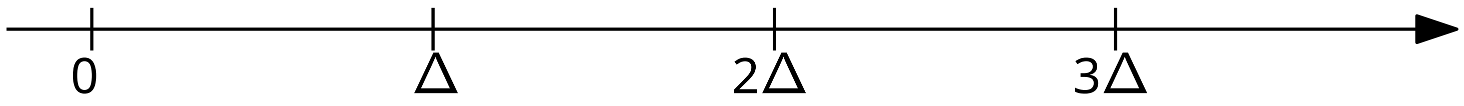
Shifting a grid over a point set

for simple and fast approximation algorithms

Shifted partition of the real line

Let $\Delta > 0$ and $b \in [0, \Delta]$ uniformly distributed.

We shift the grid G_Δ by b



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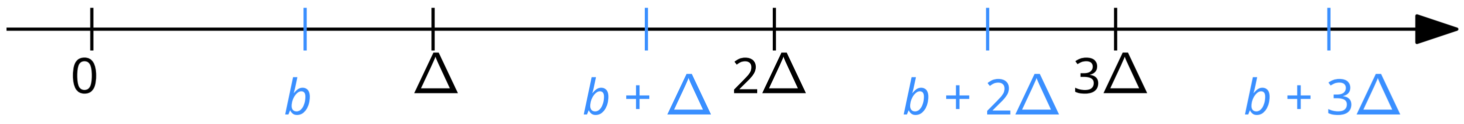

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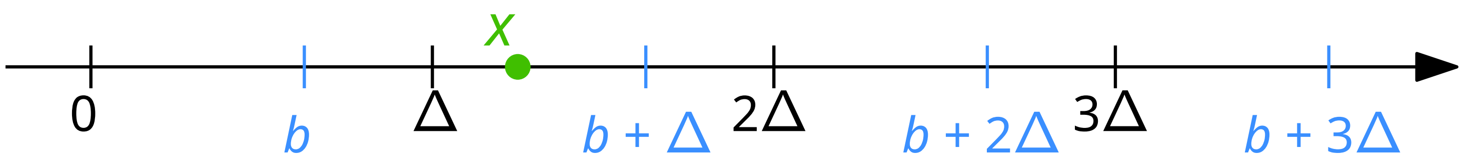

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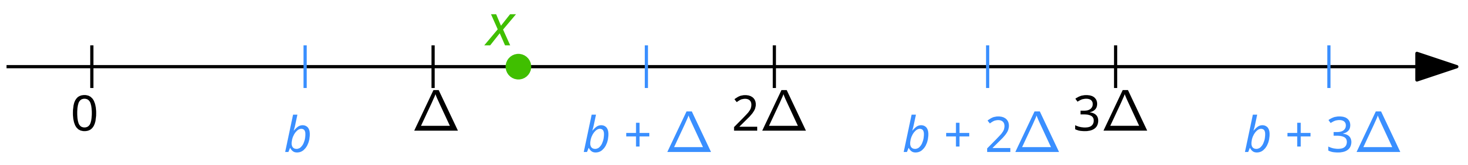

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Remark: If $|b - b'| = i\Delta$ for some $i \geq 0$, then b and b' induce the same partition.

(Later we will use this to choose $b \in [y + i\Delta, y + j\Delta]$ for $i < j$.)

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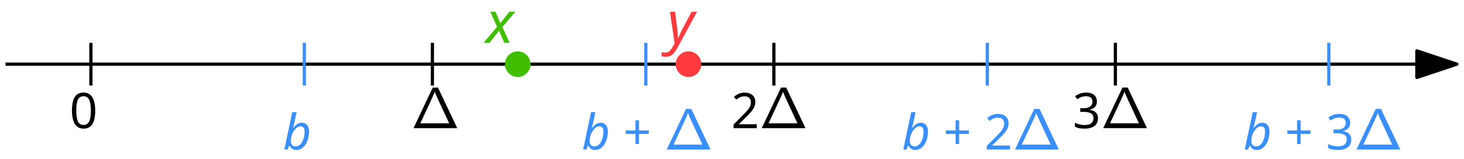

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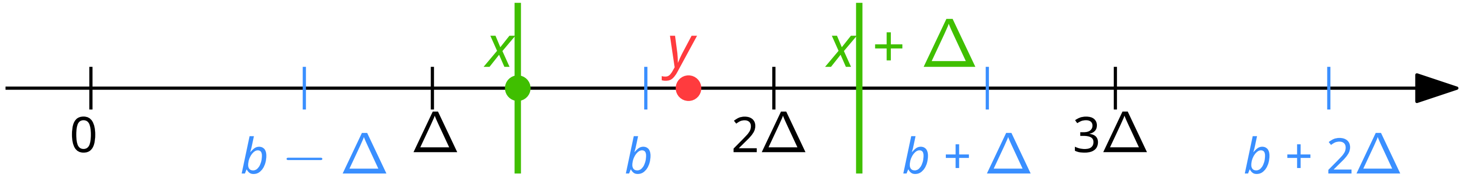

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Proof: Wlog $x < y$. Claim holds trivially if $|x - y| > \Delta$.
Otherwise assume $b \in [x, x + \Delta]$. Then $h_{b,\Delta}(x) \neq h_{b,\Delta}(y) \Leftrightarrow b \in [x, y]$.

