Computational Intelligence

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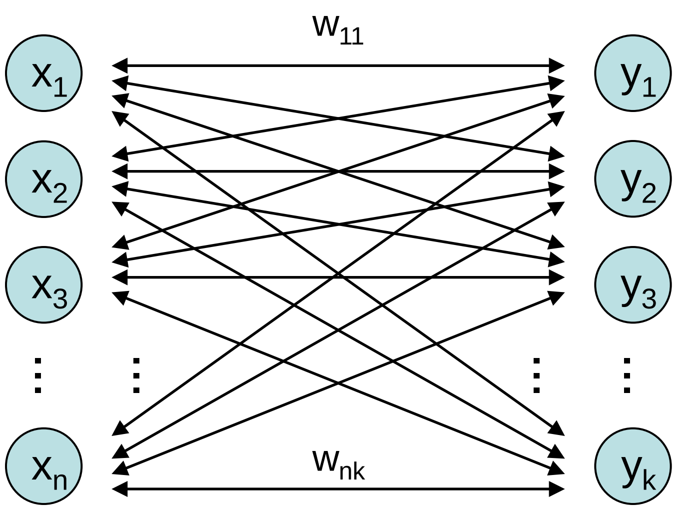
Plan for Today

- Bidirectional Associative Memory (BAM)
  - Fixed Points
  - Concept of Energy Function
  - Stable States = Minimizers of Energy Function
- Hopfield Network
  - Convergence
  - Application to Combinatorial Optimization

Network Model



- fully connected
- bidirectional edges
- synchonized:

step t : data flow from x to y
step t + 1 : data flow from y to x

x, y : row vectors

W : weight matrix

W' : transpose of W

bipolar inputs $\in \{-1,+1\}$

start: $y^{(0)} = \text{sgn}(x^{(0)} W)$

$x^{(1)} = \text{sgn}(y^{(0)} W')$

$y^{(1)} = \text{sgn}(x^{(1)} W)$

$x^{(2)} = \text{sgn}(y^{(1)} W')$

...

Fixed Points

Definition

(x, y) is ***fixed point*** of BAM iff $y = \text{sgn}(x\,W)$ and $x' = \text{sgn}(W\,y')$. □

Set $W = x'\,y$. (note: x is row vector)

$$y = \text{sgn}(\,x\,W\,) = \text{sgn}(\,x\,(x'\,y)\,) = \text{sgn}(\,(x\,x')\,y) = \text{sgn}(\,\underbrace{\|\,x\,\|^2}_{>0 \text{ (does not alter sign)}}\,y) = y$$

$$x' = \text{sgn}(\,W\,y') = \text{sgn}(\,(x'y)\,y'\,) = \text{sgn}(\,x'\,(y\,y')\,) = \text{sgn}(\,x'\,\underbrace{\|\,y\,\|^2}_{>0 \text{ (does not alter sign)}}\,) = x'$$

Theorem: If $W = x'y$ then (x,y) is fixed point of BAM. □

Concept of Energy Function

<u>given:</u> BAM with $W = x'y$ $\Rightarrow$ (x,y) is stable state of BAM

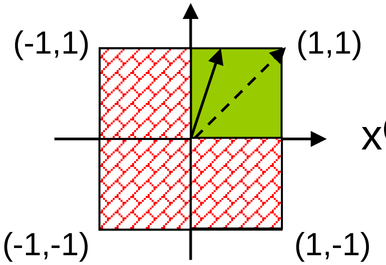
starting point $x^{(0)}$ $\Rightarrow$ $y^{(0)} = \text{sgn}(\, x^{(0)} W \,)$

$\Rightarrow$ excitation $e' = W\,(y^{(0)})'$

$\Rightarrow$ if $\text{sign}(\, e' \,) = x^{(0)}$ then $(\, x^{(0)} , y^{(0)} \,)$ stable state

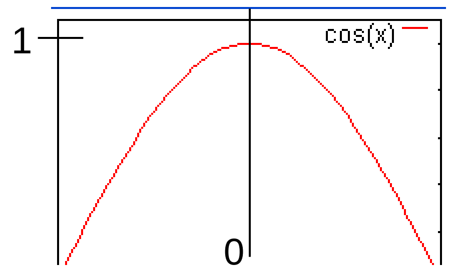
true if e' *close* to $x^{(0)}$

small angle between e' and $x^{(0)}$ $\Leftarrow$ true if e' *close* to $x^{(0)}$



$x^{(0)} = (1, 1)$

recall: $\dfrac{ab'}{\|a\| \cdot \|b\|} = \cos \angle(a, b)$



small angle $\alpha \Rightarrow$ large $\cos(\, \alpha \,)$

Concept of Energy Function

<u>required:</u>

small angle between $e' = W y^{(0)}{}'$ and $x^{(0)}$

$\Rightarrow$ larger cosine of angle indicates greater similarity of vectors

$\Rightarrow$ $\forall e'$ of equal size: try to maximize $x^{(0)} e' = \underbrace{\| x^{(0)} \|}_{\text{fixed}} \cdot \underbrace{\| e \|}_{\text{fixed}} \cdot \underbrace{\cos \angle (x^{(0)}, e)}_{\to \text{max!}}$

