# Stochastic Survivable Network Design Problems

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**Abstract**

We consider survivable network design problems under a two-stage stochastic model with recourse and finitely many scenarios (SSNDP). We propose two new cut-based formulations for SSNDP based on orientation properties and show that they are stronger than the undirected cut-based model. We use a two-stage branch&cut algorithm for solving the decomposed model to provable optimality. In order to accelerate the computations, we suggest a new cut strengthening technique for the decomposed L-shaped optimality cuts that is computationally fast and easy to implement. Computational experiments show that our cut strengthening approach significantly reduces the number of iterations required and the computational running time.

## 1 Introduction

*Motivation.* We consider the edge-connectivity version of the survivable network design problem which is one of the most fundamental problems in the design of telecommunication networks. Many of the classical network design problems like, e.g., the shortest path problem, the spanning tree problem, the Steiner tree problem, or minimum-weight edge-connected subgraphs, edge-connectivity augmentations, etc., are all special cases of the survivable network design problem.

The *(deterministic) survivable network design problem* (SNDP) is defined as follows: We are given a simple undirected graph $G = (V, E)$ with edge costs $c_e \geq 0$, $\forall e \in E$, and a symmetric $|V| \times |V|$ *connectivity requirement* matrix $\mathbf{r} = [r_{ij}]$. Thereby, $r_{ij} \in \mathbf{N} \cup \{0\}$ represents the minimal required number of *edge-disjoint paths* between two distinct vertices $i, j \in V$. The goal is to find a subset of edges $E' \subseteq E$ that satisfies the connectivity requirements and minimizes the overall solution cost being defined as $\sum_{e \in E'} c_e$. We assume that the connectivity requirements imply that each feasible solution comprises a single connected component, in which case the problem is called the *unitary* SNDP. Notice that, the higher the connectivity requirements of a node, the more important its role in the network.

In practice, however, network planners often have to deal with uncertain data, e.g., the importance of a node and the associated connectivity requirements are not known in advance, or the costs of establishing links (installing new pipes, cables, etc.) may be subject to uncertainty. One promising approach to deal with this type of uncertainty is stochastic programming (for an introduction see, e.g., [3]). Thereby, the uncertain data is modeled using random variables and a set of scenarios defines their possible outcomes.

In the two-stage stochastic network design problem the network planner wants to establish profitable connections now (*first stage*, on Monday) while taking all possible uncertain outcomes (*scenarios*) into account. In the future (*second stage*, on Tuesday) the actual scenario is revealed and additional edges can be purchased (*recourse* action) to satisfy the now known requirements.

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The objective is to optimize the *expected cost* of the solution, i.e., the sum of the first stage cost plus the expected cost of the second stage. Thereby, all connectivity requirements for all scenarios have to be satisfied. For a formal definition, see Section 2.

*Previous work.* There exists a large body of work on various variants of the deterministic survivable network design problem. We refer to [11] for a comprehensive literature overview on the SNDP. Many polyhedral studies are done in the 90's (see, e.g., [6]), and a decade later the question of deriving stronger MIP formulations by orienting the $k$-connected subgraphs has been considered (see, e.g. [1, 13]). Among the approximation algorithms for the SNDP, we point to the work of Jain [9] whose approximation factor of two remains the best one up to date. Regarding the stochastic variants of the SNDP, there are significantly less results published so far. The two-stage stochastic Steiner tree problem is a special case in which connectivity requirements are zero or one. For this problem, both approximation algorithms (see, e.g., [8]) and MIP approaches (see [4]) were developed. For the more general case in which connectivity requirements are arbitrary natural numbers, up to our knowledge, there only exists an $O(1)$ approximation algorithm (see [7]) for the following special case of the SSNDP: For each pair of distinct nodes $i$ and $j$, a single scenario (whose probability is $p_{ij}$) is given, in which nodes $i$ and $j$ need to be $k$-edge-connected.

*Our Contribution.* In this paper we study a generalization of the problem proposed in [7]. We introduce two novel mixed integer programming (MIP) models of the deterministic equivalent for solving the SSNDP on undirected graphs based on certain orientation properties of second-stage solutions. We show that the new formulations are provably stronger than the original one based on the standard undirected cuts. Directed formulations for both the SNDP and the SSNDP may lead to stronger approximation algorithms similarly as for the deterministic Steiner tree problem. For solving these models, we use a recently introduced decomposition approach—similar to Benders decomposition [2] and the integer L-shaped method [12]—that we call *two-stage branch&cut* (see [4]). In this paper we show how to strengthen the inserted $L$-shaped optimality cuts by a simple modification of the dual solution of the subproblems. Our computational experiments show that the new approach is up to ten times faster than the standard approach for separating $L$-shaped cuts. We are convinced that our simple and fast cut strengthening technique will be useful for general network optimization problems in the two-stage setting. We are able to solve all our benchmark instances with up to 75 vertices, 263 edges, and 40 scenarios to provably optimality in less than 3 minutes of computation time.

## 2 ILP models

**Problem Definition.** Let $G = (V, E)$ be an undirected network with known first-stage edge costs $c_e^0 \geq 0$, for all $e \in E$. Connectivity requirements as well as the costs of edges to be purchased in the second stage are known only in the second stage. These values together form a random variable $\xi$, for which we assume that it has a finite support. It can therefore be modeled using a finite set of scenarios $\mathcal{K} = \{1, \dots, K\}$, $K \geq 1$. The realization probability of each scenario is given by $p^k > 0$, $k \in \mathcal{K}$; we have $\sum_{k \in \mathcal{K}} p^k = 1$. Denote by $c_e^k \geq 0$ the cost of an edge $e \in E$ if it is bought in the second stage under scenario $k \in \mathcal{K}$. W.l.o.g. we assume that $\sum_{k \in \mathcal{K}} p^k c_e^k \geq c_e^0$, for all $e \in E$. Furthermore, let $\mathbf{r}^k$ be the matrix of *unitary* connectivity requirements under the $k$-th scenario. We denote by $E^0$ the set of edges purchased in the first-stage, and by $E^k$ the set of additional edges purchased under scenario $k$, $k \in \mathcal{K}$.

The *two-stage stochastic survivable network design problem* (SSNDP) can then be formulated as follows: Determine the subset of edges $E^0 \subseteq E$ to be purchased in the first stage, and the sets $E^k$ of additional edges to be purchased in each scenario $k \in \mathcal{K}$, such that the overall cost defined as $\sum_{e \in E^0} c_e^0 + \sum_{k \in \mathcal{K}} p^k \sum_{e \in E^k} c_e^k$ is minimized, while $E^0 \cup E^k$ satisfies all connectivity requirements between each pair of nodes defined by $\mathbf{r}^k$, for all $k \in \mathcal{K}$. Observe that each feasible solution of the deterministic SNDP is a connected subgraph of $G$, whereas the optimal first-stage solution of the SSNDP is not necessarily connected. In fact, the optimal solution may contain several disjoint fragments depending on the subsets of terminals throughout different scenarios or depending on the second-stage cost structure.

