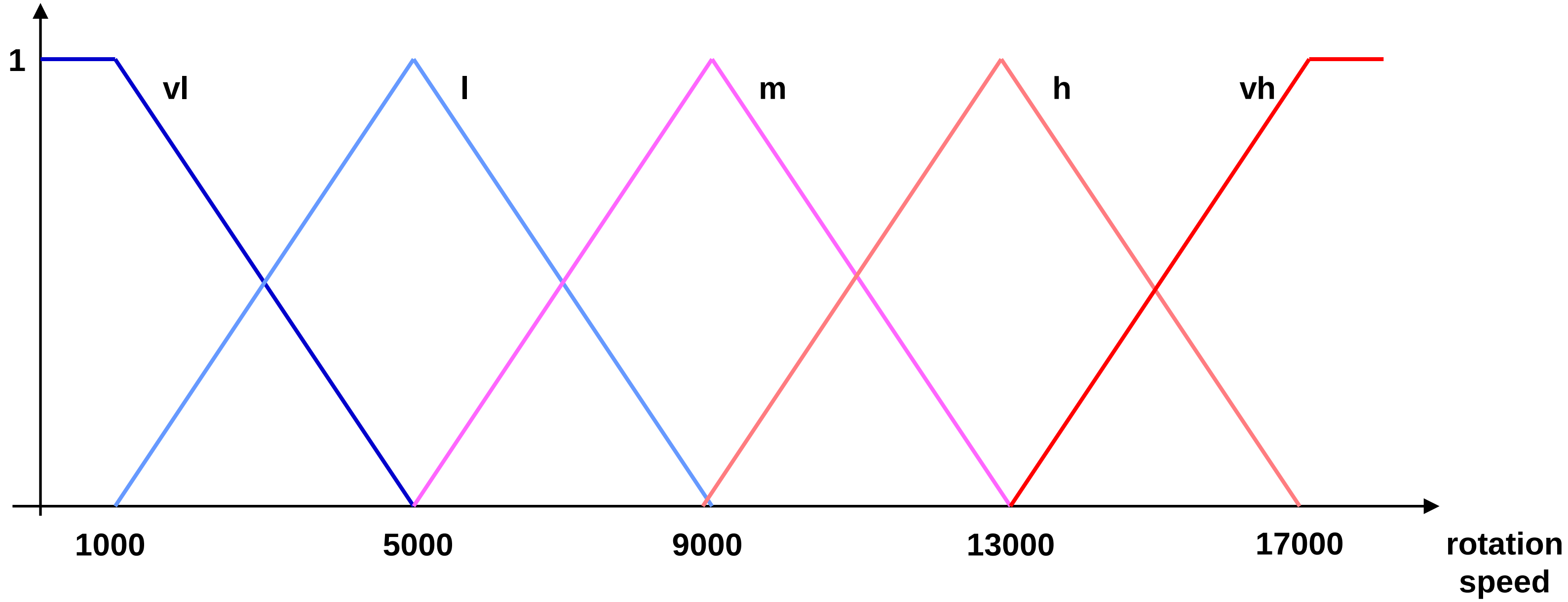
industrial drill machine → control of cooling supply

modelling

linguistic variable : **rotation speed**

linguistic terms : ***very low***, ***low***, ***medium***, ***high***, ***very high***

ground set : X with $0 \le x \le 18000$ [rpm]



