

# Computational Intelligence

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- Fuzzy relations
- Fuzzy logic
  - Linguistic variables and terms
  - Inference from fuzzy statements

relations with conventional sets  $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$ :

$$R(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$$

notice that cartesian product is a **set!**

⇒ all set operations remain valid!

crisp membership function (of  $x$  to relation  $R$ )

$$R(x_1, x_2, \dots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \dots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

### Definition

**Fuzzy relation** = fuzzy set over crisp cartesian product  $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$  ■

→ each tuple  $(x_1, \dots, x_n)$  has a degree of membership to relation

→ degree of membership expresses  
*strength of relationship* between elements of tuple

appropriate representation: n-dimensional membership matrix

**example:** Let  $X = \{ \text{New York, Paris} \}$  and  $Y = \{ \text{Beijing, New York, Dortmund} \}$ .

relation R = “very far away”

membership matrix →

relation R	New York	Paris
Beijing	1.0	0.9
New York	0.0	0.7
Dortmund	0.6	0.3

**Definition**

Let  $R(X, Y)$  be a fuzzy relation with membership matrix  $R$ . The **inverse fuzzy relation** to  $R(X, Y)$ , denoted  $R^{-1}(Y, X)$ , is a relation on  $Y \times X$  with membership matrix  $R^{-1} = R'$ . ■

**Remark:**  $R'$  is the transpose of membership matrix  $R$ .

Evidently:  $(R^{-1})^{-1} = R$  since  $(R')' = R$

**Definition**

Let  $P(X, Y)$  and  $Q(Y, Z)$  be fuzzy relations. The operation  $\circ$  on two relations, denoted  $P(X, Y) \circ Q(Y, Z)$ , is termed **max-min-composition** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}.$$
 ■

**Theorem**

- a) max-min composition is associative.
- b) max-min composition is not commutative.
- c)  $(P(X,Y) \circ Q(Y,Z))^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y, X)$ .

membership matrix of max-min composition  
determinable via “fuzzy matrix multiplication”:  $R = P \circ Q$

fuzzy matrix multiplication  $r_{ij} = \max_k \min\{p_{ik}, q_{kj}\}$

crisp matrix multiplication  $r_{ij} = \sum_k p_{ik} \cdot q_{kj}$

further methods for realizing compositions of relations:

### max-prod composition

$$(P \odot Q)(x, z) = \max_{y \in \mathcal{Y}} \{P(x, y) \cdot Q(y, z)\}$$

### generalization: sup-t composition

$$(P \circ Q)(x, z) = \sup_{y \in \mathcal{Y}} \{t(P(x, y), Q(y, z))\}, \quad \text{where } t(\dots) \text{ is a t-norm}$$

e.g.:  $t(a, b) = \min\{a, b\} \Rightarrow$  max-min-composition

$t(a, b) = a \cdot b \Rightarrow$  max-prod-composition

### Binary fuzzy relations on $X \times X$ : properties

• **reflexive**  $\Leftrightarrow \forall x \in X: R(x,x) = 1$

• **irreflexive**  $\Leftrightarrow \exists x \in X : R(x,x) < 1$

• **antireflexive**  $\Leftrightarrow \forall x \in X : R(x,x) < 1$

• **symmetric**  $\Leftrightarrow \forall (x,y) \in X \times X : R(x,y) = R(y,x)$

• **asymmetric**  $\Leftrightarrow \exists (x,y) \in X \times X : R(x,y) \neq R(y,x)$

• **antisymmetric**  $\Leftrightarrow \forall (x,y) \in X \times X : R(x,y) \neq R(y,x)$

• **transitive**  $\Leftrightarrow \forall (x,z) \in X \times X : R(x,z) \geq \max_{y \in Y} \min \{ R(x,y), R(y,z) \}$

• **intransitive**  $\Leftrightarrow \exists (x,z) \in X \times X : R(x,z) < \max_{y \in Y} \min \{ R(x,y), R(y,z) \}$