## 2.1 Undirected model

We first present a *deterministic equivalent* (in extensive form) of the SSNDP and later on show how we can strengthen the model using the ideas of *orientation*, i.e., by assigning a unique direction to each edge of a feasible second stage solution. Let binary variables $x_e^0$ indicate whether an edge $e \in E$ belongs to $E^0$, and binary second-stage variables $x_e^k$ indicate whether $e$ belongs to $E^k$, for all $k \in \mathcal{K}$. For $D \subseteq E$, let $(x^0 + x^k)(D) = \sum_{e \in D}(x_e^0 + x_e^k)$. For $S \subseteq V$, let $\delta(S) = \{\{i,j\} \in E \mid i \in S \text{ and } j \notin S\}$. A *deterministic equivalent* of the SSNDP can then be modeled using *undirected* cuts as follows:

$$(UD)\ \min \sum\nolimits_{e \in E} c_e^0 x_e^0 + \sum\nolimits_{k \in \mathcal{K}} p^k \sum\nolimits_{e \in E} c_e^k x_e^k$$

$$(x^0 + x^k)(\delta(W)) \geq \max_{i \notin W, j \in W} r_{ij}^k \qquad \forall W \subset V, \forall k \in \mathcal{K} \qquad (1)$$

$$x_e^0 + x_e^k \leq 1 \qquad \forall e \in E, \forall k \in \mathcal{K} \qquad (2)$$

$$(x^0, \ldots, x^K) \in \{0,1\}^{(K+1)|E|}$$

Constraints (1) ensure edge-connectivity between each node pair $(i,j)$ in each scenario realization. This model is an extension of one of the most prominent models from the literature for the deterministic SNDP based on undirected cuts. The associated deterministic model has been used in polyhedral studies (see, e.g., [6]) or to estimate the quality of heuristic solutions (see, e.g., [9]).

MIP models on bidirected graphs are known to provide better LP-based lower bounds for the deterministic SNDP, in particular when feasible SNDP solutions are allowed to contain two or more edge-biconnected components (see, e.g. [1, 13]). Therefore we are looking for a possibility to strengthen the model $(UD)$ by bi-directing the given graph $G$ and replacing edge- by arc-variables in the same model. The main difficulty with the SSNDP is that the edges of the first-stage solution cannot be oriented, even though the deterministic counterpart allows an orientation. We will use semi-directed models instead to overcome these difficulties and provide two MIP models that are strictly stronger than the undirected one.

## 2.2 Semi-directed model

To provide a semi-directed MIP model, we will exploit the ideas of Magnanti and Raghavan [13] used for the deterministic SNDP. If connectivity requirements are $\{0, 1, \text{even}\}$, the underlying orientation approach uses the fact that any optimal solution consists of edge-biconnected components connected with each other by bridges. Each of those edge-biconnected components can be oriented in such a way that for each pair of distinct nodes $i$ and $j$ from the same component, there exist $r_{ij}/2$ directed paths between $i$ and $j$, and $r_{ij}/2$ directed paths between $j$ and $i$ (due to the result of Nash-Williams [15]). In the orientation procedure a node $v_r$ is chosen for which we know that it is a part of an edge-biconnected component, and each bridge is oriented away from this component. In this approach we basically orient edge-biconnected components, shrink them into single nodes and orient the obtained tree away from the "root" $v_r$. The corresponding MIP model, which is stronger than the undirected one, uses binary arc variables that are associated to this orientation.

To model the general SNDP—i.e., the SNDP with arbitrary connectivity requirements $r_{ij} \in \mathbb{N} \cup \{0\}$—Raghavan and Magnanti [13] present an *extended MIP formulation*, using these orientation properties. Thereby, they use the same model from above, with the only difference that the binary arc variables are relaxed to be continuous. This change makes the model valid for arbitrary values of $r_{ij}$ and is provably stronger than its undirected counterpart.

Unfortunately, for the SSNDP, the first stage solution $E^0$ is not necessarily connected. Therefore, it is impossible to use this orientation idea for the edges from $E^0$, since each arc is not used in exactly the same direction over all scenarios. Hence, the first stage decision variables remain associated with undirected edges. However, we can provide a directed formulation once the solution gets completed in the second stage, i.e., we can "orient" the edges of $E^0 \cup E^k$—independently for each scenario. We set the root $v_r^k$ for each scenario $k \in \mathcal{K}$ to be one of the nodes with the highest connectivity requirement, and search for individual orientations of each of the $K$ scenario solutions.

By borrowing the notation from [1], let

$$\begin{aligned} \mathcal{W}_1^k =& \{W \mid W \subset V, \max_{i \in W, j \notin W} r_{ij}^k = 1, v_r^k \notin W\} \\ \hat{\mathcal{W}}^k =& \{W \cup W^c \mid W \subset V, \max_{i \in W, j \notin W} r_{ij}^k \geq 2\} \end{aligned}$$

be the set of *critical cutsets* and *regular cutsets*, respectively, with the associated values of $f^k(W)$ defined as:

$$f^k(W) = \begin{cases} 1, & W \in \mathcal{W}_1^k \\ \max_{i \in W, j \notin W} r_{ij}^k / 2, & W \in \hat{\mathcal{W}}^k \end{cases}$$

