Approximative Reasoning

So far:
• p: IF $X$ is $A$ THEN $Y$ is $B$
  \[ R(x, y) = \text{Imp}(A(x), B(y)) \]  
  rule as relation; fuzzy implication

• rule: IF $X$ is $A$ THEN $Y$ is $B$
  fact: $X$ is $A'$
  conclusion: $Y$ is $B'$
  \[ B'(y) = \sup_{x \in X} t(A'(x), \text{Imp}(A(x), B(y))) \]  
  composition rule of inference

Thus:
• $B'(y) = \sup_{x \in X} t(A'(x), R(x, y))$
  given: fuzzy rule
  input: fuzzy set $A'$
  output: fuzzy set $B'$

Plan for Today

- Approximate Reasoning
- Fuzzy Control
Lemma:

a) \( t(a, 1) = a \)

b) \( t(a, b) \leq \min \{ a, b \} \)

c) \( t(0, a) = 0 \)

Proof:

ad a) Identical to axiom 1 of t-norms.

ad b) From monotonicity (axiom 2) follows for \( b \leq 1 \), that \( t(a, b) \leq t(a, 1) = a \).

Commutativity (axiom 3) and monotonicity lead in case of \( a \leq 1 \) to \( t(a, b) = t(b, a) \leq t(b, 1) = b \) which in turn implies \( t(a, b) \leq \min \{ a, b \} \).

ad c) From b) follows \( 0 \leq t(0, a) \leq \min \{ 0, a \} = 0 \) and therefore \( t(0, a) = 0 \). ■

Approximative Reasoning

2. Equivalence of FITA and FATI?

FITA: “First inference, then aggregate!”

1. Each rule of the form IF \( X \) is \( A_k \) THEN \( Y \) is \( B_k \) must be transformed by an appropriate fuzzy implication \( \text{Imp}_k(\cdot, \cdot) \) to a relation \( R_k \):
   \[ R_k(x, y) = \text{Imp}_k( A_k(x), B_k(y) ). \]

2. Determine \( B_k'(y) = R_k(x, y) \circ A'(x) \) for all \( k = 1, \ldots, n \) (local inference).

3. Aggregate to \( B'(y) = \beta( B_1'(y), \ldots, B_n'(y) ) \).

FATI: “First aggregate, then inference!”

1. Each rule of the form IF \( X \) is \( A_k \) THEN \( Y \) is \( B_k \) must be transformed by an appropriate fuzzy implication \( \text{Imp}_k(\cdot, \cdot) \) to a relation \( R_k \):
   \[ R_k(x, y) = \text{Imp}_k( A_k(x), B_k(y) ). \]

2. Aggregate \( R_1, \ldots, R_n \) to a superrelation with aggregating function \( \alpha(\cdot) \):
   \[ R(x, y) = \alpha( R_1(x, y), \ldots, R_n(x, y) ). \]

3. Determine \( B'(y) = R(x, y) \circ A'(x) \) w.r.t. superrelation (inference).
Approximative Reasoning

Lecture 08

special case:

\[
A'(x) = \begin{cases} 
1 & \text{for } x = x_0 \\
0 & \text{otherwise} 
\end{cases}
\]

crisp input!

On the equivalence of FITA and FATI:

FITA:

\[
B'(y) = \beta( B_1'(y), \ldots, B_n'(y) ) \\
= \beta( \text{Imp}_1( A_1(x_0), B_1(y) ), \ldots, \text{Imp}_n( A_n(x_0), B_n(y) ) )
\]

FATI:

\[
B'(y) = R(x, y) \circ A'(x) \\
= \sup_{x \in X} t( A'(x), R(x, y) ) 
\text{ (from now: special case) } \\
= \alpha( \text{Imp}_1( A_1(x_0), B_1(y) ), \ldots, \text{Imp}_n( A_n(x_0), B_n(y) ) )
\]

evidently: sup-t-composition with arbitrary t-norm and \( \alpha(\cdot) = \beta(\cdot) \)

Important:

- if rules of the form IF X is A THEN Y is B interpreted as logical implication
  \[ R(x, y) = \text{Imp}( A(x), B(y) ) \]
  makes sense
- we obtain: \( B'(y) = \sup_{x \in X} \text{Imp}( A'(x), R(x, y) ) \)
- the worse the match of premise \( A'(x) \), the larger is the fuzzy set \( B'(y) \)
  \[ \Rightarrow \text{follows immediately from axiom 1: } a \leq b \Rightarrow \text{Imp}(a, z) \geq \text{Imp}(b, z) \]

interpretation of output set \( B'(y) \):