• **antitransitive**  $\Leftrightarrow \forall (x,z) \in X \times X : R(x,z) < \max_{y \in Y} \min \{ R(x,y), R(y,z) \}$

actually, here: max-min-transitivity ( $\rightarrow$  in general: sup-t-transitivity)

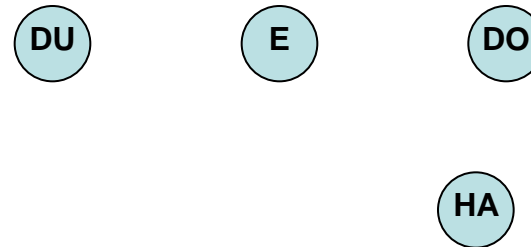


### binary fuzzy relation on $X \times X$ : example

Let  $X$  be the set of all cities in Germany.

Fuzzy relation  $R$  is intended to represent the concept of „very close to“.

- $R(x,x) = 1$ , since every city is certainly very close to itself.  
 $\Rightarrow$  **reflexive**
- $R(x,y) = R(y,x)$ : if city  $x$  is very close to city  $y$ , then also vice versa.  
 $\Rightarrow$  **symmetric**
- $R(\text{Dortmund, Essen}) = 0.8$   
 $R(\text{Essen, Duisburg}) = 0.7$   
 $R(\text{Dortmund, Duisburg}) = 0.5$   
 $R(\text{Dortmund, Hagen}) = 0.9$   
 $\Rightarrow$  **intransitive**



### crisp:

relation R is equivalence relation  $\Leftrightarrow$  R reflexive, symmetric, transitive

### fuzzy:

relation R is similarity relation  $\Leftrightarrow$  R reflexive, symmetric, (max-min-) transitive

Example:

	a	b	c	d	e	f	g
a	1,0	0,8	0,0	0,4	0,0	0,0	0,0
b	0,8	1,0	0,0	0,4	0,0	0,0	0,0
c	0,0	0,0	1,0	0,0	1,0	0,9	0,5
d	0,4	0,4	0,0	1,0	0,0	0,0	0,0
e	0,0	0,0	1,0	0,0	1,0	0,9	0,5
f	0,0	0,0	0,9	0,0	0,9	1,0	0,5
g	0,0	0,0	0,5	0,0	0,5	0,5	1,0