The following model orients the second stage solution, given the installation of (undirected) edges from the first stage. As above, we use variables $(x^0, \dots, x^K)$ to model the solution edges. In addition, we will use *continuous* variables $(d^1, \dots, d^K)$ associated to directed arcs, to "orient" the second stage solutions. Here, and in the following, $A$ is the set of arcs containing one directed arc for each undirected edge, i.e., $\forall \{i,j\} \in E : (i,j), (j,i) \in A$. For $S \subseteq V$ let $\delta^-(S) = \{(i,j) \in A | i \notin S, j \in S)\}$ and analogously $\delta^+(S) = \{(i,j) \in A | i \in S, j \notin S)\}$. The first semi-directed model will be called *SD₁*:

$$(SD_1) \ \min \sum\nolimits_{e \in E} c_e^0 x_e^0 + \sum\nolimits_{k \in \mathcal{K}} p^k \sum\nolimits_{e \in E} c_e^k x_e^k$$

$$\begin{aligned} x^0(\delta(W)) + d^k(\delta^-(W)) + d^k(\delta^+(W)) &\geq 2 f^k(W), & \forall W \in \hat{\mathcal{W}}^k, \forall k \in \mathcal{K} && (3) \\ x^0(\delta(W)) + d^k(\delta^-(W)) &\geq 1, & \forall W \in \mathcal{W}_1^k, \forall k \in \mathcal{K} && (4) \\ d_{ij}^k + d_{ji}^k &\leq x_e^k, & \forall e = \{i,j\} \in E, \forall k \in \mathcal{K} && (5) \\ x_e^k + x_e^0 &\leq 1, & \forall e \in E, \forall k \in \mathcal{K} && (6) \\ d_{ij}^k &\geq 0, & \forall (i,j) \in A, \forall k \in \mathcal{K} && (7) \\ (x^0, \dots, x^K) &\in \{0,1\}^{(K+1)|E|} \end{aligned}$$

Constraints (3) ensure that there are at least $2f^k(W)$ edge-disjoint paths between $W$ and $V \backslash W$ consisting of first- and second-stage edges. Due to constraints (4) there is at least one path to each vertex $i$ with $r_{ij}^k > 0, \forall j \neq i$, from the root node. If nothing has been purchased in the first stage, then constraints (4) associated to bridges will force the orientation of those bridges away from the root node $v_r^k$. Furthermore, since variables $d_{ij}^k$ are fractional, by using the same arguments as in [13], the model is valid for any $r_{ij}^k \in \mathbb{N} \cup \{0\}$. Hence, we have the following lemma.

**Lemma 1.** *Formulation ($SD_1$) models the deterministic equivalent of the two-stage stochastic survivable network design problem correctly.*

*Proof.* Let $\boldsymbol{x} := (x^0, x^1, \dots x^K)$ be a feasible solution to an SSNDP instance. Notice that constraints (3) can be restated as undirected cuts due to constraints (5): $x^0(\delta(W)) + x^k(\delta(W)) \geq x^0(\delta(W)) + d^k(\delta^-(W)) + d^k(\delta^+(W)) \geq 2f^k(W)$. Since $\boldsymbol{x}$ is a feasible solution there are $r_{ij}^k$ edge-disjoint path between $i$ and $j$ in scenario $k$ using edges in $x^0$ and $x^k$. Therefore, all undirected cuts and hence, all constraints (5) are satisfied. Due to the result of Nash-Williams [15] and Magnanti & Raghavan [13], it is possible to find an orientation in each scenario for the sum of the variables $x_e^0 + x_e^k$ to satisfy the constraints (4). Following this orientation the values for the directed variables $d^k$ can be set satisfying the second type of cuts. Non-negativity and all other constraints follow directly.

To show the converse, assume $(x^0, x^1, \dots x^K, d^1, \dots d^K)$ is a feasible solution to ($SD_1$). Due to the previous discussion concerning constraints (3) it is clear that all edge connectivity requirements with connectivity $\geq 2$ are satisfied. Furthermore, constraints (4) ensure that there is at least one (semi-directed) path from the root node to each vertex with requirement one. Hence, all connectivity requirements are satisfied and each solution to ($SD_1$) is a valid solution to SSNDP. □

Let $\text{Proj}_{(x^0,\dots,x^K)}(\mathcal{P}_{SD_1})$ denote the projection of the polytope defined by the LP-relaxation of ($SD_1$) onto the space of $(x^0, x^1, \dots, x^K)$ variables given as $x^0 := \hat{x}^0$, and $x^k := \hat{x}^k, \forall k \in \mathcal{K}$. Let $\mathcal{P}_{UD}$ be the polytope of the LP-relaxation of ($UD$).

**Lemma 2.** *The semi-directed formulation ($SD_1$) is provably stronger than the undirected formulation ($UD$), i.e., $\text{Proj}_{(x^0,\dots,x^K)}(\mathcal{P}_{SD_1}) \subsetneq \mathcal{P}_{UD}$ and there exist instances for which the strict inequality holds.*

*Proof.* Let $(\hat{x}^0, \hat{x}^1 \dots, \hat{x}^K, d^1, \dots, d^K)$ be a feasible solution of ($SD_1$) with objective value $z_{SD_1}$. Obviously, the projected vector $(x_e^0, x^1, \dots, x_e^K)$ is a valid solution to ($UD$) with the same objective value as $z_{SD_1}$. The feasibility of the undirected cuts follows directly since $(x^0 + x^k)(\delta(W)) = \sum_{e \in \delta(W)}(x_e^0 + x_e^k) = \sum_{e \in \delta(W)}(\hat{x}_e^0 + \hat{x}_e^k) \geq \sum_{e \in \delta(W)}(\hat{x}_e^0 + d_{ij}^k + d_{ji}^k) = \hat{x}^0(\delta(W)) + d^k(\delta^-(W)) + d^k(\delta^+(W)) \geq 2f^k(W) \geq \max_{i \in W, j \notin W} r_{ij}^k$, for each $W \subset V$, $k \in \mathcal{K}$.

The strict inequality of the previous lemma is shown by the example given in Figure 1(a). Assume that we have 2 scenarios with equal probability, $r_{01}^1 = r_{02}^1 = 1$, $r_{03}^2 = 1$, and $c_e^0 = 10, c_e^k = 12, \forall e \in E, k \in \{1, 2\}$. The optimum solution of ($UD$) buys edges only in the second stage with a total objective value of 15: $x_{01}^1 = x_{12}^1 = x_{02}^1 = 0.5$ and $x_{03}^1 = 1$. On the other hand, this solution is infeasible for the relaxed model of ($SD_1$), i.e., there is no solution to ($SD_1$) with the same objective value. □

## 2.3 Stronger semi-directed formulation

In the following model, which represents an alternative model to ($SD_1$), binary variables $y_e^k$ are used to model the second-stage solution. These variables include the edges that are already bought in the first stage, i.e., we have $y_e^k = 1$ if $e \in E^0 \cup E^k$, and $y_e^k = 0$, otherwise. To "orient" the edges from $E_0 \cup E_k$, as above, continuous variables $z_{ij}^k$ are used. The model will be called *SD2*:

$$(SD_2)\ \min \sum\nolimits_{e \in E} c_e^0 x_e^0 + \sum\nolimits_{k \in \mathcal{K}} p^k \sum\nolimits_{e \in E} c_e^k (y_e^k - x_e^0)$$

$$\begin{aligned}
z^k(\delta^-(W)) &\geq f^k(W), && \forall W \in \mathcal{W}_1^k \cup \hat{\mathcal{W}}^k, \forall k \in \mathcal{K} && (8)\\
z_{ij}^k + z_{ji}^k &\geq x_e^0, && \forall e = \{i,j\} \in E, \forall k \in \mathcal{K} && (9)\\
z_{ij}^k + z_{ji}^k &\leq y_e^k, && \forall e = \{i,j\} \in E, \forall k \in \mathcal{K} && (10)\\
z_{ij}^k &\geq 0, && \forall (i,j) \in A, \forall k \in \mathcal{K} && (11)\\
(x^0, y^1, \ldots, y^K)^T &\in \{0,1\}^{(K+1)|E|}
\end{aligned}$$