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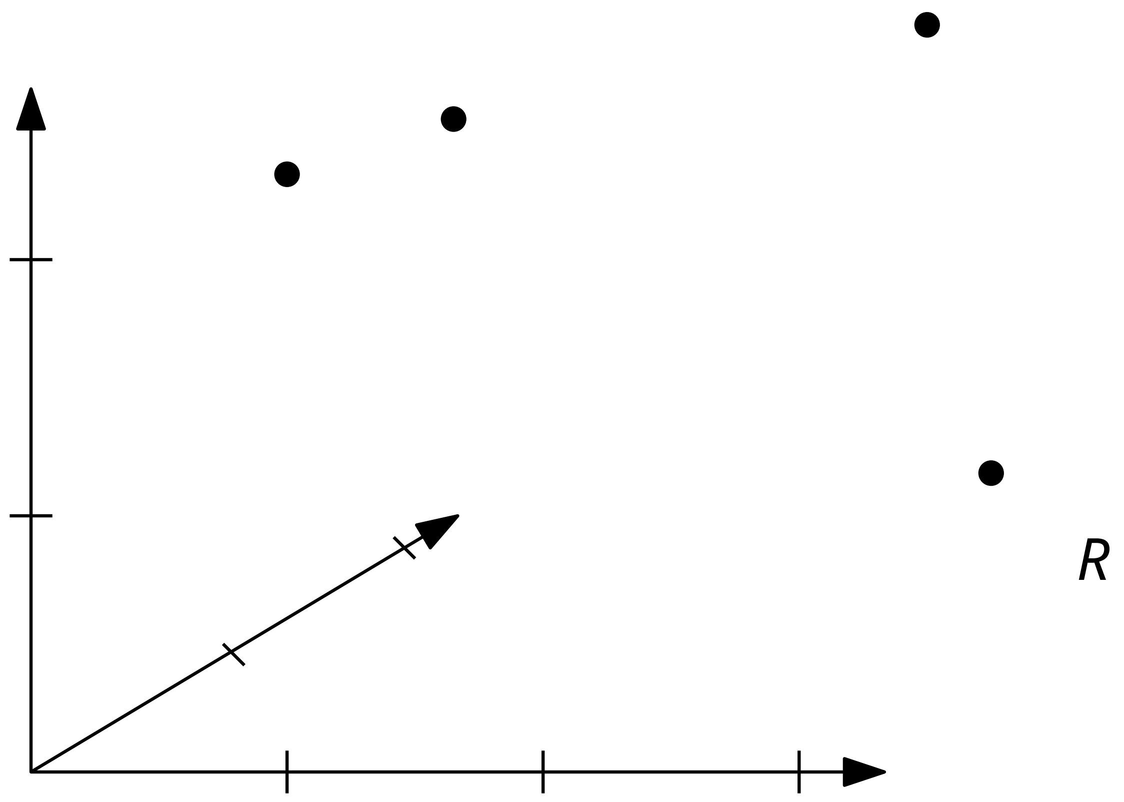

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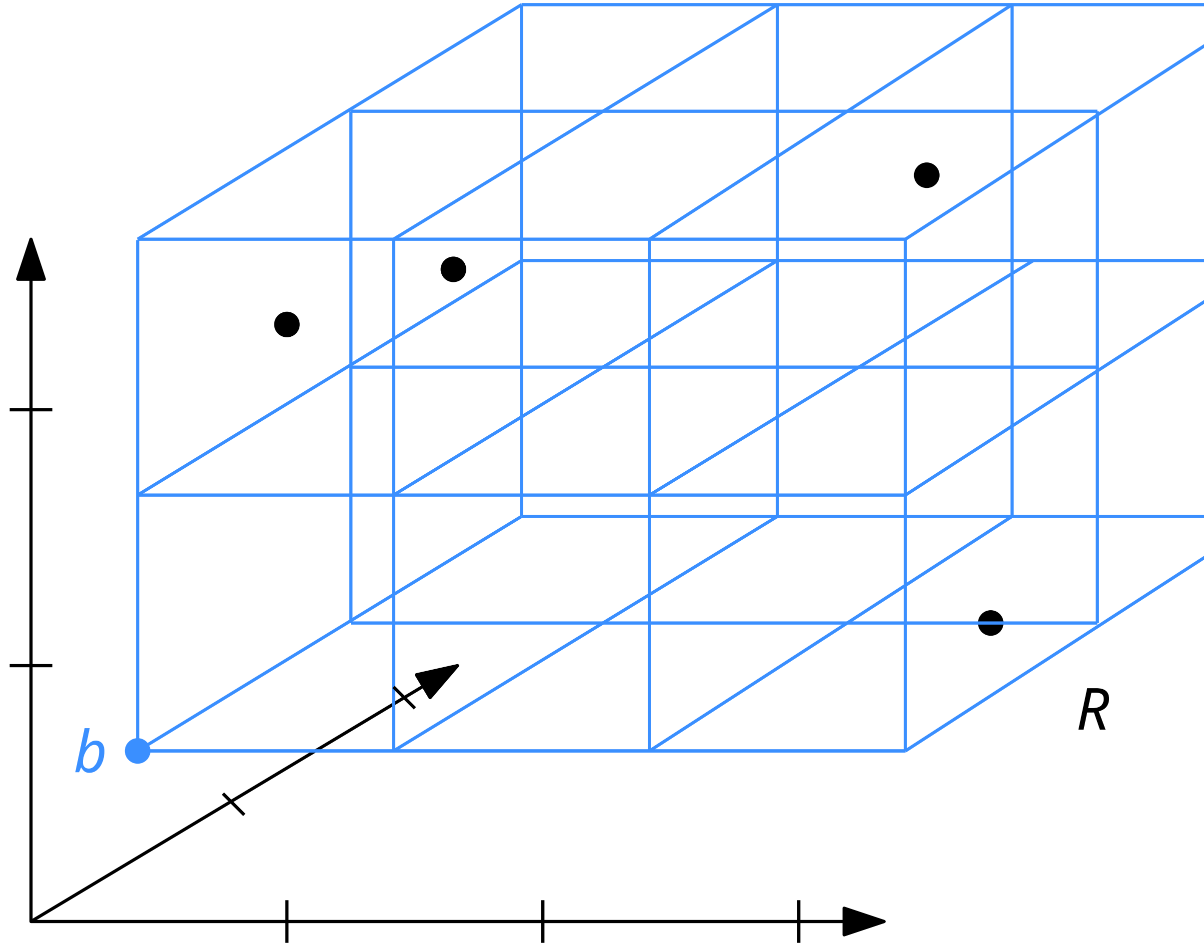