**example:** (continued)

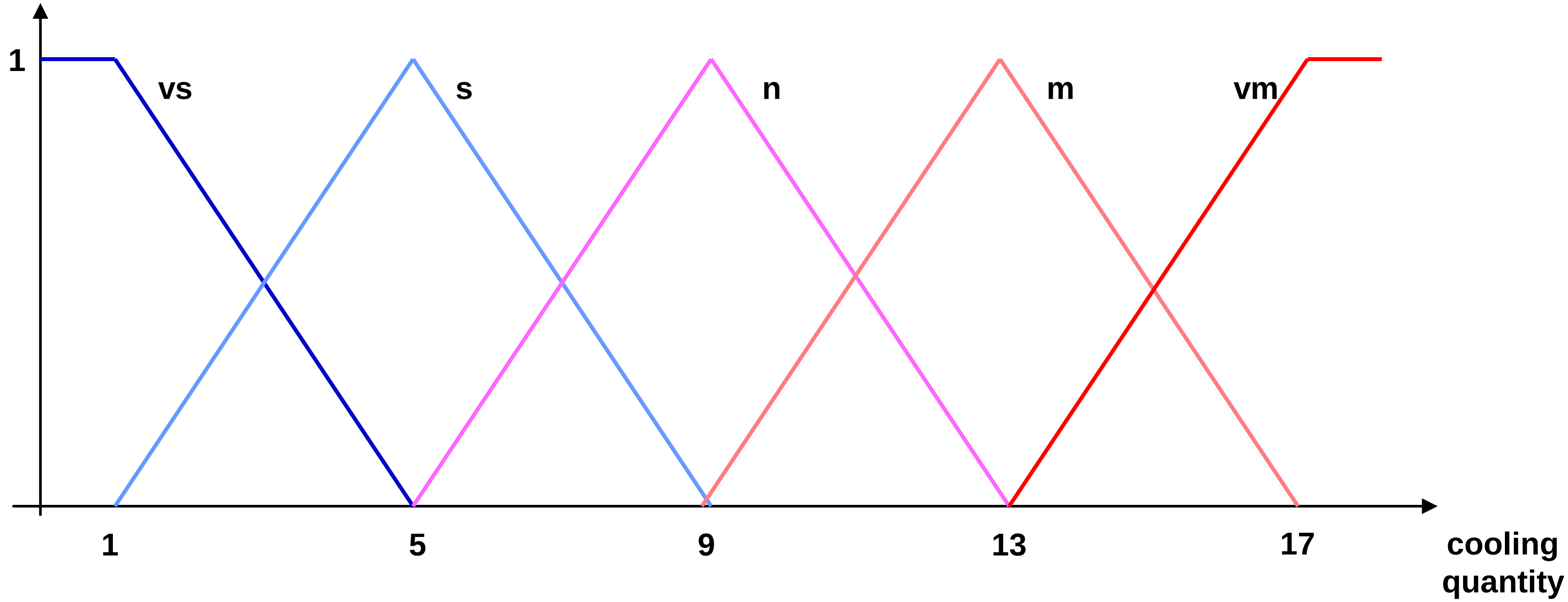
industrial drill machine → control of cooling supply

modelling

linguistic variable : **cooling quantity**

linguistic terms : ***very small***, ***small***, ***normal***, ***much***, ***very much***

ground set : Y with $0 \le y \le 18$ [liter / time unit]



**example:** (continued)

industrial drill machine → control of cooling supply

rule base

| | | | |
|---|---|---|---|
| **IF rotation speed IS** | ***very low*** | **THEN cooling quantity IS** | ***very small*** |
| | ***low*** | | ***small*** |
| | ***medium*** | | ***normal*** |
| | ***high*** | | ***much*** |
| | ***very high*** | | ***very much*** |

↑ sets $S_{vl}$, $S_l$, $S_m$, $S_h$, $S_{vh}$ "rotation speed"

↑ sets $C_{vs}$, $C_s$, $C_n$, $C_m$, $C_{vm}$ "cooling quantity"

**example:** (continued)

industrial drill machine → control of cooling supply

1. **input**: crisp value $x_0 = 10000 \text{ min}^{-1}$ (not a fuzzy set!)

   → **fuzzyfication** = determine membership for each fuzzy set over X

   → yields S' = (0, 0, ¾, ¼, 0) via x $\alpha$ ( $S_{vl}(x_0)$, $S_l(x_0)$, $S_m(x_0)$, $S_h(x_0)$, $S_{vh}(x_0)$ )

2. FITA: locale **inference** ⇒ since Imp(0,a) = 0 we only need to consider:

   $S_m$: $C'_n(y) = \text{Imp}(\,¾, C_n(y)\,)$

   $S_h$: $C'_m(y) = \text{Imp}(\,¼, C_m(y)\,)$

3. **aggregation**:

   ?