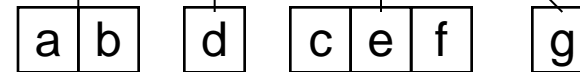
$\alpha = 0,4$



$\alpha = 0,5$



$\alpha = 0,8$



$\alpha = 0,9$



$\alpha = 1,0$



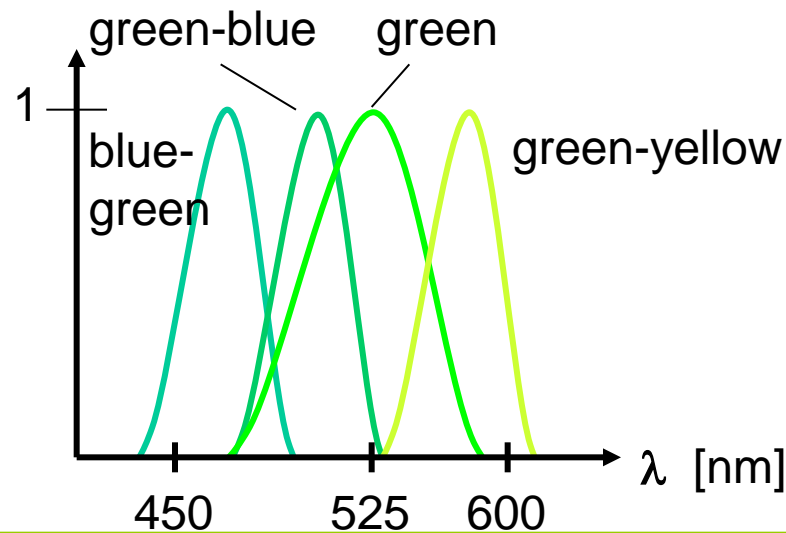
**linguistic variable:**

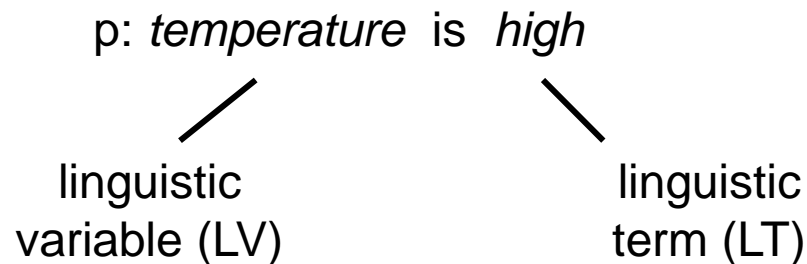
variable that can attain several values of linguistic / verbal nature

e.g.: **color** can attain values **red, green, blue, yellow, ...**

values (red, green, ...) of linguistic variable are called **linguistic terms**

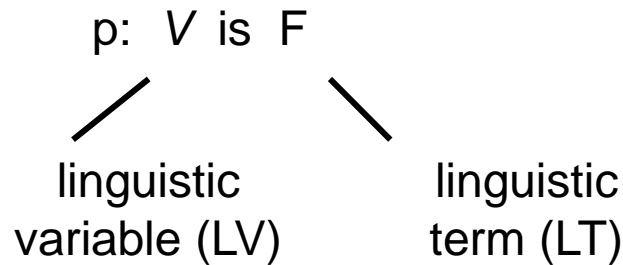
linguistic terms are associated with fuzzy sets



*fuzzy proposition*

- LV may be associated with several LT : *high, medium, low, ...*
- *high, medium, low* temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given **concrete crisp** temperature value  $v$  is interpreted as equal to the degree of membership  $high(v)$  of the fuzzy set *high*

fuzzy proposition



actually:

$p: V \text{ is } F(v)$

and

$T(p) = F(v)$  for a concrete crisp value  $v$

$\backslash$   
 trueness( $p$ )

establishes connection between *degree of membership* of a fuzzy set and the *degree of trueness* of a fuzzy proposition

**fuzzy proposition**

p: IF *heating* is *hot*, THEN *energy consumption* is *high*

LV	LT	LV	LT

expresses relation between

- a) temperature of heating and
- b) quantity of energy consumption

p: (*heating*, *energy consumption*)  $\in R$  relation

*fuzzy proposition*

p: IF X is A, THEN Y is B  
    |      |          |      |  
    LV   LT          LV   LT

How can we determine / express degree of trueness  $T(p)$  ?

- For crisp, given values  $x, y$  we know  $A(x)$  and  $B(y)$
- $A(x)$  and  $B(y)$  must be processed to single value via relation  $R$
- $R(x, y) = \text{function}(A(x), B(y))$  is fuzzy set over  $X \times Y$
- as before: interpret  $T(p)$  as degree of membership  $R(x,y)$

**fuzzy proposition**

p: IF  $X$  is A, THEN  $Y$  is B

A is fuzzy set over  $X$

B is fuzzy set over  $Y$

R is fuzzy set over  $X \times Y$

$\forall (x,y) \in X \times Y: R(x, y) = \text{Imp}( A(x), B(y) )$

What is  $\text{Imp}(\cdot, \cdot)$  ?

$\Rightarrow$  „appropriate“ fuzzy implication  $[0,1] \times [0,1] \rightarrow [0,1]$



**assumption:** we know an „appropriate“  $\text{Imp}(a,b)$ .

How can we determine the degree of trueness  $T(p)$  ?