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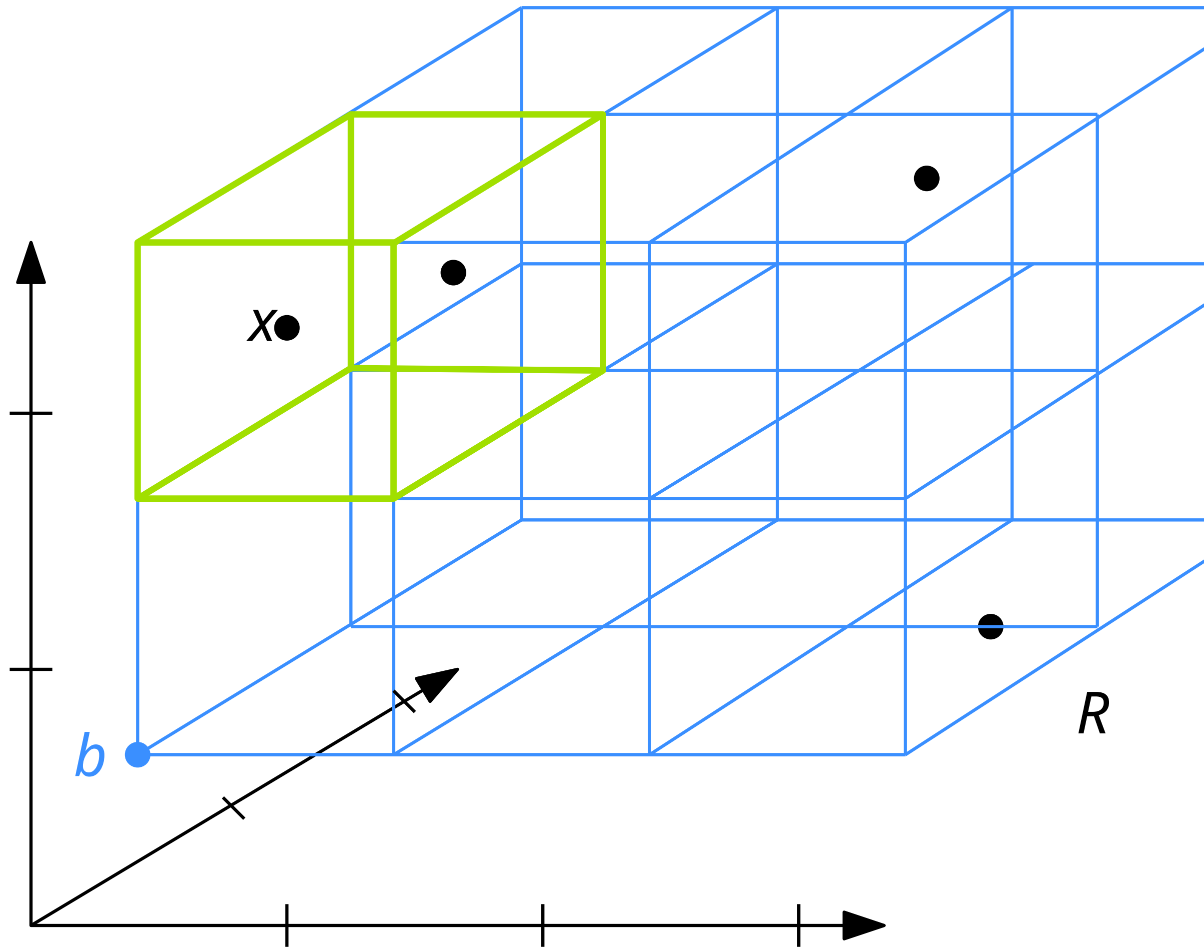

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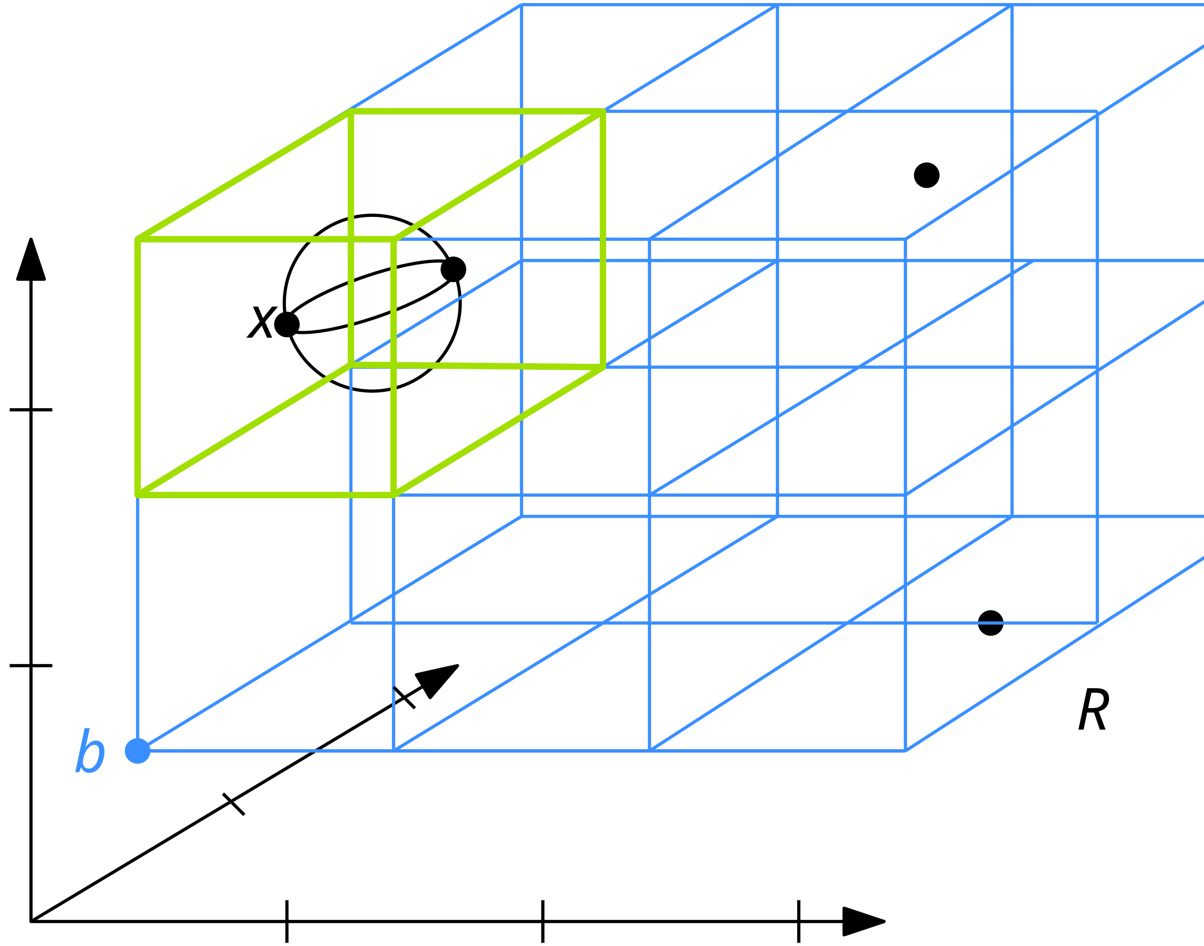

