Computational Intelligence

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- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Fuzzy Complement: Axioms

Definition

A function c: $[0,1] \to [0,1]$ is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall\, a, b \in [0,1]: a \le b \Rightarrow c(a) \ge c(b)$. — monotone decreasing

“nice to have”:

(A3) $c(\cdot)$ is continuous.

(A4) $\forall\, a \in [0,1]: c(c(a)) = a$ — involutive

Examples:

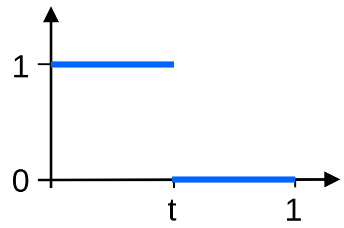
a) standard fuzzy complement $c(a) = 1 - a$

ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$
ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)

ad (A3): ☑
ad (A4): $1 - (1 - a) = a$

Fuzzy Complement: Examples

b) $c(a) = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$



ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t

☑

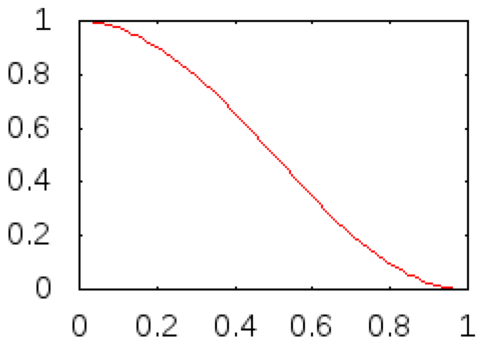
ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(¼)) = c(1) = 0 \neq ¼$ for $t = ½$

Fuzzy Complement: Examples

c) $c(a) = \dfrac{1 + \cos(\pi\, a)}{2}$



ad (A1): c(0) = 1 and c(1) = 0

ad (A2): c'(a) = –½ π sin(π a) < 0 since sin(π a) > 0 for a ∈ (0,1)

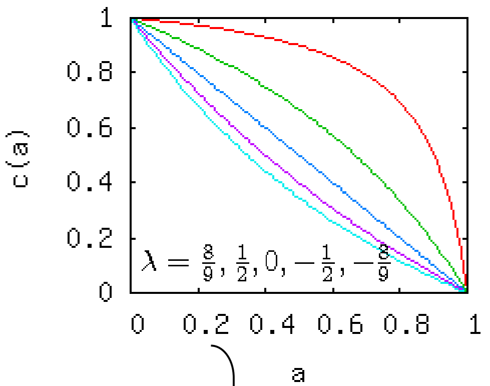
☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$$c\left(c\left(\tfrac{1}{3}\right)\right) = c\left(\tfrac{3}{4}\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) \neq \tfrac{1}{3}$$

d) c(a) = $\frac{1-a}{1+\lambda a}$ for $\lambda > -1$ **Sugeno class**



ad (A1): c(0) = 1 and c(1) = 0

ad (A2): $c(a) \geq c(b) \;\Leftrightarrow\; \frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b} \;\Leftrightarrow$

$$(1-a)\,(1+\lambda b) \geq (1-b)\,(1+\lambda a) \;\Leftrightarrow$$

$$b\,(\lambda+1) \geq a\,(\lambda+1) \;\Leftrightarrow\; b \geq a$$

☑

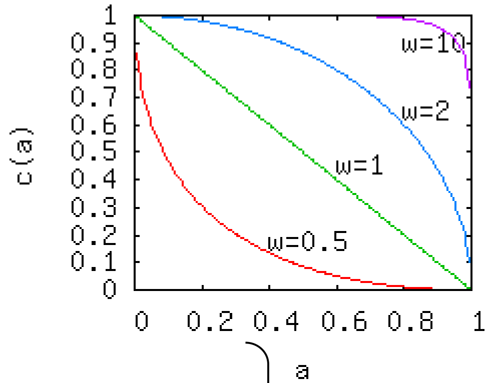
ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) \;=\; c\left(\frac{1-a}{1+\lambda a}\right) \;=\; \frac{1-\frac{1-a}{1+\lambda a}}{1+\lambda\frac{1-a}{1+\lambda a}} \;=\; \frac{a\,(\lambda+1)}{\lambda+1} \;=\; a$

☑

Fuzzy Complement: Examples

e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$

$a^w \leq b^w \Leftrightarrow a \leq b$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $$c(c(a)) = c\left((1 - a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1 - a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$$