   $C'(y) =$ **aggr** $\{ C'_n(y), C'_m(y) \} = \max\{ \text{Imp}(\,¾, C_n(y)\,), \text{Imp}(\,¼, C_m(y)\,) \}$

**example:** (continued)

industrial drill machine → control of cooling supply

$C'(y) = \max \{ \mathrm{Imp}( \frac{3}{4}, C_n(y) ), \mathrm{Imp}( \frac{1}{4}, C_m(y) ) \}$

in case of control task typically no logic-based interpretation:

→ max-aggregation and

→ relation R(x,y) not interpreted as implication.

often: R(x,y) = min(a, b) „Mamdani controller“

**thus:**

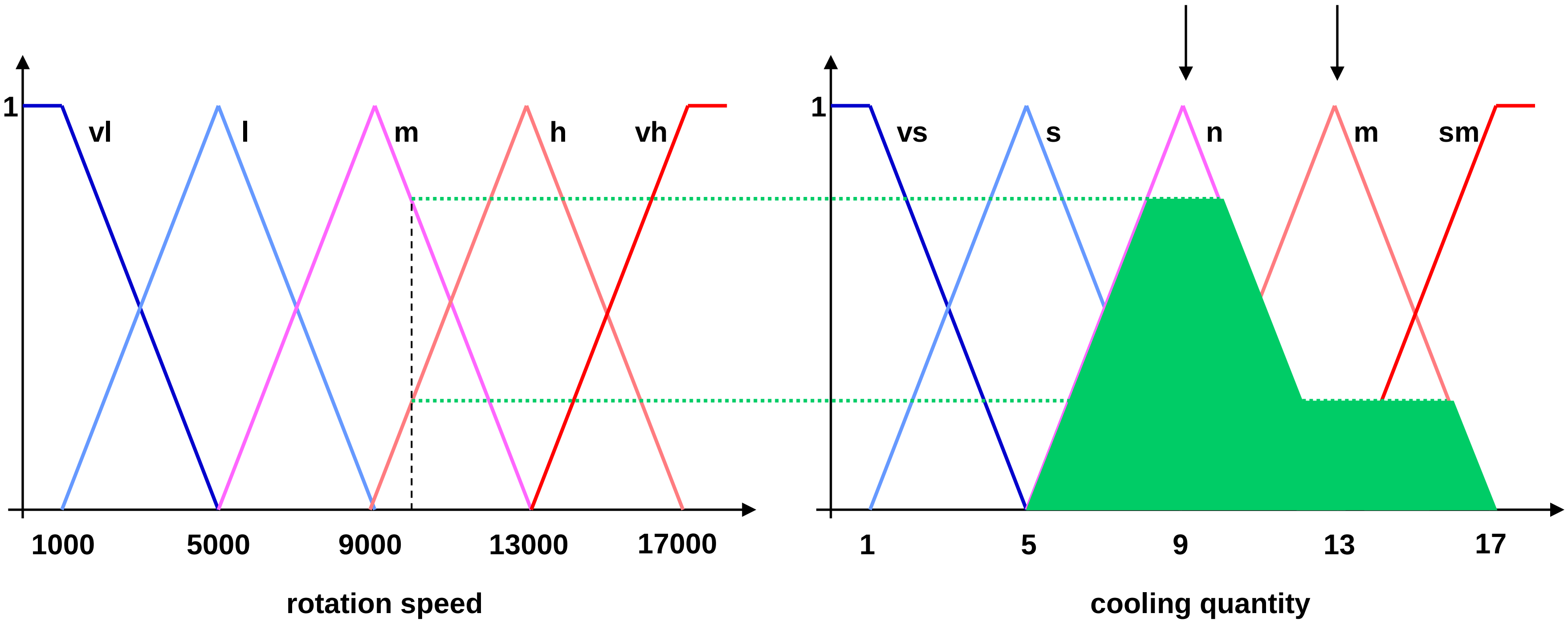
$C'(y) = \max \{ \min \{ \frac{3}{4}, C_n(y) \}, \min \{ \frac{1}{4}, C_m(y) \} \}$