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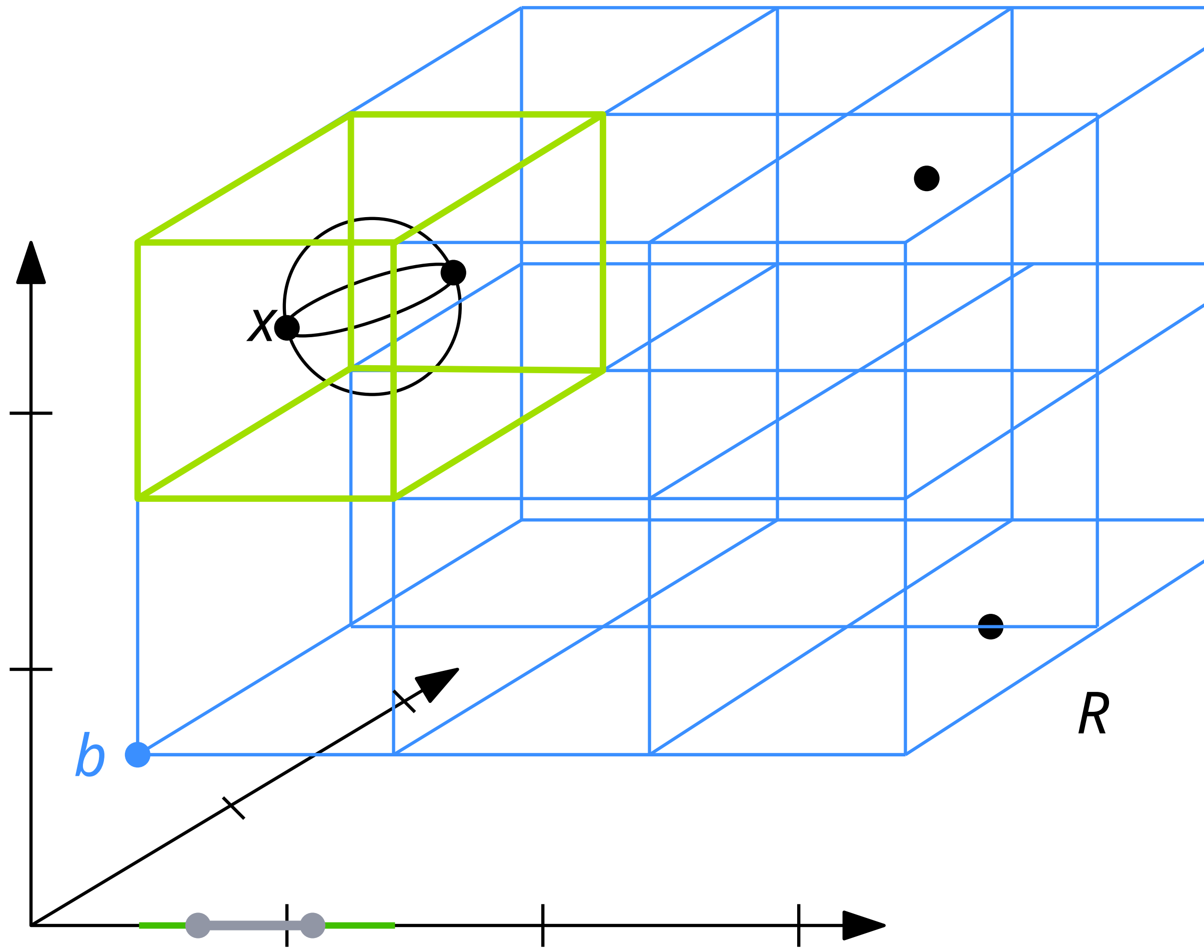

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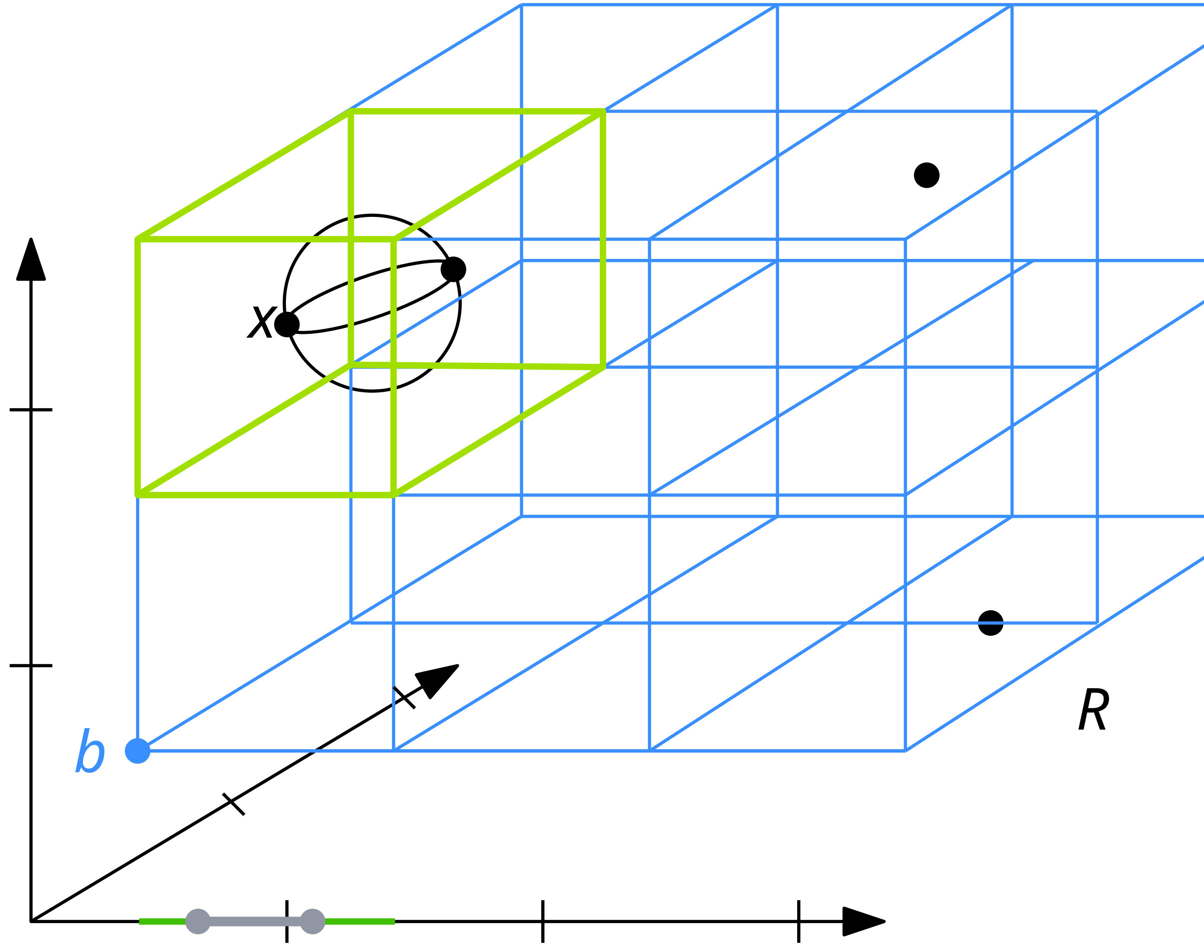