$\Rightarrow$ maximize $x^{(0)} e' = x^{(0)} W y^{(0)}{}'$

$\Rightarrow$ identical to minimize $-x^{(0)} W y^{(0)}{}'$

Definition

Energy function of BAM at iteration t is $E(x^{(t)}, y^{(t)}) = -\frac{1}{2} x^{(t)} W y^{(t)}{}'$ □

Stable States

Theorem

An asynchronous BAM with arbitrary weight matrix W reaches steady state in a finite number of updates.

Proof:

$$E(x,y) = -\frac{1}{2} x W y' = \begin{cases} -\frac{1}{2} x (W y') = -\frac{1}{2} x b' = -\frac{1}{2} \sum_{i=1}^{n} b_i \, x_i \\ -\frac{1}{2} (x W) y' = -\frac{1}{2} a y' = -\frac{1}{2} \sum_{i=1}^{k} a_i \, y_i \end{cases}$$

excitations (b_i, a_i)

BAM asynchronous $\Rightarrow$ select neuron at random from left or right layer, compute its excitation and change state if necessary (states of other neurons not affected) neuron i of left layer has changed $\Rightarrow \text{sgn}(x_i) \neq \text{sgn}(b_i)$

$\Rightarrow x_i$ was updated to $\tilde{x}_i = -x_i$

$$E(x,y) - E(\tilde{x},y) = -\frac{1}{2} \underbrace{b_i\,(x_i - \tilde{x}_i)}_{< 0} > 0$$

x_i	b_i	$x_i - \tilde{x}_i$
-1	> 0	< 0
+1	< 0	> 0

use analogous argumentation if neuron of right layer has changed

$\Rightarrow$ every update (change of state) decreases energy function

$\Rightarrow$ since number of different bipolar vectors is finite
update stops after finite #updates

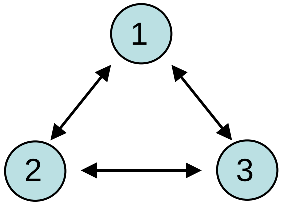
remark: dynamics of BAM get stable in local minimum of energy function!

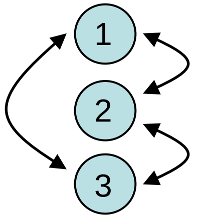
q.e.d.

special case of BAM but proposed earlier (1982)

characterization:

- neurons preserve state until selected at random for update
- n neurons fully connected
- symmetric weight matrix
- no self-loops (→ zero main diagonal entries)
- thresholds θ , neuron i fires if excitations larger than θ_i





transition: select index k at random, new state is $\tilde{x} = \text{sgn}(\, x\, W - \theta\,)$

where $\tilde{x} = (x_1, \ldots, x_{k-1}, \tilde{x}_k, x_{k+1}, \ldots, x_n)$

energy of state x is $E(x) \;=\; -\frac{1}{2}\, x W x' + \theta\, x'$

Theorem:

Hopfield network converges to local minimum of energy function after a finite number of updates. □

Proof: assume that x_k has been updated $\Rightarrow \tilde{x}_k = -x_k$ and $\tilde{x}_i = x_i$ for $i \neq k$

$$E(x) - E(\tilde{x}) = -\tfrac{1}{2}\, x W x' + \theta\, x' + \tfrac{1}{2}\, \tilde{x} W \tilde{x}' - \theta\, \tilde{x}'$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, x_i\, x_j + \sum_{i=1}^{n} \theta_i\, x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, \tilde{x}_i\, \tilde{x}_j - \sum_{i=1}^{n} \theta_i\, \tilde{x}_i$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, (x_i\, x_j - \tilde{x}_i\, \tilde{x}_j) + \sum_{i=1}^{n} \theta_i\, \underbrace{(x_i - \tilde{x}_i)}_{= 0 \text{ if } i \neq k}$$

$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{j=1}^{n} w_{ij}\, (x_i\, x_j - \underbrace{\tilde{x}_i}_{= x_i}\, \tilde{x}_j) - \frac{1}{2} \sum_{j=1}^{n} \underbrace{w_{kj}}_{0 \text{ if } j = k}\, (x_k\, x_j - \tilde{x}_k\, \underbrace{\tilde{x}_j}_{x_j \text{ if } j \neq k}) + \theta_k\, (x_k - \tilde{x}_k)$$

$$= -\frac{1}{2}\sum_{\substack{i=1\\i\neq k}}^{n}\sum_{j=1}^{n} w_{ij}\,x_i\,\underbrace{(x_j - \tilde{x}_j)}_{=0 \text{ if } j\neq k} - \frac{1}{2}\sum_{\substack{j=1\\j\neq k}}^{n} w_{kj}\,x_j\,(x_k - \tilde{x}_k) + \theta_k\,(x_k - \tilde{x}_k)$$

$$= -\frac{1}{2}\sum_{\substack{i=1\\i\neq k}}^{n} w_{ik}\,x_i\,(x_k - \tilde{x}_k) - \frac{1}{2}\sum_{\substack{j=1\\j\neq k}}^{n} w_{kj}\,x_j\,(x_k - \tilde{x}_k) + \theta_k\,(x_k - \tilde{x}_k)$$