**example:**

let  $\text{Imp}(a, b) = \min\{ 1, 1 - a + b \}$  and consider fuzzy sets

A:

$x_1$	$x_2$	$x_3$
0.1	0.8	1.0

B:

$y_1$	$y_2$
0.5	1.0

$\Rightarrow$

<b>R</b>	$x_1$	$x_2$	$x_3$
$y_1$	1.0	0.7	0.5
$y_2$	1.0	1.0	1.0

z.B.

$$R(x_2, y_1) = \text{Imp}(A(x_2), B(y_1)) = \text{Imp}(0.8, 0.5) = \min\{1.0, 0.7\} = 0.7$$

and  $T(p)$  for  $(x_2, y_1)$  is  $R(x_2, y_1) = 0.7$  ■

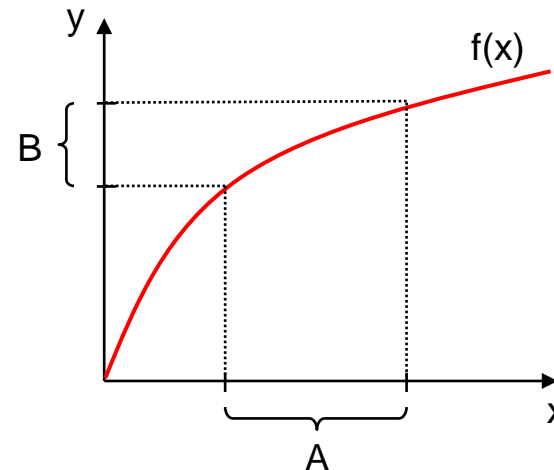
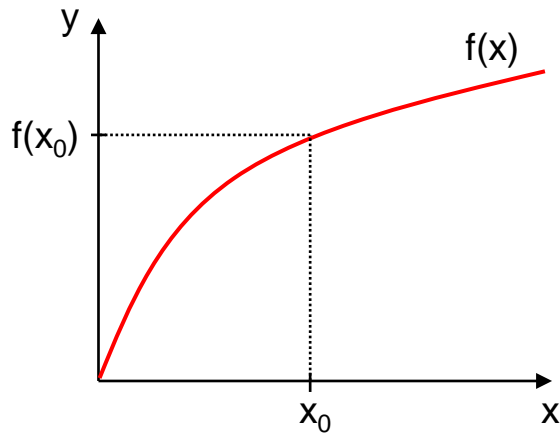
## toward inference from fuzzy statements:

- let  $\forall x, y: y = f(x)$ .

IF  $X = x_0$  THEN  $Y = f(x_0)$

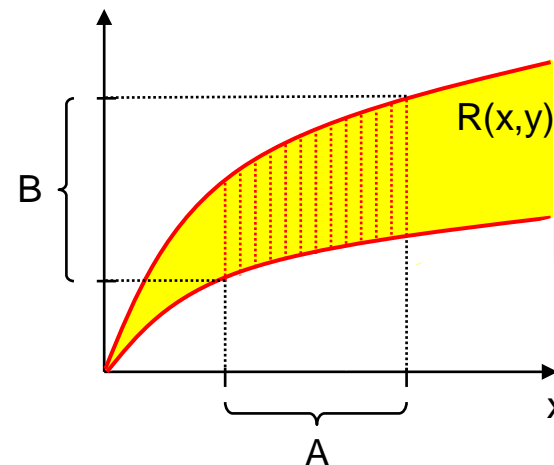
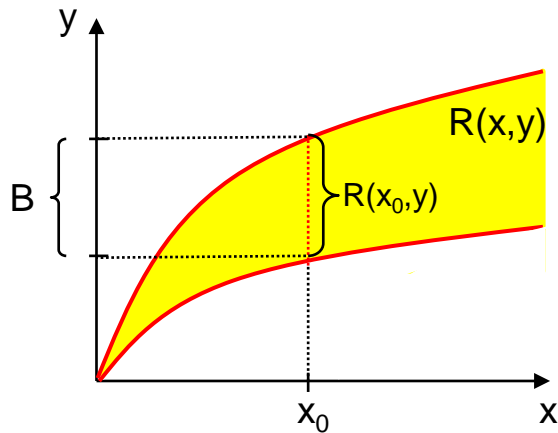
- IF  $X \in A$  THEN  $Y \in B = \{y \in \mathcal{Y} : y = f(x), x \in A\}$

crisp case:  
functional  
relationship



## toward inference from fuzzy statements:

- let relationship between  $x$  and  $y$  be a relation  $R$  on  $\mathcal{X} \times \mathcal{Y}$
  - IF  $X = x_0$  THEN  $Y \in B = \{ y \in \mathcal{Y} : (x_0, y) \in R \}$
  - IF  $X \in A$  THEN  $Y \in B = \{ y \in \mathcal{Y} : (x, y) \in R, x \in A \}$
- } crisp case:  
relational  
relationship