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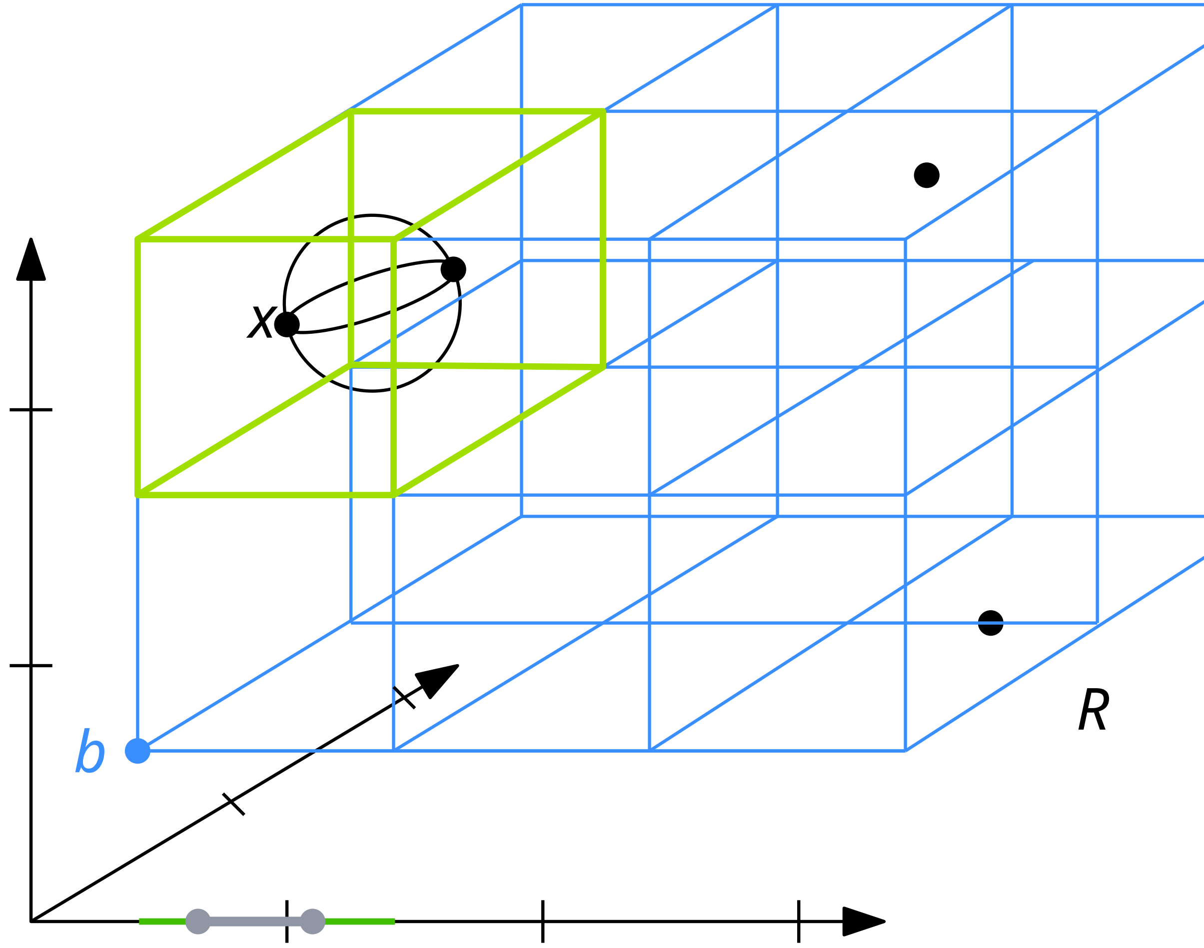