→ graphical illustration

**example:** (continued)

industial drill machine → control of cooling supply

$C'(y) = \max \{ \min \{ \frac{3}{4}, C_n(y) \}, \min \{ \frac{1}{4}, C_m(y) \} \}, x_0 = 10000$ [rpm]



## open and closed loop control:

affect the dynamical behavior of a system
in a desired manner

- **open loop control**

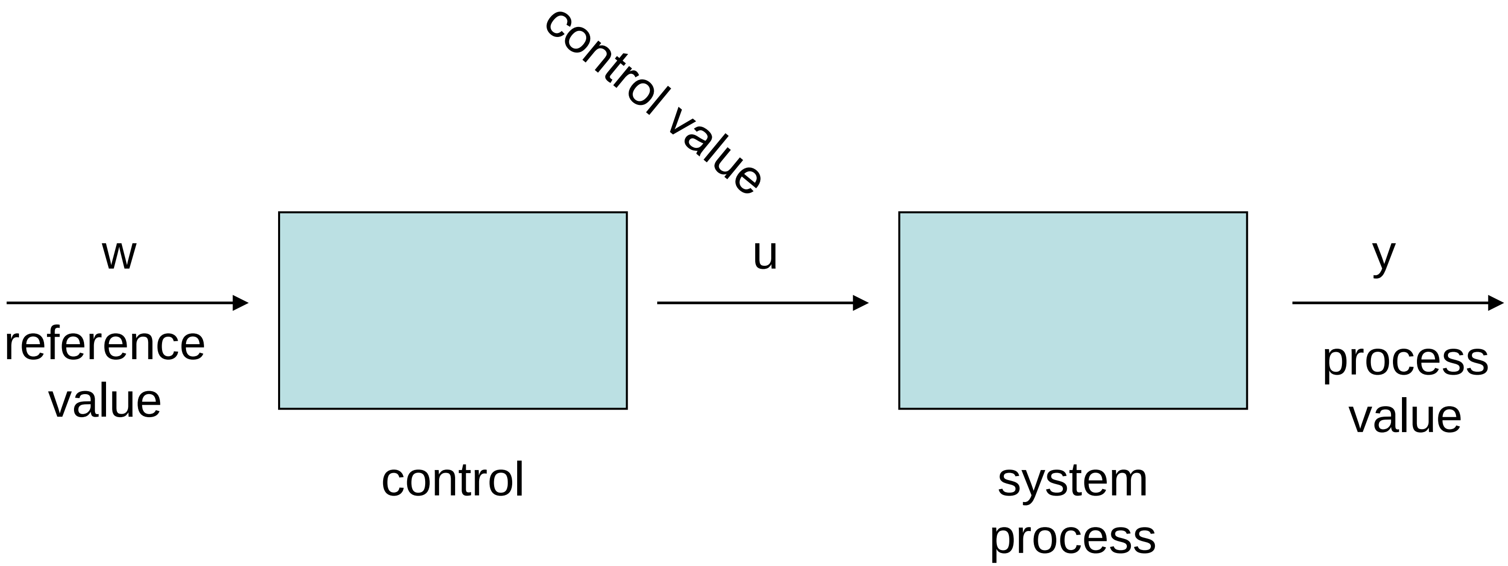
control is aware of reference values and has a model of the system
$\Rightarrow$ control values can be adjusted,
such that process value of system is equal to reference value