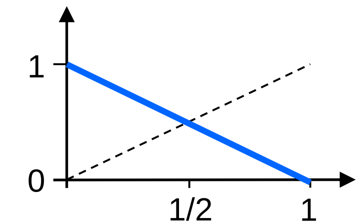
$$= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$$ ☑

Theorem

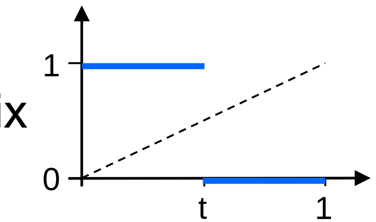
If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix



no fixed point → see example (b) → no intersection with bisectrix



assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b* (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2) ■

Theorem

If function $c:[0,1] \to [0,1]$ satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a^* with $c(a^*) = a^*$.

Proof:

Intermediate value theorem →

If $c(\cdot)$ continuous (A3) and $c(0) \geq c(1)$ (A1/A2)

then $\forall\, v \in [c(1), c(0)] = [0,1]: \exists\, a \in [0,1]: c(a) = v$.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points! ■

Examples:

(a) $c(a) = 1 - a$ ⇒ $a = 1 - a$ ⇒ $a^* = ½$

(b) $c(a) = (1 - a^w)^{1/w}$ ⇒ $a = (1 - a^w)^{1/w}$ ⇒ $a^* = (½)^{1/w}$

Theorem

c: [0,1] $\rightarrow$ [0,1] is involutive fuzzy complement iff

$\exists$continuous function g: [0,1] $\rightarrow$ $\mathbb{R}$ with

- g(0) = 0
- strictly monotone increasing
- $\forall$ a $\in$ [0,1]: c(a) = $g^{(-1)}$(g(1) – g(a)). ■

defines an **increasing generator**

$g^{(-1)}(x)$ pseudo-inverse

Examples

a) g(x) = x $\Rightarrow$ $g^{-1}(x) = x$ $\Rightarrow$ c(a) = 1 – a (Standard)

b) $g(x) = x^w$ $\Rightarrow$ $g^{-1}(x) = x^{1/w}$ $\Rightarrow$ $c(a) = (1 - a^w)^{1/w}$ (Yager class, w > 0)

c) g(x) = log(x+1) $\Rightarrow$ $g^{-1}(x) = e^x - 1$ $\Rightarrow$ c(a) = exp(log(2) – log(a+1)) – 1

$= \dfrac{1-a}{1+a}$ (Sugeno class. λ = 1)

Examples

d) $g(a) = \frac{1}{\lambda} \log_e(1 + \lambda\, a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda\, a} > 0$ for $a \in [0,1]$
- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda\, a) - 1}{\lambda}$, thus

$$
\begin{aligned}
c(a) &= g^{-1}\left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda\, a)}{\lambda}\right) \\
&= \frac{\exp(\log(1+\lambda) - \log(1+\lambda\, a)) - 1}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda\, a} - 1\right) = \frac{1-a}{1+\lambda\, a} \quad \text{(Sugeno Complement)}
\end{aligned}
$$

Theorem

c: $[0,1] \to [0,1]$ is involutive fuzzy complement iff

$\exists$continuous function f: $[0,1] \to \mathbb{R}$ with

- $f(1) = 0$
- strictly monotone decreasing
- $\forall\, a \in [0,1]$: $c(a) = f^{(-1)}(\, f(0) - f(a)\,)$. ■

defines a **decreasing generator**

$f^{(-1)}(x)$ pseudo-inverse

Examples

a) $f(x) = k - k \cdot x \;\; (k > 0)$ $\quad f^{(-1)}(x) = 1 - x/k$ $\quad c(a) = 1 - \frac{k-(k-ka)}{k} = 1 - a$

b) $f(x) = 1 - x^w$ $\quad f^{(-1)}(x) = (1 - x)^{1/w}$ $\quad c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w}$ (Yager)

Fuzzy Intersection: t-norm

Definition

A function $t:[0,1] \times [0,1] \rightarrow [0,1]$ is a ***fuzzy intersection*** or ***t-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $t(a, 1) = a$ (boundary condition)

(A2) $b \leq d \Rightarrow t(a, b) \leq t(a, d)$ (monotonicity)

(A3) $t(a,b) = t(b, a)$ (commutative)

(A4) $t(a, t(b, d)) = t(t(a, b), d)$ (associative) ■

“nice to have”

(A5) $t(a, b)$ is continuous (continuity)

(A6) $t(a, a) < a$ for $0 < a < 1$ (subidempotent)

(A7) $a_1 < a_2$ and $b_1 \leq b_2 \Rightarrow t(a_1, b_1) < t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Theorem:

The only idempotent t-norm is the standard fuzzy intersection.