## toward inference from fuzzy statements:

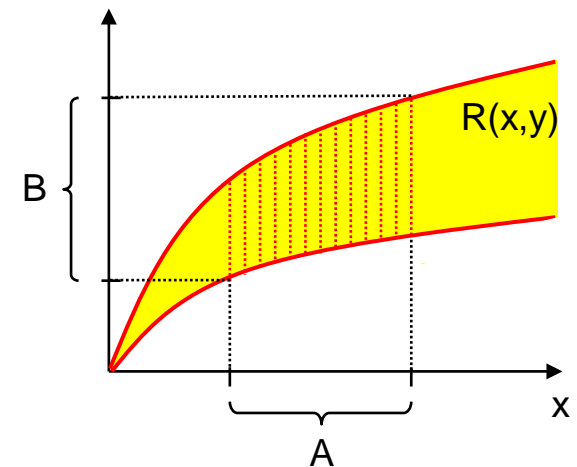
IF  $X \in A$  THEN  $Y \in B = \{ y \in \mathcal{Y} : (x, y) \in R, x \in A \}$

also expressible via characteristic functions of sets A, B, R:

$B(y) = 1$  iff  $\exists x: A(x) = 1$  and  $R(x, y) = 1$

$\Leftrightarrow \exists x: \min\{ A(x), R(x, y) \} = 1$

$\Leftrightarrow \max_{x \in \mathcal{X}} \min\{ A(x), R(x, y) \} = 1$



$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$$

inference from fuzzy statements

**Now:**  $A'$ ,  $B'$  fuzzy sets over  $\mathcal{X}$  resp.  $\mathcal{Y}$

Assume:  $R(x,y)$  and  $A'(x)$  are given.

Idea: Generalize characteristic function of  $B(y)$  to membership function  $B'(y)$

$$\forall y \in \mathcal{Y}: B(y) = \max_{x \in \mathcal{X}} \min \{ A(x), R(x, y) \}$$

characteristic functions

$$\forall y \in \mathcal{Y}: B'(y) = \sup_{x \in \mathcal{X}} \min \{ A'(x), R(x, y) \}$$

membership functions

**composition rule of inference (in matrix form):  $\mathbf{B}^T = \mathbf{A} \circ \mathbf{R}$**

inference from fuzzy statements

- conventional:  
modus ponens

$$\begin{array}{l} a \Rightarrow b \\ a \\ \hline b \end{array}$$

- fuzzy:  
generalized modus ponens (GMP)

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ X \text{ is } A' \\ \hline Y \text{ is } B' \end{array}$$

e.g.:      IF *heating* is hot, THEN *energy consumption* is high  
              *heating* is warm  
              

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              *energy consumption* is normal

## example: GMP

consider

A:

$x_1$	$x_2$	$x_3$
0.5	1.0	0.6

B:

$y_1$	$y_2$
1.0	0.4

with the rule: IF  $X$  is A THEN  $Y$  is B

given fact

A':

$x_1$	$x_2$	$x_3$
0.6	0.9	0.7

$\Rightarrow$

<b>R</b>	$x_1$	$x_2$	$x_3$
$y_1$	1.0	1.0	1.0
$y_2$	0.9	0.4	0.8

with  $\text{Imp}(a,b) = \min\{1, 1-a+b\}$

thus:  $A' \circ R = B'$

$$\begin{pmatrix} 0.6 & 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.7 \end{pmatrix}$$

with max-min-composition

## inference from fuzzy statements

- conventional:  
modus tollens

$$\begin{array}{r} a \Rightarrow b \\ \bar{b} \\ \hline \bar{a} \end{array}$$

- fuzzy:  
generalized modus tollens (GMT)

$$\begin{array}{r} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ Y \text{ is } B' \\ \hline X \text{ is } A' \end{array}$$

e.g.: IF *heating* is hot, THEN *energy consumption* is high  
*energy consumption* is normal  
*heating* is warm