- \( B'(y) \) is the set of values that are still possible
- each rule leads to an additional restriction of the values that are still possible
  \[ \Rightarrow \text{aggregation via } B'(y) = \min \{ B_1'(y), \ldots, B_n'(y) \} \]

Approximative Reasoning

Lecture 08

important:

- if rules of the form IF X is A THEN Y is B are not interpreted as logical implications, then the function \( \text{Fct}(\cdot) \) in
  \[
  R(x, y) = \text{Fct}( A(x), B(y) )
  \]
  can be chosen as required for desired interpretation.
- frequent choice (especially in fuzzy control):
  - \( R(x, y) = \min \{ A(x), B(x) \} \) Mamdani – “implication”
  - \( R(x, y) = A(x) \cdot B(x) \) Larsen – “implication”
  \[ \Rightarrow \text{of course, they are no implications but specific t-norms!} \]
  \[ \Rightarrow \text{thus, if relation } R(x, y) \text{ is given,} \]
  then the composition rule of inference
  \[
  B'(y) = A'(x) \circ R(x, y) = \sup_{x \in X} \min \{ A(x), R(x, y) \}
  \]
  still can lead to a conclusion via fuzzy logic.
Approximative Reasoning

example: [JM96, S. 244ff.]

industrial drill machine → control of cooling supply

modelling
linguistic variable: rotation speed
linguistic terms: very low, low, medium, high, very high
ground set: X with 0 ≤ x ≤ 18000 [rpm]

example: (continued)

industrial drill machine → control of cooling supply

modelling
linguistic variable: cooling quantity
linguistic terms: very small, small, normal, much, very much
ground set: Y with 0 ≤ y ≤ 18 [liter / time unit]

rule base
IF rotation speed IS very low THEN cooling quantity IS very small

low small
medium normal
high much
very high very much

sets S_{vl}, S_{l}, S_{m}, S_{h}, S_{vh}

sets C_{v}, C_{s}, C_{m}, C_{nm}, C_{vm} “rotation speed”

“cooling quantity”

1. input: crisp value x_0 = 10000 min^{-1} (not a fuzzy set!)

→ fuzzyfication = determine membership for each fuzzy set over X

→ yields S' = (0, 0, \frac{3}{4}, \frac{1}{4}, 0) via x \mapsto (S_{vl}(x_0), S_{l}(x_0), S_{m}(x_0), S_{n}(x_0), S_{vh}(x_0))

2. FITA: locale inference ⇒ since Imp(0,a) = 0 we only need to consider:

S_{m}': C_{n}'(y) = \text{Imp}(\frac{3}{4}, C_{n}(y))

S_{h}': C_{m}'(y) = \text{Imp}(\frac{1}{4}, C_{m}(y))

3. aggregation:

C'(y) = \text{aggr}( C_{n}'(y), C_{m}'(y) ) = \text{max}( \{ \text{Imp}(\frac{3}{4}, C_{n}(y)), \text{Imp}(\frac{1}{4}, C_{m}(y)) \} )

approximation
Approximative Reasoning

**example:** (continued)

industrial drill machine $\rightarrow$ control of cooling supply

$$C'(y) = \max \{ \text{Imp} \left( \frac{3}{4}, C_n(y) \right), \text{Imp} \left( \frac{1}{4}, C_m(y) \right) \}$$

in case of control task typically no logic-based interpretation:

$\rightarrow$ max-aggregation and

$\rightarrow$ relation $R(x,y)$ not interpreted as implication.

often: $R(x,y) = \min(a, b)$ „Mamdani controller“

thus:

$$C'(y) = \max \{ \min \left( \frac{3}{4}, C_n(y) \right), \min \left( \frac{1}{4}, C_m(y) \right) \}$$

$\rightarrow$ graphical illustration

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Fuzzy Control

**open and closed loop control:**

affect the dynamical behavior of a system in a desired manner

- **open loop control**
  
  control is aware of reference values and has a model of the system
  $\Rightarrow$ control values can be adjusted,
  such that process value of system is equal to reference value
  
  problem: noise! $\Rightarrow$ deviation from reference value not detected

- **closed loop control**
  
  now: detection of deviations from reference value possible
  (by means of measurements / sensors)
  and new control values can take into account the amount of deviation