Proof:

Assume there exists a t-norm with $t(a,a) = a$ for all $a \in [0,1]$.

- If $0 \leq a \leq b \leq 1$ then

$$a \underbrace{=}_{\text{by assumption}} t(a,a) \underbrace{\leq}_{\text{by monotonicity}} t(a,b) \underbrace{\leq}_{\text{by monotonicity}} t(a,1) \underbrace{=}_{\text{by boundary condition}} a$$

  and hence $t(a,b) = a$.

- If $0 \leq b \leq a \leq 1$ then

$$b \underbrace{=}_{\text{by assumption}} t(b,b) \underbrace{\leq}_{\text{by monotonicity}} t(b,a) \underbrace{\leq}_{\text{by monotonicity}} t(b,1) \underbrace{=}_{\text{by boundary condition}} b$$

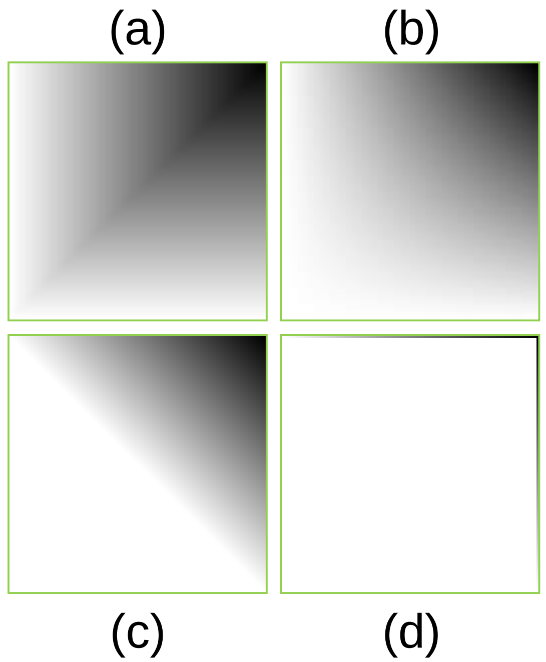
  and hence $t(a,b) \underbrace{=}_{\text{by commutativity}} t(b,a) = b$.

$t(a,b) = \min(a,b)$ is the only possible solution!

q.e.d.

Examples:

Name	Function
(a) Standard	$t(a, b) = \min \{ a, b \}$
(b) Algebraic Product	$t(a, b) = a \cdot b$
(c) Bounded Difference	$t(a, b) = \max \{ 0, a + b - 1 \}$
(d) Drastic Product	$t(a, b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$



Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): $t(a, 1) = a \cdot 1 = a$ ☑

ad (A2): $a \cdot b \le a \cdot d \Leftrightarrow b \le d$ ☑

ad (A3): $t(a, b) = a \cdot b = b \cdot a = t(b, a)$ ☑

ad (A4): $a \cdot (b \cdot d) = (a \cdot b) \cdot d$ ☑

Theorem

Function t: [0,1] x [0,1] → [0,1] is a t-norm ,

∃decreasing generator f:[0,1] → $\mathbb{R}$ with t(a, b) = f^{-1}(min{ f(0), f(a) + f(b) }). ■

Example:

f(x) = 1/x – 1 is decreasing generator since

- f(x) is continuous ☑
- f(1) = 1/1 – 1 = 0 ☑
- f'(x) = $-1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \frac{1}{x+1}$; f(0) = ∞ ⇒ min{ f(0), f(a) + f(b) } = f(a) + f(b)

$$\Rightarrow t(a,b) = f^{-1}\left(\frac{1}{a}+\frac{1}{b}-2\right) = \frac{1}{\frac{1}{a}+\frac{1}{b}-1} = \frac{ab}{a+b-a\,b}$$

Definition

A function $s:[0,1] \times [0,1] \to [0,1]$ is a ***fuzzy union*** or ***s-norm*** iff $\forall a,b,d \in [0,1]$

(A1) $s(a, 0) = a$ (boundary condition)

(A2) $b \le d \Rightarrow s(a, b) \le s(a, d)$ (monotonicity)

(A3) $s(a, b) = s(b, a)$ (commutative)