(rename j to i, recall W = W‘, $w_{kk} = 0$)

$$= -\sum_{i=1}^{n} w_{ik}\,x_i\,(x_k - \tilde{x}_k) + \theta_k\,(x_k - \tilde{x}_k)$$

$$\underbrace{= -(x_k - \tilde{x}_k)\left[\underbrace{\sum_{i=1}^{n} w_{ik}\,x_i}_{\text{excitation } e_k} - \theta_k\right]}_{>0 \text{ if } x_k<0 \text{ and vice versa}} > 0$$

since:

x_k	$x_k - \tilde{x}_k$	$e_k - \theta_k$	ΔE
$+1$	> 0	< 0	> 0
-1	< 0	> 0	> 0

q.e.d.

Application to Combinatorial Optimization

Idea:

- transform combinatorial optimization problem as objective function with $x \in \{-1,+1\}^n$
- rearrange objective function to look like a Hopfield energy function
- extract weights W and thresholds θ from this energy function
- initialize a Hopfield net with these parameters W and θ
- run the Hopfield net until reaching stable state (= local minimizer of energy function)
- stable state is local minimizer of combinatorial optimization problem

Example I: Linear Functions

$$f(x) = \sum_{i=1}^{n} c_i \, x_i \quad \to \min! \qquad (\, x_i \in \{-1, +1\} \,)$$

Evidently: $E(x) = f(x)$ with $W = 0$ and $\theta = c$

$\Downarrow$

choose $x^{(0)} \in \{-1, +1\}^n$
set iteration counter $t = 0$
repeat
  choose index k at random
  $x_k^{(t+1)} = \text{sgn}(x^{(t)} \cdot W_{\cdot,k} - \theta_k) = \text{sgn}(x^{(t)} \cdot 0 - c_k) = -\text{sgn}(c_k) = \begin{cases} -1 & \text{if } c_k > 0 \\ +1 & \text{if } c_k < 0 \end{cases}$
  increment t
until reaching fixed point

$\Rightarrow$ fixed point reached after Θ(n log n) iterations on average

[proof: → black board]

Example II: MAXCUT

given: graph with n nodes and symmetric weights $\omega_{ij} = \omega_{ji}$, $\omega_{ii} = 0$, on edges

task: find a partition $V = (V_0, V_1)$ of the nodes such that the weighted sum of edges with one endpoint in V_0 and one endpoint in V_1 becomes maximal

encoding: $\forall$ i=1,...,n: $y_i = 0 \Leftrightarrow$ node i in set V_0; $y_i = 1 \Leftrightarrow$ node i in set V_1

objective function: $$f(y) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij} \left[y_i (1-y_j) + y_j (1-y_i) \right] \quad \to \max!$$

preparations for applying Hopfield network

step 1: conversion to minimization problem

step 2: transformation of variables

step 3: transformation to “Hopfield normal form“

step 4: extract coefficients as weights and thresholds of Hopfield net

Example II: MAXCUT (continued)

step 1: conversion to minimization problem

⇒ multiply function with -1 ⇒ E(y) = -f(y) → min!

step 2: transformation of variables

⇒ $y_i = (x_i+1) / 2$

$$\Rightarrow f(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij} \left[\frac{x_i+1}{2} \left(1 - \frac{x_j+1}{2}\right) + \frac{x_j+1}{2} \left(1 - \frac{x_i+1}{2}\right) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij} \left[1 - x_i x_j\right]$$

$$= \underbrace{\frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij}}_{\text{constant value}} - \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij} x_i x_j$$

constant value (does not affect location of optimal solution)

Example II: MAXCUT (continued)

step 3: transformation to “Hopfield normal form“

$$E(x) = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij}\, x_i\, x_j \quad = -\frac{1}{2} \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n}}_{i \neq j} \underbrace{\left(-\frac{1}{2}\omega_{ij}\right)}_{w_{ij}} x_i\, x_j$$

$$= -\frac{1}{2} x' W x + \underbrace{\theta' x}_{0'}$$

step 4: extract coefficients as weights and thresholds of Hopfield net

$$w_{ij} = -\frac{\omega_{ij}}{2} \text{ for } i \neq j, \quad w_{ii} = 0, \quad \theta_i = 0$$

remark: ω_{ij} : weights in graph — w_{ij} : weights in Hopfield net