problem: noise! $\Rightarrow$ deviation from reference value not detected

- **closed loop control**

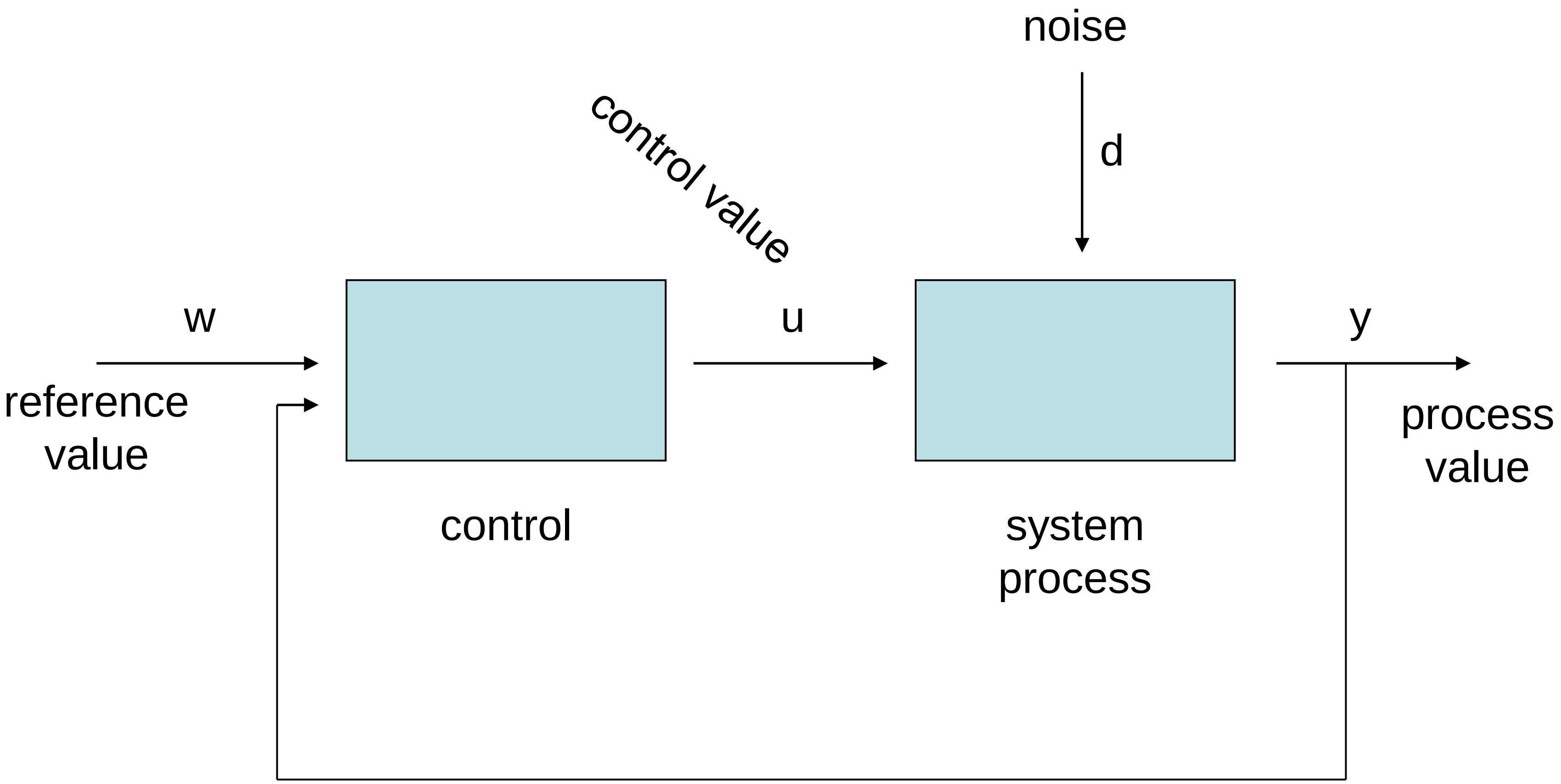
now: detection of deviations from reference value possible
(by means of measurements / sensors)
and new control values can take into account the amount of deviation

## open loop control




assumption: undisturbed operation $\Rightarrow$ process value = reference value

## closed loop control




control deviation = reference value – process value

## required:

model of system / process

→ as differential equations or difference equations (DEs)

→ well developed theory available

## so, why fuzzy control?

- there exists no process model in form of DEs etc.
  (operator/human being has realized control by hand)
- process with high-dimensional nonlinearities → no classic methods available
- control goals are vaguely formulated („soft“ changing gears in cars)

**fuzzy description of control behavior**

IF X is $A_1$, THEN Y is $B_1$
IF X is $A_2$, THEN Y is $B_2$
IF X is $A_3$, THEN Y is $B_3$
…
IF X is $A_n$, THEN Y is $B_n$
X is A‘