(A4) $s(a, s(b, d)) = s(s(a, b), d)$ (associative) ■

“nice to have”

(A5) $s(a, b)$ is continuous (continuity)

(A6) $s(a, a) > a$ for $0 < a < 1$ (superidempotent)

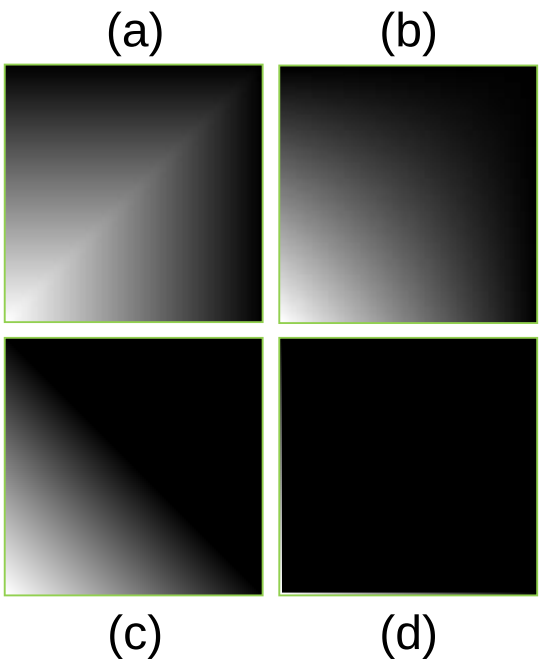
(A7) $a_1 < a_2$ and $b_1 \le b_2 \Rightarrow s(a_1, b_1) < s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Fuzzy Union: s-norm

Examples:

Name	Function
Standard	$s(a, b) = \max \{ a, b \}$
Algebraic Sum	$s(a, b) = a + b - a \cdot b$
Bounded Sum	$s(a, b) = \min \{ 1, a + b \}$
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$



Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): $s(a, 0) = a + 0 - a \cdot 0 = a$ ☑

ad (A2): $a + b - a \cdot b \le a + d - a \cdot d \Leftrightarrow b\,(1 - a) \le d\,(1 - a) \Leftrightarrow b \le d$ ☑

ad (A3): ☑

ad (A4): ☑

Theorem

Function s: [0,1] x [0,1] → [0,1] is a s-norm ⇔

∃increasing generator g:[0,1] → $\mathbb{R}$ with s(a, b) = g^{-1}(min{ g(1), g(a) + g(b) }). ■

Example:

g(x) = –log(1 – x) is increasing generator since

- g(x) is continuous ☑
- g(0) = –log(1 – 0) = 0 ☑
- g‘(x) = 1/(1 – x) > 0 (monotone increasing) ☑

inverse function is $g^{-1}(x)$ = 1 – exp(–x) ; g(1) = ∞ ⇒ min{g(1), g(a) + g(b)} = g(a) + g(b)

$$
\Rightarrow s(a, b) = g^{-1}(-\log(1-a) - \log(1-b))
$$

$$
= 1 - \exp(\log(1-a) + \log(1-b))
$$

$$
= 1 - (1-a)(1-b) = a + b - a\,b \quad \text{(algebraic sum)}
$$

Background from classical set theory:

$\cap$ and $\cup$ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition

A pair of t-norm $t(\cdot, \cdot)$ and s-norm $s(\cdot, \cdot)$ is said to be ***dual with regard to the fuzzy complement*** $c(\cdot)$ iff

- $c(\, t(a, b)\,) \;=\; s(\, c(a), c(b)\,)$
- $c(\, s(a, b)\,) = \; t(\, c(a), c(b)\,)$

for all $a, b \in [0,1]$. ■

Definition

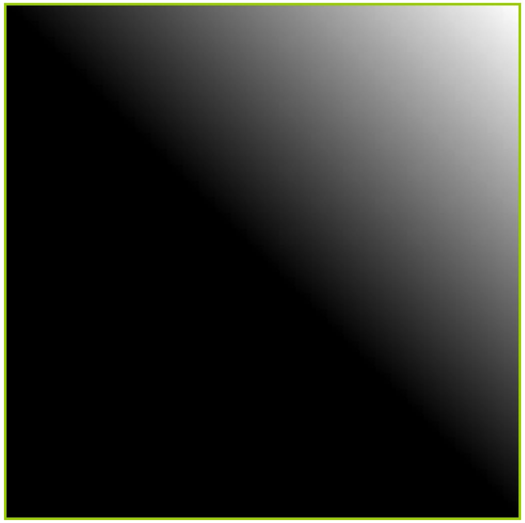
Let (c, s, t) be a tripel of fuzzy complement $c(\cdot)$, s- and t-norm.