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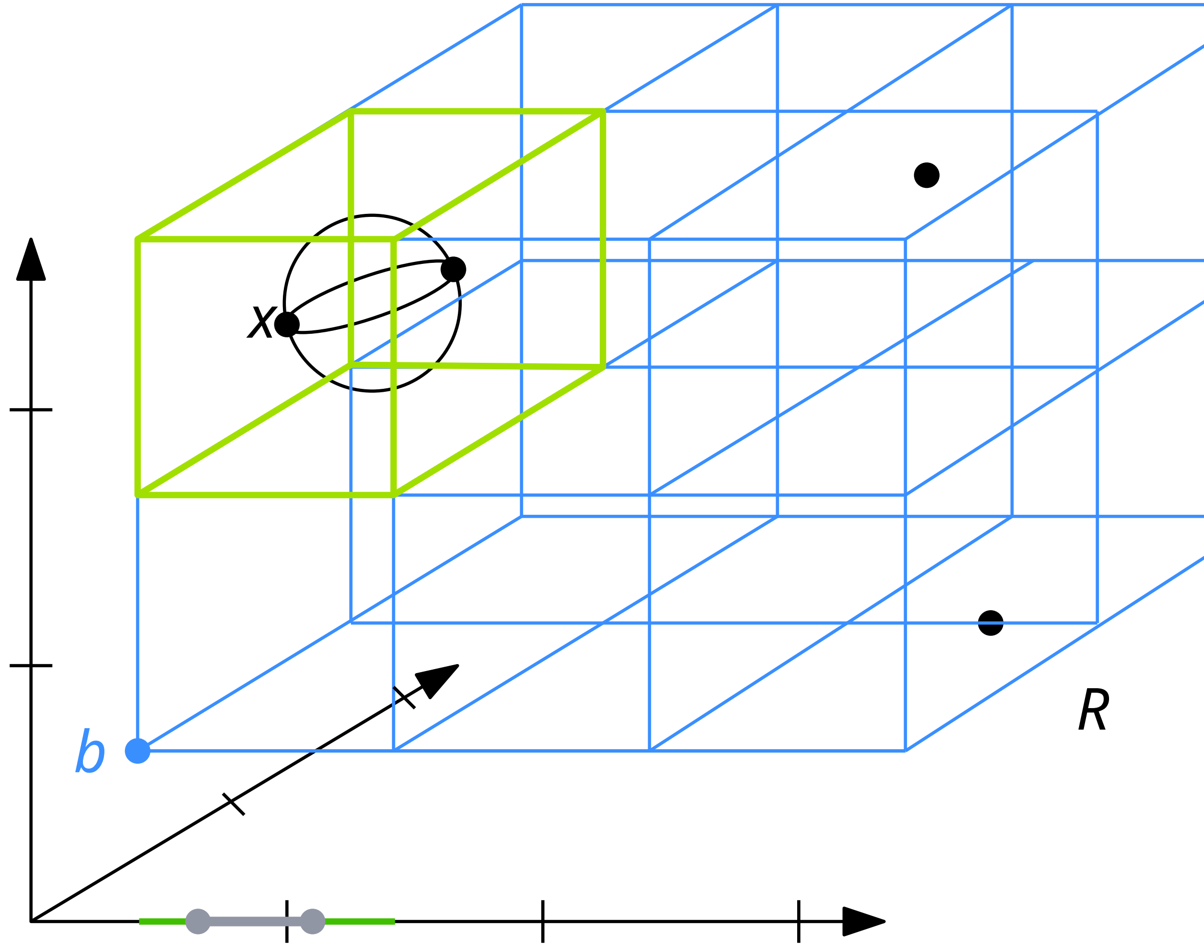
bound: $\mathbb{P}\left[\cup_{i=1}^d E_i\right] \leq \sum_{i=1}^d \mathbb{P}[E_i] \leq 2dr/\Delta$



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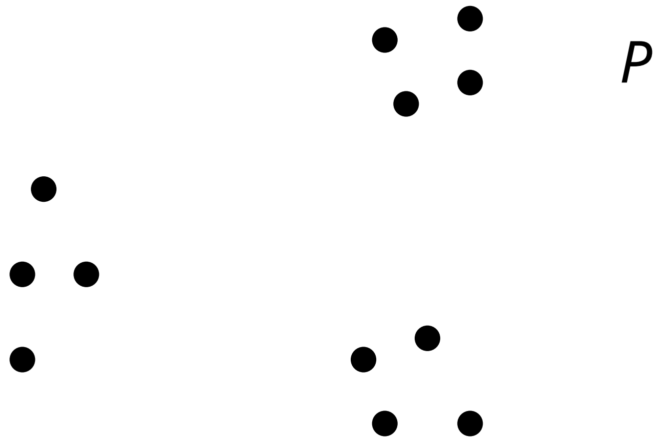

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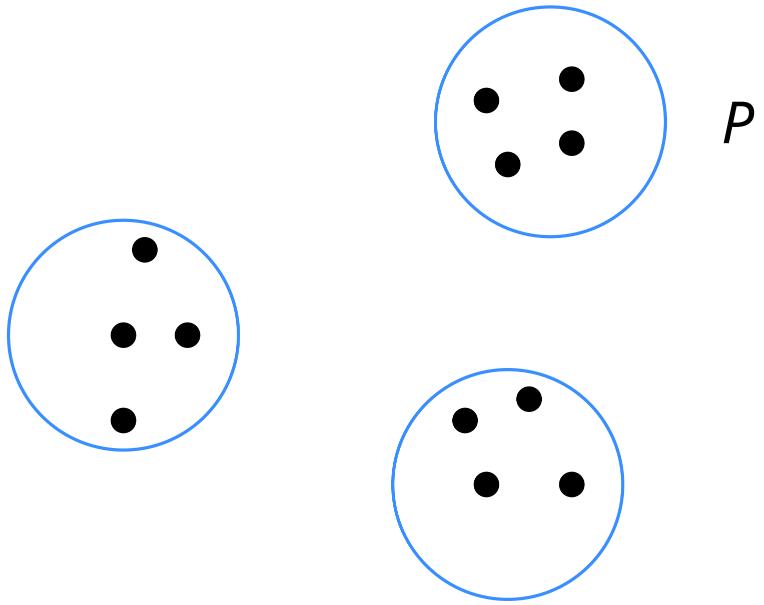