---

Y is B‘

} similar to approximative reasoning

but fact A‘ is not a fuzzy set but a crisp input

→ actually, it is the current process value

fuzzy controller executes inference step

→ yields fuzzy output set B‘(y)

but crisp control value required for the process / system

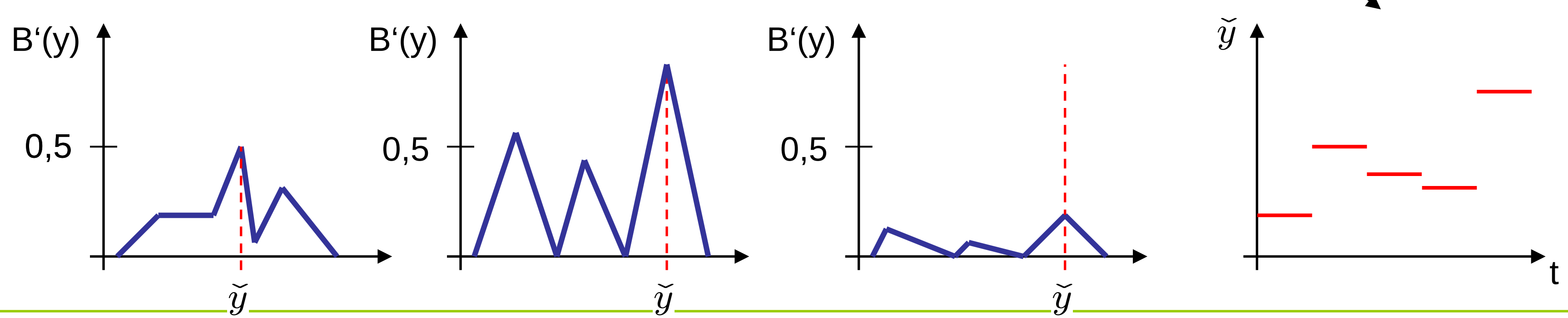
→ defuzzification (= “condense” fuzzy set to crisp value)

## defuzzification

**Def**: rule k active $\Leftrightarrow A_k(x_0) > 0$

- maximum method
  - only active rule with largest activation level is taken into account
    - → suitable for pattern recognition / classification
    - → decision for a single alternative among finitely many alternatives
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - if used for control: incontinuous curve of output values (leaps)

$$\breve{y} = \text{argmax } B'(y)$$

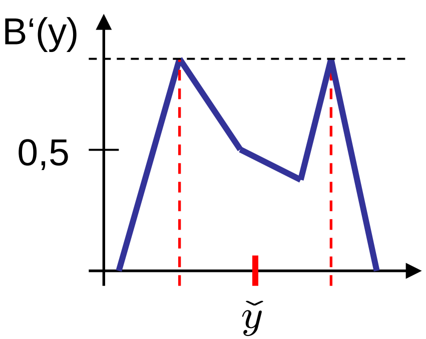


## defuzzification

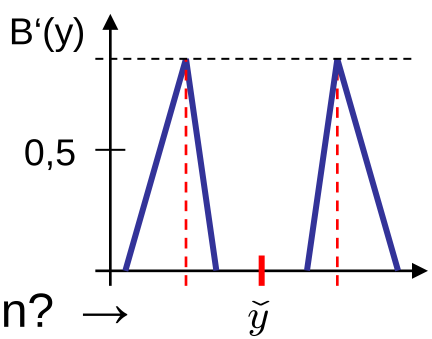
$Y^* = \{ y \in Y: B'(y) = hgt(B') \}$

- <u>maximum mean value method</u>
  - all active rules with largest activation level are taken into account
    - → interpolations possible, but need not be useful
    - → obviously, only useful for neighboring rules with max. activation
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - if used in control: incontinuous curve of output values (leaps)

$$\breve{y} = \frac{1}{|Y^*|} \sum_{y^* \in Y^*} y^*$$




useful solution? →



## defuzzification

$Y^* = \{ y \in Y: B'(y) = hgt(B') \}$

- <u>center-of-maxima method (COM)</u>

  - only **extreme** active rules with largest activation level are taken into account