Constraints (8) model the "orientation" of the solution, independently for each of the scenarios. Variables $z_{ij}^k$ are fractional, and therefore, the model is valid for any $r_{ij}^k \in \mathbb{N} \cup \{0\}$. Finally, constraints (9) and (10) ensure that variables $z_{ij}^k$ can be used only along the edges that are either purchased in the first stage, or added in the second stage.

**Lemma 3.** *Formulation ($SD_2$) models the deterministic equivalent of the two-stage stochastic survivable network design problem correctly.*

*Proof.* Again, let $(x^0, x^1, \ldots x^K)$ be a feasible solution to an SSNDP instance. Following the ideas of the proof of Lemma 1 it is possible to find an orientation for the directed variables $z^k$ using edge capacities $(x^0 + x^k)$ for each scenario and hence create a valid solution for ($SD_2$).

Conversely, a feasible solution $(x^0, y^1, \ldots, y^K, z^1, \ldots z^K)$ to ($SD_2$) satisfies all edge connectivity requirements in each scenario due to the directed formulation and variables $z^k$. Thereby, $z_{ij}^k \leq y_e^k$ and analogously to the deterministic case $(x^0, (x^k = x^0 - y^k)_{k=1,\ldots,K})$ implies a feasible solution to SSNDP. □

Let $\mathrm{Proj}_{(x^0,\ldots,x^K)}(\mathcal{P}_{SD_2})$ denote the projection of the polytope of the LP-relaxation of ($SD_2$) onto the space of $(x^0, x^1, \ldots, x^K)$ variables with $x^k := y^k - x^0$.

**Lemma 4.** *The semi-directed formulation ($SD_2$) is provably stronger than the semi-directed formulation ($SD_1$), i.e., $Proj_{(x^0,\ldots,x^K)}(\mathcal{P}_{SD_2}) \subsetneq Proj_{(x^0,\ldots,x^K)}(\mathcal{P}_{SD_1})$ and there exist instances for which the strict inequality holds.*

*Proof.* Let $(\hat{x}^0, y^1 \ldots, y^K, z^1, \ldots, z^K)$ be a feasible solution of ($SD_2$) with objective value $z_{SD_2}$. For each $k \in \mathcal{K}$, $(i,j) \in A, e = \{i,j\} \in E$ let $\lambda_{ij}^k := 0$ if $z_{ij}^k + z_{ji}^k = 0$ and $\lambda_{ij}^k := z_{ij}^k/(z_{ij}^k + z_{ji}^k)$, otherwise. Hence, $\lambda_{ij}^k + \lambda_{ji}^k = 1, \forall (i,j) \in A$ with $z_{ij}^k + z_{ji}^k > 0$. Moreover, $x^0 := \hat{x}^0, x^k := y^k - x^0$ and $d_{ij}^k := z_{ij}^k - \lambda_{ij}^k x_e^0$.

Obviously, the objective value of this ($SD_1$)-solution is equal to $z_{SD_2}$. Connectivity constraints (3) are satisfied since for each $W \in \hat{\mathcal{W}}^k$, $k \in \mathcal{K}$, we have: $x^0(\delta(W)) + d^k(\delta^-(W)) + d^k(\delta^+(W)) = \hat{x}^0(\delta(W)) + \sum_{e=\{i,j\} \in \delta(W)} (z_{ij}^k - \lambda_{ij}^k x_e^0 + z_{ji}^k - \lambda_{ji}^k x_e^0) = \hat{x}^0(\delta(W)) + z^k(\delta^-(W)) + z^k(\delta^+(W)) - \hat{x}^0(\delta(W)) \geq 2f^k(W)$. The 1-connectivity constraints (4) are also fulfilled since for each $W \in \mathcal{W}_1^k$, $k \in \mathcal{K}$, we have: $x^0(\delta(W)) + d^k(\delta^-(W)) = \hat{x}^0(\delta(W)) + \sum_{(i,j) \in \delta^-(W)} (z_{ij}^k - \lambda_{ij}^k x_e^0) \geq z^k(\delta^-(W)) \geq f^k(W)$. All remaining constraints (5)–(7) are also satisfied: $\forall k \in \mathcal{K}, e \in E, (i,j) \in A$: (5): $d_{ij}^k + d_{ji}^k = z_{ij}^k + z_{ji}^k - \hat{x}_e^0 \leq y_e^k - x_e^0 = x_e^k$, (6): $x_e^k + x_e^0 = y_e^k - \hat{x}_e^0 + \hat{x}_e^0 \leq 1$, and

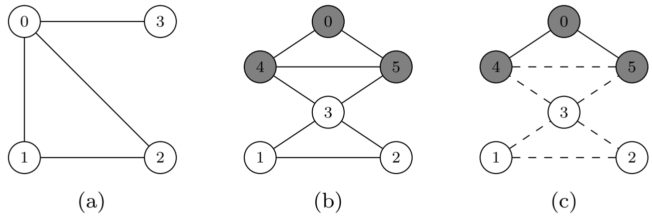


Figure 1: Two counterexamples that prove the strength of the new formulations. 1(a) instance with $LP(SD_1) > LP(UD)$, 1(b) instance with $LP(SD_2) > LP(SD_1)$, and 1(c) the optimal LP-solution of $(SD_1)$: A solid line represents an LP value of 1, a dashed line a value of 0.5. Gray nodes have connectivity requirement two, all other nodes connectivity requirement one.

$d^k$ variables are non-negative: $d^k_{ij} = z^k_{ij} - (z^k_{ij}/(z^k_{ij}+z^k_{ji}))\hat{x}^0_e \geq z^k_{ij} - (z^k_{ij}/(z^k_{ij}+z^k_{ji}))(z^k_{ij}+z^k_{ji}) = 0$. Hence, $(x^0, x^1, \ldots, x^K, d^1, \ldots, d^K)$ is a feasible solution for $(SD_1)$ with the same objective value.