## example: GMT

consider

A:

$x_1$	$x_2$	$x_3$
0.5	1.0	0.6

B:

$y_1$	$y_2$
1.0	0.4

with the rule: IF  $X$  is A THEN  $Y$  is B

given fact

B':

$y_1$	$y_2$
0.9	0.7

$\Rightarrow$

<b>R</b>	$x_1$	$x_2$	$x_3$
$y_1$	1.0	1.0	1.0
$y_2$	0.9	0.4	0.8

with  $\text{Imp}(a,b) = \min\{1, 1-a+b\}$

thus:  $B' \circ R^{-1} = A'$

$$\begin{pmatrix} 0.9 & 0.7 \end{pmatrix} \circ \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.9 & 0.9 & 0.9 \end{pmatrix}$$

with max-min-composition

inference from fuzzy statements

- conventional:  
hypothetic syllogism

$$a \Rightarrow b$$

$$b \Rightarrow c$$

---


$$a \Rightarrow c$$

- fuzzy:  
generalized HS

IF X is A, THEN Y is B

IF Y is B, THEN Z is C

---

IF X is A, THEN Z is C

- e.g.:
- IF *heating* is hot, THEN *energy consumption* is high
- IF *energy consumption* is high, THEN *living* is expensive
- 
- IF *heating* is hot, THEN *living* is expensive

**example: GHS**

let fuzzy sets  $A(x)$ ,  $B(x)$ ,  $C(x)$  be given

⇒ determine the three relations

$$R_1(x,y) = \text{Imp}(A(x),B(y))$$

$$R_2(y,z) = \text{Imp}(B(y),C(z))$$

$$R_3(x,z) = \text{Imp}(A(x),C(z))$$

and express them as matrices  $R_1$ ,  $R_2$ ,  $R_3$

**We say:**

GHS is valid if  $R_1 \circ R_2 = R_3$

So, ... what makes sense for  $\text{Imp}(\cdot, \cdot)$  ?

$\text{Imp}(a,b)$  ought to express fuzzy version of implication ( $a \Rightarrow b$ )

conventional:  $a \Rightarrow b$  identical to  $\bar{a} \vee b$

But how can we calculate with fuzzy “boolean” expressions?

**request:** must be compatible to crisp version (and more) for  $a,b \in \{ 0, 1 \}$

a	b	$a \wedge b$	$t(a,b)$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

a	b	$a \vee b$	$s(a,b)$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

a	$\bar{a}$	$c(a)$
0	1	1
1	0	0

So, ... what makes sense for  $\text{Imp}(\cdot, \cdot)$  ?

### 1st approach: S implications

conventional:  $a \Rightarrow b$  identical to  $\bar{a} \vee b$

fuzzy:  $\text{Imp}(a, b) = s(c(a), b)$

### 2nd approach: R implications

conventional:  $a \Rightarrow b$  identical to  $\max\{x \in \{0, 1\} : a \wedge x \leq b\}$

fuzzy:  $\text{Imp}(a, b) = \max\{x \in [0, 1] : t(a, x) \leq b\}$

### 3rd approach: QL implications

conventional:  $a \Rightarrow b$  identical to  $\bar{a} \vee b \equiv \bar{a} \vee (a \wedge b)$  law of absorption

fuzzy:  $\text{Imp}(a, b) = s(c(a), t(a, b))$  (dual tripel ?)