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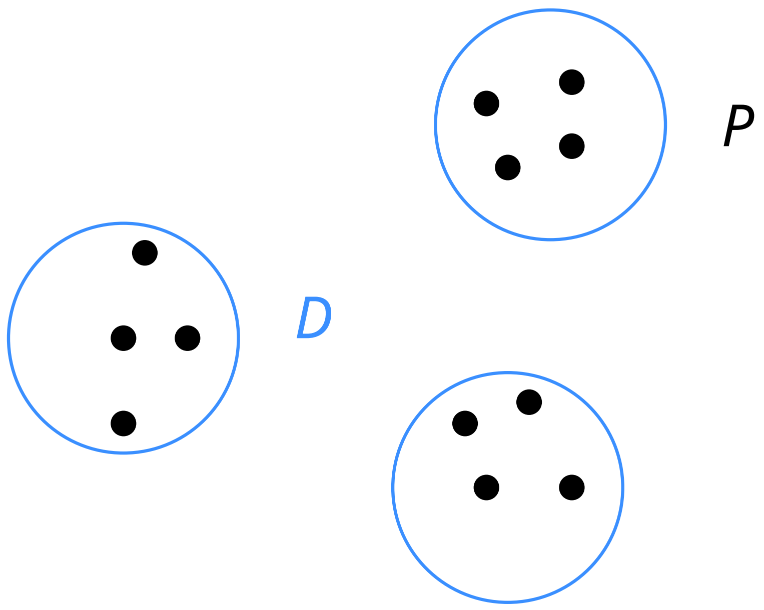

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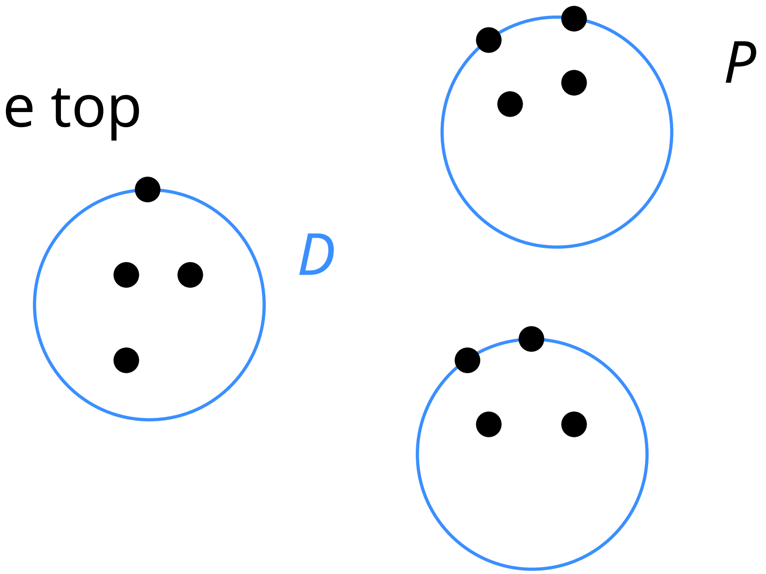

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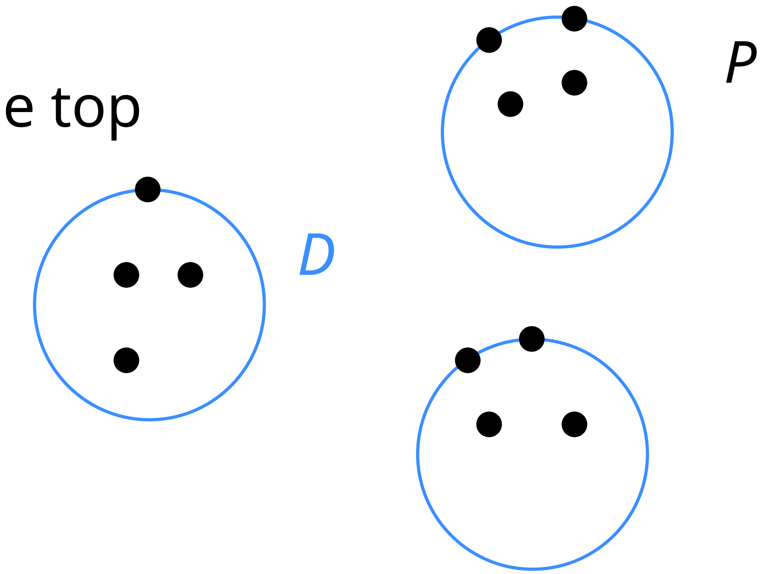

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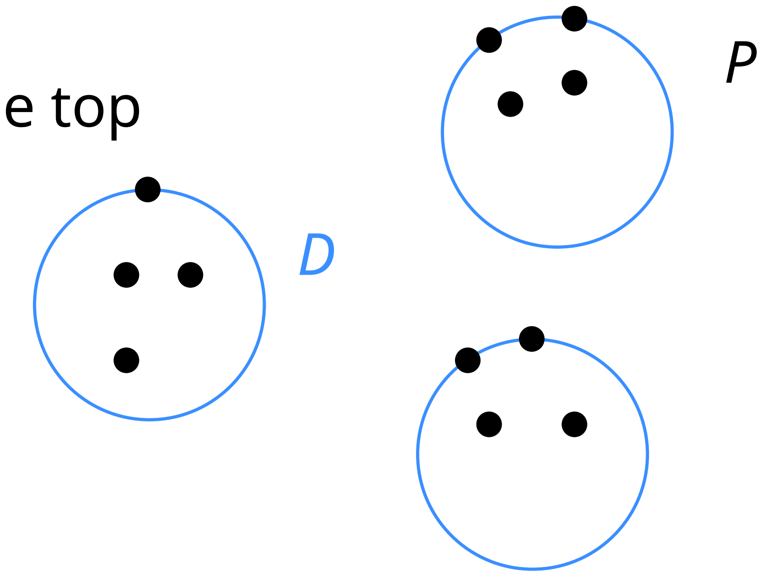

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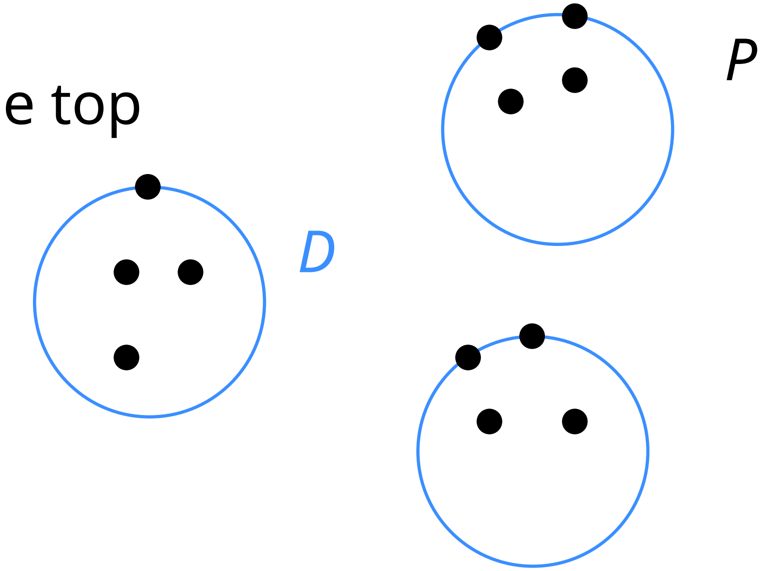