To show that there exist instances for which the strict inequality holds, consider the input graph shown in Figure 1(b). We assume that the input consists of a single scenario in which the gray nodes require two-connectivity and the remaining ones only simple connectivity. Furthermore, all edge costs are 1 in the first stage and 10 in the second stage. The LP solution shown in Figure 1(c) shows the first-stage solution (nothing needs to be purchased in the second stage) with a total objective value of 5. This solution is valid for the model $(SD_1)$ but it is impossible to "orient" this solution such that it becomes feasible for the model $(SD_2)$. □

# 3 Decomposition

Notice that, even for a constant number of scenarios, our model contains an exponential number of constraints that can be typically separated in a cutting-plane fashion (e.g., in a single-stage branch&cut approach). The main drawback of such a branch&cut approach is that we still have to deal with a large set of variables. Alternatively, in [4] we have proposed to combine the cutting plane algorithm with a Benders-like decomposition approach in which the variables of the first stage are kept in the master problem, and the second stage variables are projected out and replaced by a single variable per scenario ($\Theta^k$). The objective function of the decomposed model becomes: $\min c^0 x^0 + \sum_{k\in\mathcal{K}} p_k \Theta^k$, where $\Theta^k$ represents the lower bound on the value of the second stage subproblem in scenario $k$. For a fixed first stage decision $\tilde{x}^0$, the problem decomposes into $K$ subproblems, each of which can be independently solved using a branch&cut approach. Dual variables of the LP-relaxations of these subproblems impose $L$-shaped cuts that are added to the master while the exact solutions of the subproblems impose integer $L$-shaped cuts [12, 19]. This new decomposition approach that combines the branch&cut in the master problem with the branch&cut in the subproblems is called *two-stage branch&cut algorithm*. Computational results of [4], applied to the stochastic Steiner tree problem, have shown that two-stage branch&cut significantly outperforms the branch&cut applied directly to the deterministic equivalent in extended formulation.

Hence, in this paper we propose to solve the SSNDP using the two-stage branch&cut from [4] applied to the stronger of the two semi-directed models, namely $(SD_2)$. The main details of this decomposition approach are provided below.

For each fixed—and possibly fractional—first-stage solution $\tilde{x}^0$, the second-stage problem decomposes into $K$ independent subproblems, which we will refer to as *restricted deterministic SNDP's*. For each $k \in \mathcal{K}$, these subproblems are given as follows:

$$(P{:}SD_2)\ \min \sum\nolimits_{e\in E} c_e^k(y_e^k - \tilde{x}_e^0)$$

$$\begin{aligned}
z^k(\delta^-(W)) &\geq f^k(W), && \forall W \in \mathcal{W}_1^k \cup \hat{\mathcal{W}}^k c && (12)\\
z_{ij}^k + z_{ji}^k &\geq \tilde{x}_e^0, && \forall e = \{i,j\} \in E && (13)\\
y_e^k - z_{ij}^k - z_{ji}^k &\geq 0 && \forall e = \{i,j\} \in E && (14)\\
z_{ij}^k &\geq 0 && \forall (i,j) \in A && (15)\\
-y_e^k &\geq -1, && \forall e \in E && (16)\\
y^k &\in \{0,1\}^{|E|}
\end{aligned}$$

By removing the integrality constraints and using dual variables $\alpha_W$, $\beta_e$, $\gamma_e$, $\eta_{ij}$ and $\tau_e$ associated to constraints (12), (13), (14), (15), and (16), respectively, we obtain the following dual problem, for each fixed $k \in \mathcal{K}$ and the first stage solution $\tilde{x}^0$:

$$(D{:}SD_2)\ \max \sum\nolimits_{W\in\mathcal{W}_1^k\cup\hat{\mathcal{W}}^k} f^k(W)\alpha_W + \sum\nolimits_{e\in E}(\tilde{x}_e^0\beta_e - c_e^k\tilde{x}_e^0 - \tau_e)$$

$$\begin{aligned}
\gamma_e - \tau_e &\leq c_e^k, \quad \forall e \in E && (17)\\
\sum_{W\in\mathcal{W}_1^k\cup\hat{\mathcal{W}}^k:(i,j)\in\delta^-(W)} \alpha_W + \beta_e - \gamma_e + \eta_{ij} &\leq 0, \quad \forall (i,j) \in A && (18)\\
(\alpha,\beta,\gamma,\eta,\tau) &\geq 0 && (19)
\end{aligned}$$

Let $\tilde{\alpha}_W, \tilde{\beta}_e, \tilde{\gamma}_e, \tilde{\eta}_{ij}, \tilde{\tau}_e$ be an optimal solution to $(D{:}SD_2)$. A (decomposed) *L-shaped optimality cut* is then defined as follows:

$$\Theta^k + \sum\nolimits_{e\in E} x_e^0(c_e^k - \tilde{\beta}_e) \geq \sum\nolimits_{W:W\in\mathcal{W}_1^k\cup\hat{\mathcal{W}}^k} f^k(W)\tilde{\alpha}_W - \sum\nolimits_{e\in E}\tilde{\tau}_e \qquad (20)$$

Rounded $L$-shaped cuts are obtained by replacing the coefficients of $x_e^0$ by $\min(c_e^k - \tilde{\beta}_e, \sum_{W:W\in\mathcal{W}_1^k\cup\hat{\mathcal{W}}^k} f^k(W)\tilde{\alpha}_W - \sum_{e\in E}\tilde{\tau}_e)$, for each $e \in E$ and $k \in \mathcal{K}$.

### 3.1 Two-stage branch&cut algorithm

Let *RMP* denote the relaxed master problem, i.e., $\min c^0x^0 + \sum_{k\in\mathcal{K}} p_k\Theta^k$ s.t. additional separated L-shaped and integer optimality cuts. Furthermore, $RSP^k$ denotes the relaxed subproblem (restricted deterministic SNDP) of scenario $k$, $k \in \mathcal{K}$. A brief description of the algorithm is given as follows:

**Step 0: Initialization.** $UB = +\infty$ (global upper bound, corresponding to a feasible solution), $\nu = 0$. Create the first pendant node. In the initial *RMP* the set of (integer) L-shaped cuts is empty.

**Step 1: Selection.** Select a pendant node from the b&b tree, if such a node exists, otherwise STOP.

**Step 2: Separation.** Solve the *RMP* at the current node. $\nu = \nu + 1$. Let $(x^\nu, \Theta_1^\nu, \ldots, \Theta_K^\nu)$ be the current optimal solution, $\Theta^\nu = \sum_{k\in\mathcal{K}} p_k \Theta_k^\nu$.