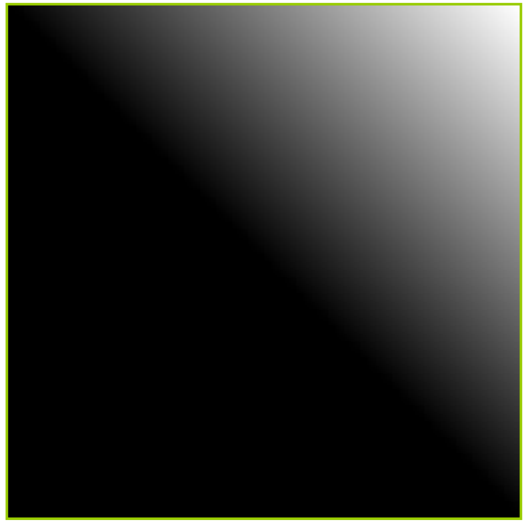
If t and s are dual to c then the tripel (c,s, t) is called a ***dual tripel***. ■

Examples of dual tripels

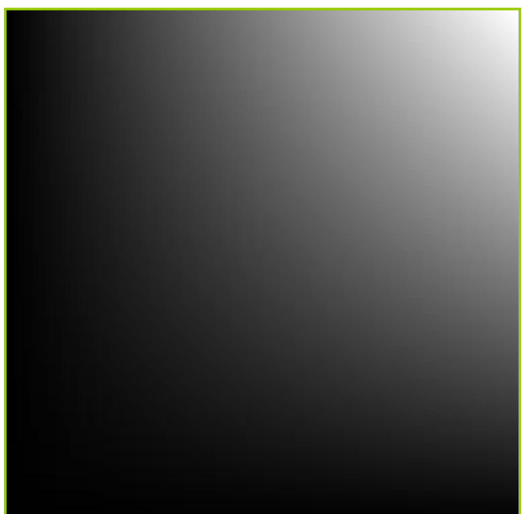
t-norm	s-norm	complement
$\min \{ a, b \}$	$\max \{ a, b \}$	$1 - a$
$a \cdot b$	$a + b - a \cdot b$	$1 - a$
$\max \{ 0, a + b - 1 \}$	$\min \{ 1, a + b \}$	$1 - a$

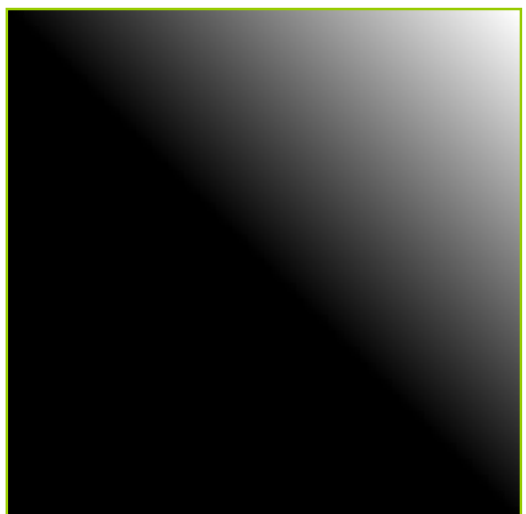
Dual Triples vs. Non-Dual Triples





c(t(a, b))   s(c(a), c(b))





Dual Triple:

- bounded difference
- bounded sum
- standard complement

⇒ left image = right image

Non-Dual Triple:

- algebraic product
- bounded sum
- standard complement

⇒ left image ≠ right image

Why are dual triples so important?

⇒ allow equivalence transformations of fuzzy set expressions

⇒ required to transform into some equivalent normal form (standardized input)

⇒ e.g. two stages: intersection of unions $$\bigcap_{i=1}^{n} (A_i \cup B_i)$$

or union of intersections $$\bigcup_{i=1}^{n} (A_i \cap B_i)$$

Example:

$A \cup (B \cap (C \cap D)^c) \ =$	← not in normal form
$A \cup (B \cap (C^c \cup D^c)) \ =$	← equivalent if DeMorgan's law valid (dual triples!)
$A \cup (B \cap C^c) \cup (B \cap D^c)$	← equivalent (distributive lattice!)