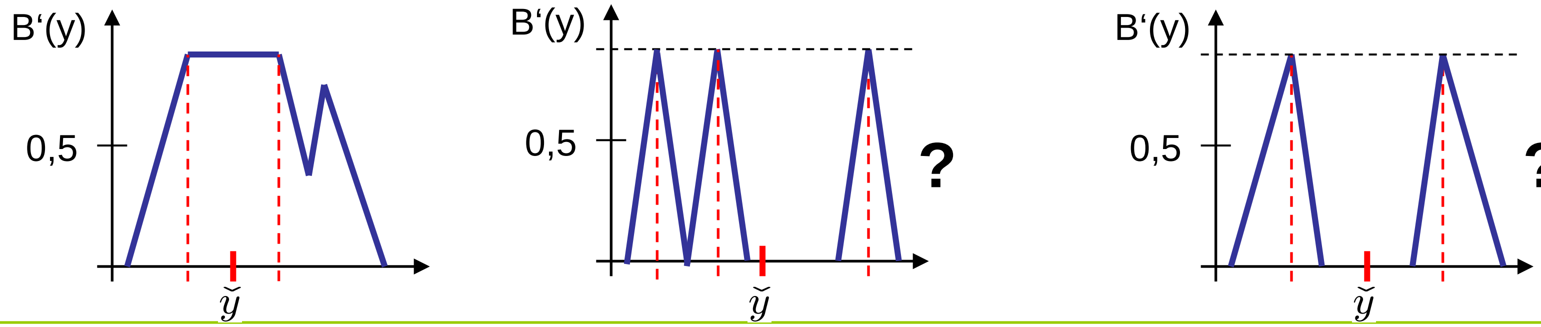
    → interpolations possible, but need not be useful

    → obviously, only useful for neighboring rules with max. activation level

  - selection indepependent from activation level of rule (0.05 vs. 0.95)

  - in case of control: incontinuous curve of output values (leaps)

$$\check{y} = \frac{\inf Y^* + \sup Y^*}{2}$$



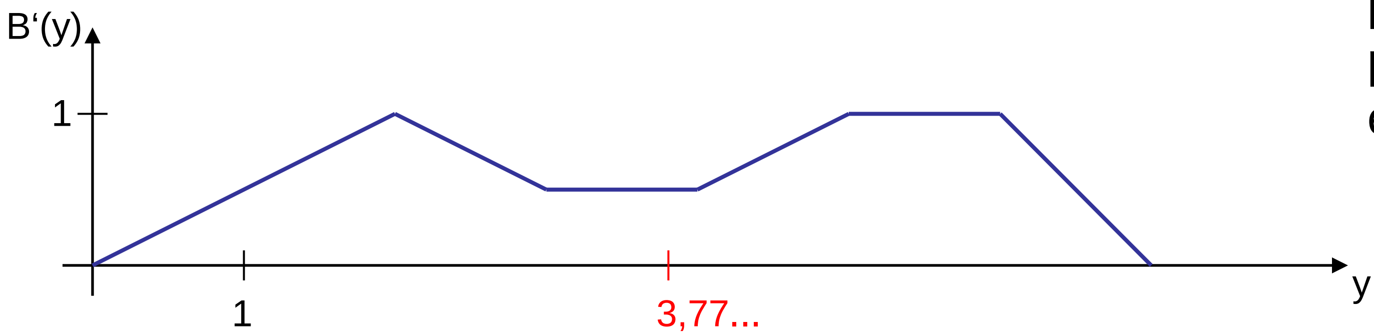
## defuzzification

- Center of Gravity (COG)
  - all active rules are taken into account
    - → but numerically expensive … …only valid for HW solution, today!
    - → borders cannot appear in output ( ∃ work-around )
  - if only single active rule: independent from activation level
  - continuous curve for output values

$$\check{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$

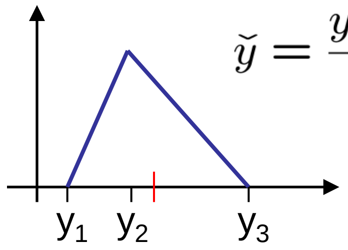
## Excursion: COG

$$\breve{y} = \frac{\int y \cdot B'(y)\,dy}{\int B'(y)\,dy}$$



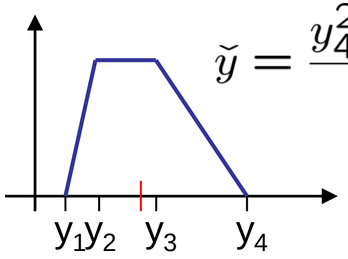

pendant in probability theory: expectation value

triangle:

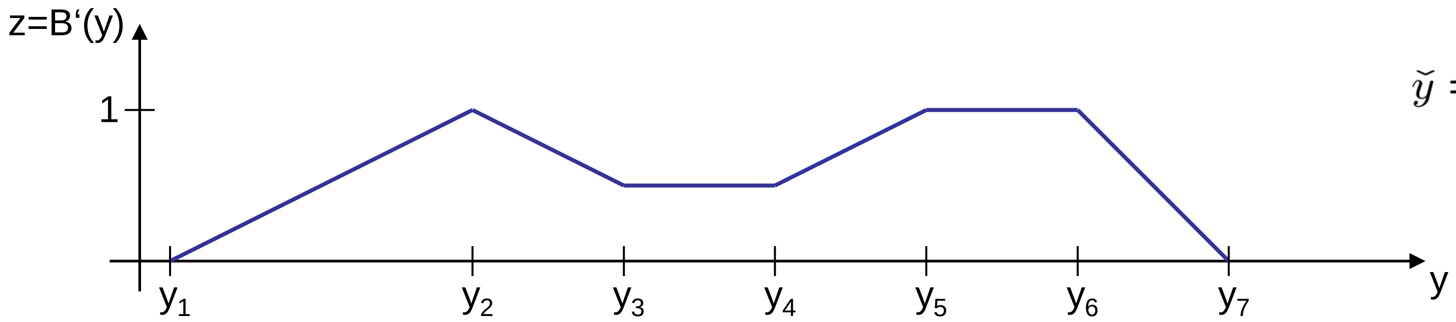



$$\breve{y} = \frac{y_1 + y_2 + y_3}{3}$$

trapezoid:




$$\breve{y} = \frac{y_4^2 + y_3^2 - y_2^2 - y_1^2 + y_3 y_4 - y_1 y_2}{3\,(y_4 + y_3 - y_2 - y_1)}$$




$$\breve{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$

assumption: fuzzy membership functions piecewise linear

output set B'(y) represented by sequence of points $(y_1, z_1), (y_2, z_2), \ldots, (y_n, z_n)$

⇒ area under B'(y) and weighted area can be determined additively piece by piece

⇒ linear equation z = m y + b ⇒ insert $(y_i, z_i)$ and $(y_{i+1}, z_{i+1})$

⇒ yields m and b for each of the n-1 linear sections

$$\Rightarrow F_i = \int_{y_i}^{y_{i+1}} (m\, y + b)\, dy = \frac{m}{2}(y_{i+1}^2 - y_i^2) + b(y_{i+1} - y_i)$$

$$\Rightarrow G_i = \int_{y_i}^{y_{i+1}} y\,(m\, y + b)\, dy = \frac{m}{3}(y_{i+1}^3 - y_i^3) + \frac{b}{2}(y_{i+1}^2 - y_i^2)$$

$$\breve{y} = \frac{\sum_i G_i}{\sum_i F_i}$$

## Defuzzification

- Center of Area (COA)
  - developed as an approximation of COG
  - let $\hat{y}_k$ be the COGs of the output sets $B'_k(y)$:

$$\check{y} = \frac{\sum_k A_k(x_0) \cdot \widehat{y}_k}{\sum_k A_k(x_0)}$$

**how to:**
assume that fuzzy sets $A_k(x)$ and $B_k(x)$ are triangles or trapezoids
let $x_0$ be the crisp input value
for each fuzzy rule "IF $A_k$ is X THEN $B_k$ is Y"
  determine $B'_k(y) = R(A_k(x_0), B_k(y))$, where R(.,.) is the relation
  find $\hat{y}_k$ as center of gravity of $B'_k(y)$