**(2.1)** If $c^t x^\nu + \Theta^\nu > UB$ fathom the current node and goto Step 1.

**(2.2)** Search for **violated L-shaped cuts**:
For all $k \in \mathcal{K}$, compute the LP-relaxation value $R(x^\nu, k)$ of $RSP^k$. If $R(x^\nu, k) > \Theta_k^\nu$: insert L-shaped cut (20) into *RMP*.
If at least one L-shaped cut was inserted goto Step 2.

**(2.3)** If $x$ is binary, search for **violated integer L-shaped cuts**:

**(2.3.1)** For all $k \in \mathcal{K}$ s.t. $z^k$ is not binary in the previously computed LP-relaxation, solve $RSP^k$ to integer optimality. Let $Q(x^\nu, k)$ be the optimal $RSP^k$ value.
If $\sum_{k\in\mathcal{K}} p_k Q(x^\nu, k) > \Theta^\nu$ insert integer L-shaped cut (21) into *RMP*. Goto Step 2.

**(2.3.2)** $UB = \min(UB, c^t x^\nu + \Theta^\nu)$. Fathom the current node and goto Step 1.

**Step 3: Branching.** Using a branching criterion, create two nodes, append them to the list of pendant nodes, goto Step 1.

**Integer L-shaped cuts.** Let $x^\nu$ be a binary first stage solution with its corresponding optimal second stage value $Q(x^\nu) = \sum_{k\in\mathcal{K}} p_k Q(x^\nu, k)$. Let $\mathcal{I}^\nu := \{e \in E : x_e^\nu = 1\}$ be the index set of the edge variables chosen in the first stage, and the constant $L$ be a known lower bound of the recourse function (before branching: $L = 0$). We want to explicitly cut off the solution $(x^\nu, \Theta^\nu)$ by inserting the general integer optimality cuts of the L-shaped scheme [12]:

$$\Theta \geq (Q(x^\nu) - L)\left(\sum\nolimits_{e\in\mathcal{I}^\nu} x_e - \sum\nolimits_{e\in E\setminus\mathcal{I}^\nu} x_e - |\mathcal{I}^\nu| + 1\right) + L. \tag{21}$$

# 4 Strengthening $L$-shaped cuts

Notice that the number of master iterations of our decomposition approach —and hence, the overall running time—is highly influenced by the strength of the generated $L$-shaped cuts. In this paper we propose a new and fast way of generating strengthened $L$-shaped cuts. In contrast to the previously proposed strengthening approaches (cf. [5, 14, 16, 17, 20]), we do not require solving an auxiliary LP in order to generate a stronger cut, but rather, we are able to find it in linear time.

Instead of solving additional LPs, the L-shaped cuts for the formulation ($SD_2$) of the SSNDP can be strengthened as follows: If for an edge $e \in E$ the first stage solution $\tilde{x}^0$ is such that $\tilde{x}_e^0 = 0$, then the corresponding $\beta_e$ variable does not appear in the objective function of the dual ($SD_2$). Furthermore, variables $\gamma_e$ and $\eta_{ij}$ do not appear in the objective function neither, and therefore, LP-optimal solutions frequently have a positive slack in constraints (18). Our idea is to reduce this slack to zero, and to thereby increase the value of $\beta_e$ as follows:
Let $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\eta}, \tilde{\tau})$ be an optimal solution to ($D$:$SD_2$) as before. Let

$$\hat{\beta}_e := \tilde{\gamma}_e - \max_{a\in\{(i,j),(j,i)\}}\{\sum_{W:a\in\delta^-(W)} \tilde{\alpha}_W - \tilde{\eta}_a\}, \quad \forall e = \{i,j\} \in E \text{ s.t. } \tilde{x}_e^0 = 0.$$

If $\hat{\beta}_e > \tilde{\beta}_e$ holds for at least one $e \in E$ the *strengthened L-shaped cut* is given as:

$$\Theta^k + \sum\nolimits_{e\in E} x_e^0 (c_e^k - \hat{\beta}_e) \geq \sum\nolimits_{W:W\in\mathcal{W}_1^k\cup\hat{\mathcal{W}}^k} f^k(W)\tilde{\alpha}_W - \sum_{e\in E} \tilde{\tau}_e. \tag{22}$$

**Lemma 5.** *The strengthened L-shaped* (22) *cuts are valid and strictly stronger than the standard L-shaped cuts* (21).

*Proof.* Consider two L-shaped cuts: the standard one, implied by the dual solution $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\eta}, \tilde{\tau})$ and the strengthened one $(\tilde{\alpha}, \hat{\beta}, \tilde{\gamma}, \tilde{\eta}, \tilde{\tau})$ with $\hat{\beta}$ being set as described above. Obviously, $(\tilde{\alpha}, \hat{\beta}, \tilde{\gamma}, \tilde{\eta}, \tilde{\tau})$ is a feasible (and LP-optimal) solution to the dual subproblem $(D{:}SD_2)$ since $\hat{\beta}$ is set without violating the dual constraints. Furthermore, notice that $\hat{\beta}_e \geq \tilde{\beta}_e$, for all $e \in E$. The right-hand-side of both cuts is identical and since there exist $e' \in E$ such that $\hat{\beta}_{e'} > \tilde{\beta}_{e'}$, the coefficient of $x^0_{e'}$ is strictly smaller for the strengthened $L$-shaped cut than for the standard one. □

# 5 Computational Results

We implemented the decomposition of model $(SD_2)$ using Abacus 3.0 as a generic branch&cut framework. We use IBM CPLEX (version 12.1) via the interface COIN-Osi 0.102 as LP solver. All experiments were performed on an Intel Core-i7 2.67 GHz Quad Core machine with 12 GB RAM under Ubuntu 11.4. Each run was performed on a single core.

Deterministic instances were generated by adopting the idea of Johnson, Minkoff, and Phillips [10], which is frequently used as benchmark in the network design community. After randomly distributing $n \in \{20, 30, 40, 50, 75\}$ points in the unit square, a minimum spanning tree is computed using the points as vertices and the Euclidean distances between all vertex pairs as edge costs. This MST is extended by adding all edges for which the Euclidean length is less than or equal to $1.6\alpha/\sqrt{n}$. We have introduced $\alpha$ in order to control the density of the graph[1]. In our experiments we use $\alpha = 0.9$ which leads to graphs with average density 2.77. The edge connectivity requirements are set as follows. We have randomly drawn $\rho\%$ of the vertices as $R_1$ and $R_2$ customers each with edge connectivity requirement 1 and 2, respectively. Then, for two vertices $i$ and $j$ let $r_{ij} = 2$ if both $i$ and $j$ are in $R_2$, $r_{ij} = 1$ if one of them is in $R_1$ and the other in $R_1 \cup R_2$, and $r_{ij} = 0$, otherwise. Here, we use $\rho = 40$ and we additionally introduce a random root node that is contained in $R_2$.