---
**Fuzzy Control**

### Lecture 08

- **Closed Loop Control**

![Diagram of closed loop control system](Diagram.png)

- **Control Deviation**

\[ \text{control deviation} = \text{reference value} - \text{process value} \]

- **Model of System/Process**

- As differential equations or difference equations (DEs)
- Well developed theory available

- **Why Fuzzy Control?**

- No process model in form of DEs etc. (operator/human being has realized control by hand)
- Process with high-dimensional nonlinearities → no classic methods available
- Control goals are vaguely formulated ("soft" changing gears in cars)

- **Fuzzy Description of Control Behavior**

\[
\begin{align*}
\text{IF } X & \text{ is } A_1, \text{ THEN } Y \text{ is } B_1 \\
\text{IF } X & \text{ is } A_2, \text{ THEN } Y \text{ is } B_2 \\
\text{IF } X & \text{ is } A_3, \text{ THEN } Y \text{ is } B_3 \\
\ldots & \\
\text{IF } X & \text{ is } A_n, \text{ THEN } Y \text{ is } B_n \\
X & \text{ is } A' \\
Y & \text{ is } B'
\end{align*}
\]

- Similar to approximative reasoning
- Fact \( A' \) is not a fuzzy set but a crisp input
- Actually, it is the current process value

- Defuzzification

- Maximum method
  - Only active rule with largest activation level is taken into account
  - Suitable for pattern recognition / classification
  - Decision for a single alternative among finitely many alternatives
  - Selection independent from activation level of rule (0.05 vs. 0.95)
  - If used for control: incontinuous curve of output values (leaps)

- Def: rule k active \( \Leftrightarrow A_k(x_0) > 0 \)
defuzzification

- maximum mean value method
  - all active rules with largest activation level are taken into account
    → interpolations possible, but need not be useful
    → obviously, only useful for neighboring rules with max. activation
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - if used in control: incontinuous curve of output values (leaps)

\[ Y^* = \{ y \in Y : B'(y) = \text{hgt}(B') \} \]

![Graph](image1)

- center-of-maxima method (COM)
  - only extreme active rules with largest activation level are taken into account
    → interpolations possible, but need not be useful
    → obviously, only useful for neighboring rules with max. activation level
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - in case of control: incontinuous curve of output values (leaps)

\[ \bar{y} = \frac{\inf Y^* + \sup Y^*}{2} \]

![Graph](image2)

- Center of Gravity (COG)
  - all active rules are taken into account
    → but numerically expensive … …only valid for HW solution, today!
    → borders cannot appear in output (∃ work-around )
  - if only single active rule: independent from activation level
  - continuous curve for output values

\[ \bar{y} = \frac{\int y \cdot B'(y) \, dy}{\int B'(y) \, dy} \]

Excursion: COG

- triangle:
  \[ \bar{y} = \frac{y_1 + y_2 + y_3}{3} \]

- trapezoid:
  \[ \bar{y} = \frac{y_1^2 + y_2^2 - y_2^2 - y_1^2 + y_3y_4 - y_1y_2}{3(y_4 + y_3 - y_2 - y_1)} \]
assumption: fuzzy membership functions piecewise linear

output set $B'(y)$ represented by sequence of points $(y_1, z_1), (y_2, z_2), \ldots, (y_n, z_n)$

$\Rightarrow$ area under $B'(y)$ and weighted area can be determined additively piece by piece

$\Rightarrow$ linear equation $z = my + b$ $\Rightarrow$ insert $(y_i, z_i)$ and $(y_{i+1}, z_{i+1})$

$\Rightarrow$ yields $m$ and $b$ for each of the $n-1$ linear sections

$\Rightarrow$

Defuzzification

- Center of Area (COA)
  - developed as an approximation of COG
  - let $\tilde{y}_k$ be the COGs of the output sets $B'_k(y)$:

$$\tilde{y} = \frac{\sum_k A_k(x_0) \cdot \tilde{y}_k}{\sum_k A_k(x_0)}$$

how to:

assume that fuzzy sets $A_k(x)$ and $B_k(x)$ are triangles or trapezoids

let $x_0$ be the crisp input value

for each fuzzy rule “IF $A_k$ is X THEN $B_k$ is Y”

determine $B'_k(y) = R(A_k(x_0), B_k(y))$, where $R(.,.)$ is the relation

find $\tilde{y}_k$ as center of gravity of $B'_k(y)$