**example: S implication**

$$\text{Imp}(a, b) = s(c_s(a), b) \quad (c_s : \text{std. complement})$$

## 1. Kleene-Dienes implication

$$s(a, b) = \max\{ a, b \} \quad (\text{standard}) \quad \text{Imp}(a, b) = \max\{ 1-a, b \}$$

## 2. Reichenbach implication

$$s(a, b) = a + b - ab \quad (\text{algebraic sum}) \quad \text{Imp}(a, b) = 1 - a + ab$$

## 3. Łukasiewicz implication

$$s(a, b) = \min\{ 1, a + b \} \quad (\text{bounded sum}) \quad \text{Imp}(a, b) = \min\{ 1, 1 - a + b \}$$

**example: R implicationen**

$$\text{Imp}(a, b) = \max\{ x \in [0, 1] : t(a, x) \leq b \}$$

## 1. Gödel implication

$$t(a, b) = \min\{ a, b \} \quad (\text{std.})$$

$$\text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ b & , \text{ else} \end{cases}$$

## 2. Goguen implication

$$t(a, b) = ab \quad (\text{algeb. product})$$

$$\text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ \frac{b}{a} & , \text{ else} \end{cases}$$

## 3. Łukasiewicz implication

$$t(a, b) = \max\{ 0, a + b - 1 \} \quad (\text{bounded diff.})$$

$$\text{Imp}(a, b) = \min\{ 1, 1 - a + b \}$$

**example: QL implication**

$$\text{Imp}(a, b) = s( c(a), t(a, b) )$$

## 1. Zadeh implication

$$t(a, b) = \min \{ a, b \} \quad (\text{std.})$$

$$s(a,b) = \max\{ a, b \} \quad (\text{std.})$$

$$\text{Imp}(a, b) = \max\{ 1 - a, \min\{a, b\} \}$$

## 2. „NN“ implication ☺ (Klir/Yuan 1994)

$$t(a, b) = ab \quad (\text{algebr. prd.})$$

$$s(a,b) = a + b - ab \quad (\text{algebr. sum})$$

$$\text{Imp}(a, b) = 1 - a + a^2b$$

## 3. Kleene-Dienes implication

$$t(a, b) = \max\{ 0, a + b - 1 \} \quad (\text{bounded diff.}) \quad \text{Imp}(a, b) = \max\{ 1-a, b \}$$

$$s(a,b) = \min \{ 1, a + b \} \quad (\text{bounded sum})$$



## axioms for fuzzy implications

1.  $a \leq b$  implies  $\text{Imp}(a, x) \geq \text{Imp}(b, x)$  monotone in 1st argument
2.  $a \leq b$  implies  $\text{Imp}(x, a) \leq \text{Imp}(x, b)$  monotone in 2nd argument
3.  $\text{Imp}(0, a) = 1$  dominance of falseness
4.  $\text{Imp}(1, b) = b$  neutrality of trueness
5.  $\text{Imp}(a, a) = 1$  identity
6.  $\text{Imp}(a, \text{Imp}(b, x)) = \text{Imp}(b, \text{Imp}(a, x))$  exchange property
7.  $\text{Imp}(a, b) = 1$  iff  $a \leq b$  boundary condition
8.  $\text{Imp}(a, b) = \text{Imp}(c(b), c(a))$  contraposition
9.  $\text{Imp}(\cdot, \cdot)$  is continuous continuity

## characterization of fuzzy implication

### Theorem:

Imp:  $[0,1] \times [0,1] \rightarrow [0,1]$  satisfies axioms 1-9 for fuzzy implications for a certain fuzzy complement  $c(\cdot)$   $\Leftrightarrow$

$\exists$  strictly monotone increasing, continuous function  $f: [0,1] \rightarrow [0, \infty)$  with

- $f(0) = 0$
- $\forall a, b \in [0,1]: \text{Imp}(a, b) = f^{-1}(\min\{f(1) - f(a) + f(b), f(1)\})$
- $\forall a \in [0,1]: c(a) = f^{-1}(f(1) - f(a))$

**Proof:** Smets & Magrez (1987), p. 337f. ■

**examples:** (in tutorial)

## choosing an „appropriate“ fuzzy implication ...

**apt quotation:** (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem.“

### **guideline:**

GMP, GMT, GHS should be compatible with MP, MT, HS

for fuzzy implication in calculations with relations:

$$B(y) = \sup \{ t( A(x), \text{Imp}( A(x), B(y) ) ) : x \in X \}$$

### **example:**

Gödel implication for t-norm = bounded difference