To transform these instances into stochastic ones we randomly and independently generate $\bar{k}$ scenarios. The probabilities are set by randomly distributing 10,000 points over the scenarios, where each point corresponds to a probability of 0.01%. Edge costs $c^0$ in the first stage are Euclidean distances and in the second stage for each edge $e$ and scenario $k \in \mathcal{K}$ randomly drawn from $[1.1c^0_e, 1.3c^0_e]$. Edge connectivities are generated by randomly drawing $\rho_k\%$ from the vertex sets $R_1$ and $R_2$ each as $R^k_1$ and $R^k_2$ customers, respectively, for scenario $k$. Here, we use $\rho_k = 30$ for all scenarios $k$. The special root node was set to be an $R^k_2$ node in each scenario $k$.

For each deterministic instance we generated a stochastic instance with $\bar{k} = 40$ scenarios and take the first $k$ to obtain an SSNDP instance, $k \in \{5, 10, 20, 30, 40\}$; probabilities for the scenarios of the instances with $k < 40$ are scaled appropriately. Overall, we generated 5 instances for each pair of $n$ and $k$[2].

To analyze the benefit of using strengthened L-shaped cuts we compare the computation time as well as the number of master iterations. Figure 2 depicts the computation time of the two-stage branch&cut algorithm with standard and strengthened L-shaped cuts, respectively. Each data point is the average over 5 runs and 5 instances per $k$ scenarios, $k \in \{5, 10, 20, 30, 40\}$ for each number of nodes $n \in \{20, 30, 40, 50, 75\}$. We considered only feasible SSNDP instances which are solved with the standard L-shaped cuts in less than 1 hour computation time.

[1] The original parameter used by [10] was 1.6 and corresponds to $\alpha = 1$ in our setting

[2] These 125 instances can be downloaded from our SSNDP webpage, see [18]

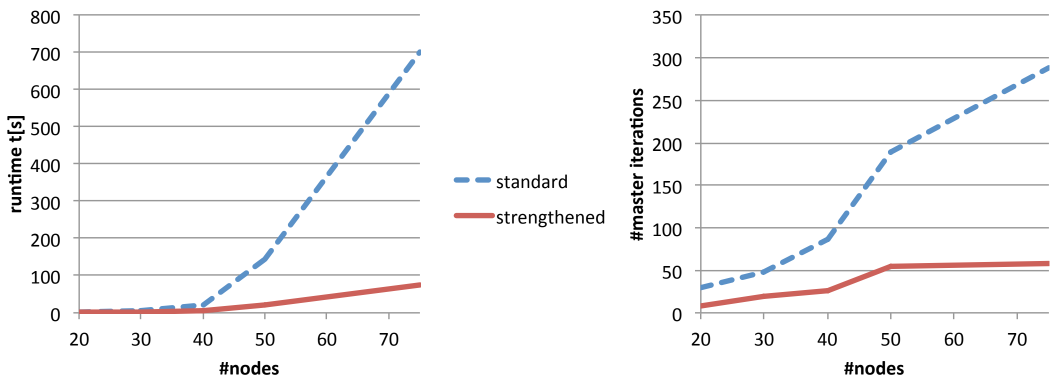


Figure 2: Comparison of the runtime (grouped by the number of nodes) and number of master iterations between two-stage branch&cut with standard and with strengthened L-shaped cuts, respectively.

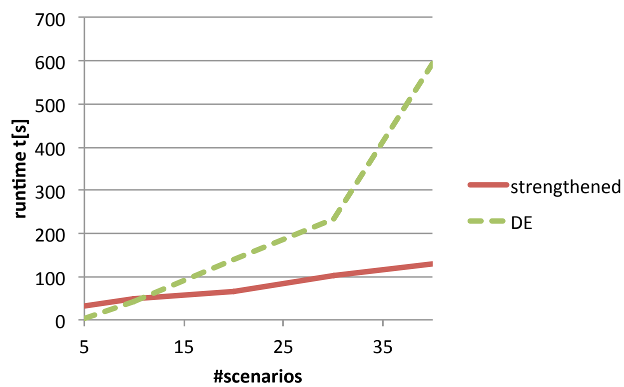


Figure 3: Comparison of the runtime (grouped by the number of scenarios) between the two-stage branch&cut with strengthened L-shaped cuts and the extended formulation of the deterministic equivalent, respectively for the graphs with $n = 75$ nodes. Only running times less than the time limit of 2 hours are considered.

The gained speedup in running time by using the strengthened L-shaped cuts is significant: It is about 4 (for $n \in \{20, 30, 40\}$), 6 ($n = 50$), and over 9 ($n = 75$) times, respectively. For example, instances with 75 vertices are solved in approx. 12 minutes without and in 1:30 minutes with the strengthening (on average). The number of master iterations also decreases rapidly, e.g., for $n = 75$ from 288 to 58 on average.

Table 1 shows detailed results for instances of size $n = 75$. The first three columns give a description of the instance (number of edges $e$, scenarios $K$ and optimum solution value OPT*). The 4th to 19th column give the main results of the two-stage branch&cut algorithm with standard and strengthened L-shaped cuts, respectively, and the extended formulation of the deterministic equivalent (DE): the running time in seconds (t[s]) with a time-limit of two hours (2h), the number of branch&bound nodes (b&b), the number of iterations of the master problem (iter), the number of inserted L-shaped cuts (cuts), the value of the relaxed LP in the root LP root, and the gap between the optimum solution and the LP root.

It can be observed that the number of iterations, required L-shaped cuts, and the running time decreases drastically when the strengthened cuts are used in comparison to the standard cuts. Furthermore, the number of branch&bound nodes (b&b) is mostly only decreasing slightly and the relaxed solution in the root node is almost identical.

The benefit of using the decomposition over the extended formulation (EF) is similar to the results presented for the stochastic Steiner tree problem [4]. For small instances ($k \leq 20$ or $n \leq 40$) the EF has a faster computation time. But with an increasing number of scenarios or graph size the decomposition clearly outperforms the EF which, e.g., was not able to solve all of the instances with $n \geq 75$ in two hours, c.f. Figure 3.

| instance | | | standard | | | | | | strengthened | | | | | | DE | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| e | $K$ | OPT* | $t[s]$ | iter | cuts | b&b | LP root | gap | $t[s]$ | iter | cuts | b&b | LP root | gap | $t[s]$ | b&b | LP root | gap |
| 208 | 5 | 5989.54 | 134.66 | 470 | 2287 | 5 | 5989.51 | $< 0.1$ | 20.98 | 112 | 555 | 1 | 5989.54 | - | 0.50 | 1 | 5989.54 | - |
| 208 | 10 | 6144.79 | 371.23 | 546 | 3951 | 33 | 6143.86 | $< 0.1$ | 38.69 | 99 | 944 | 3 | 6143.9 | $< 0.1$ | 3.76 | 5 | 6143.93 | $< 0.1$ |
| 208 | 20 | 6161.15 | 422.55 | 346 | 4865 | 3 | 6161.13 | $< 0.1$ | 90.00 | 117 | 1602 | 9 | 6161.12 | $< 0.1$ | 4.99 | 1 | 6161.15 | - |
| 208 | 30 | 6222.57 | 376.07 | 197 | 5621 | 3 | 6221.56 | $< 0.1$ | 94.91 | 76 | 2045 | 5 | 6221.58 | $< 0.1$ | 23.44 | 3 | 6221.59 | $< 0.1$ |
| 208 | 40 | 6171.61 | 613.38 | 236 | 7444 | 5 | 6171.61 | - | 79.97 | 49 | 1920 | 1 | 6171.61 | - | 23.58 | 1 | 6171.61 | - |
| 241 | 5 | 4715.84 | 841.46 | 659 | 3290 | 1 | 4715.84 | - | 74.13 | 128 | 635 | 1 | 4715.84 | - | 5.91 | 1 | 4715.84 | - |
| 241 | 10 | 4701.92 | 1027.81 | 427 | 4210 | 7 | 4701.15 | $< 0.1$ | 117.22 | 99 | 930 | 7 | 4701.15 | $< 0.1$ | 177.34 | 147 | 4701.15 | $< 0.1$ |
| 241 | 20 | 4550.35 | 918.52 | 223 | 4216 | 3 | 4549.92 | $< 0.1$ | 79.86 | 41 | 740 | 3 | 4549.92 | $< 0.1$ | 415.58 | 75 | 4549.92 | $< 0.1$ |
| 241 | 30 | 4545.76 | 1685.30 | 368 | 6585 | 11 | 4541.96 | $< 0.1$ | 174.26 | 61 | 1400 | 11 | 4541.96 | $< 0.1$ | 2h | - | 4541.96 | $< 0.1$ |
| 241 | 40 | 4568.98 | 2125.82 | 299 | 8022 | 19 | 4567.56 | $< 0.1$ | 279.61 | 71 | 1772 | 15 | 4567.56 | $< 0.1$ | 2h | - | 4567.56 | $< 0.1$ |
| 244 | 5 | 4390.57 | 381.66 | 463 | 2295 | 3 | 4390.25 | $< 0.1$ | 25.45 | 81 | 385 | 3 | 4390.25 | $< 0.1$ | 2.75 | 3 | 4390.25 | $< 0.1$ |
| 244 | 10 | 4466.16 | 404.20 | 273 | 2720 | 1 | 4466.16 | - | 32.52 | 49 | 480 | 1 | 4466.16 | - | 9.72 | 1 | 4466.16 | - |
| 244 | 20 | 4476.55 | 612.82 | 206 | 4080 | 1 | 4476.55 | - | 56.45 | 40 | 760 | 1 | 4476.55 | - | 146.45 | 19 | 4476.55 | - |
| 244 | 30 | 4473.63 | 838.42 | 182 | 5190 | 5 | 4472.65 | $< 0.1$ | 110.27 | 49 | 1230 | 5 | 4472.65 | $< 0.1$ | 448.31 | 27 | 4472.65 | $< 0.1$ |
| 244 | 40 | 4458.77 | 886.81 | 146 | 5560 | 1 | 4458.77 | - | 97.70 | 32 | 1200 | 1 | 4458.77 | - | 732.41 | 19 | 4458.77 | - |
| 251 | 5 | 4041.37 | 399.93 | 394 | 1945 | 5 | 4041.37 | 0.14 | 24.17 | 72 | 334 | 5 | 4035.51 | 0.14 | 3.90 | 7 | 4035.51 | 0.14 |
| 251 | 10 | 4240.28 | 497.65 | 257 | 2540 | 3 | 4236.44 | $< 0.1$ | 35.65 | 45 | 420 | 3 | 4236.44 | $< 0.1$ | 15.11 | 3 | 4236.44 | $< 0.1$ |
| 251 | 20 | 4349.39 | 666.28 | 161 | 3200 | 1 | 4349.39 | - | 49.78 | 24 | 460 | 1 | 4349.39 | - | 73.74 | 1 | 4349.39 | - |
| 251 | 30 | 4363.77 | 1023.44 | 176 | 4411 | 1 | 4363.77 | - | 79.12 | 25 | 661 | 1 | 4363.77 | - | 345.86 | 7 | 4363.77 | - |
| 251 | 40 | 4381.79 | 1087.63 | 129 | 4979 | 1 | 4381.79 | - | 106.61 | 23 | 807 | 1 | 4381.79 | - | 1277.92 | 59 | 4381.79 | - |
| 259 | 5 | 4109.29 | 195.69 | 339 | 1680 | 3 | 4108.47 | $< 0.1$ | 12.08 | 43 | 200 | 3 | 4108.47 | $< 0.1$ | 1.91 | 3 | 4108.47 | $< 0.1$ |
| 259 | 10 | 4286.03 | 250.11 | 220 | 2170 | 3 | 4285.34 | $< 0.1$ | 19.18 | 32 | 290 | 3 | 4285.34 | $< 0.1$ | 8.56 | 3 | 4285.34 | $< 0.1$ |
| 259 | 20 | 4418.09 | 505.51 | 189 | 3720 | 3 | 4416.08 | $< 0.1$ | 54.81 | 35 | 640 | 3 | 4416.08 | $< 0.1$ | 64.18 | 3 | 4416.08 | $< 0.1$ |
| 259 | 30 | 4490.25 | 577.74 | 154 | 4590 | 1 | 4490.25 | - | 58.42 | 25 | 720 | 1 | 4490.25 | - | 121.65 | 1 | 4490.25 | - |
| 259 | 40 | 4517.68 | 733.39 | 144 | 5639 | 3 | 4517.04 | $< 0.1$ | 88.37 | 30 | 1080 | 3 | 4517.04 | $< 0.1$ | 337.25 | 3 | 4517.04 | $< 0.1$ |

Table 1: Detailed results for the instances with $n = 75$

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