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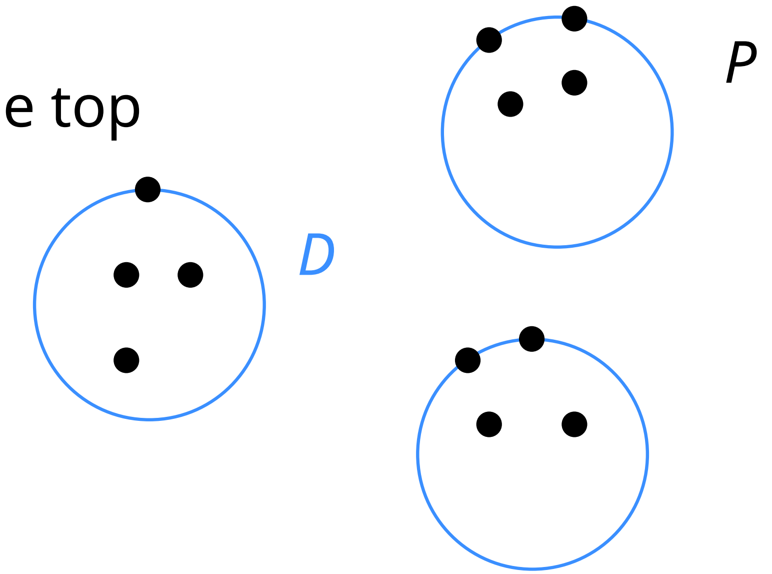

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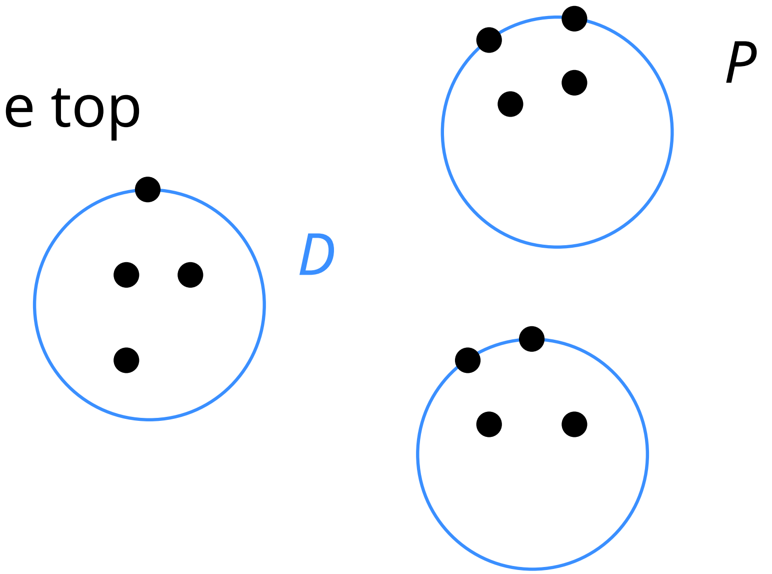

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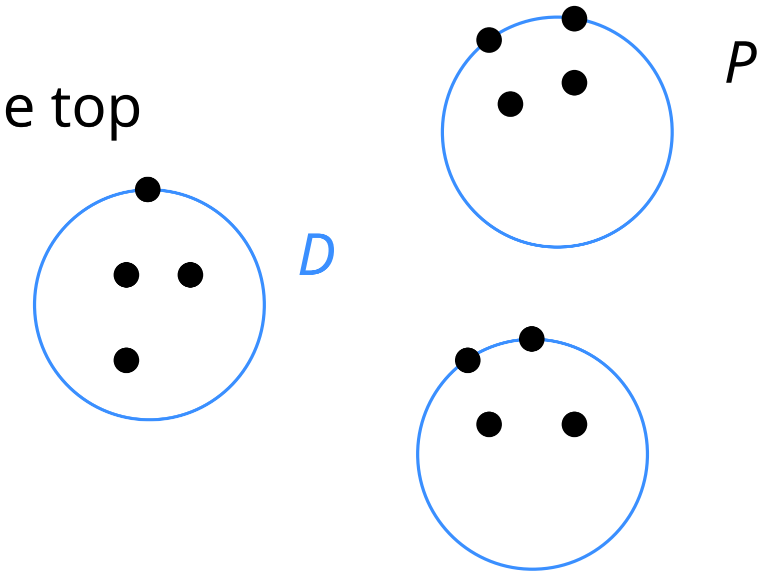

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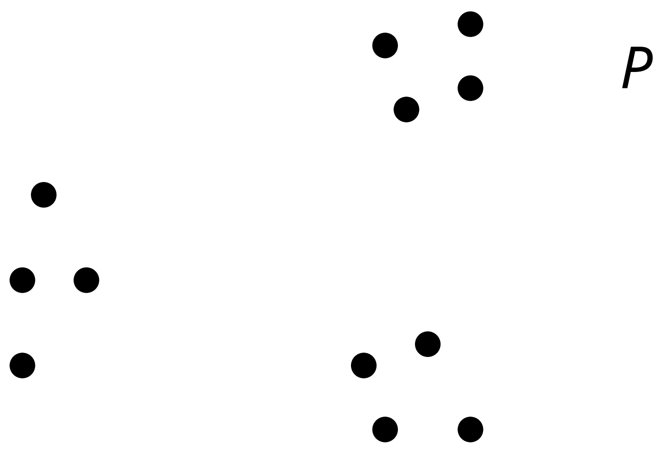
Faster approximate covering with unit disks

Let $\Delta = 12/\varepsilon$ and consider shifted grid $G^2(b, \Delta)$



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Algorithm

- compute all grid cells containing points in P
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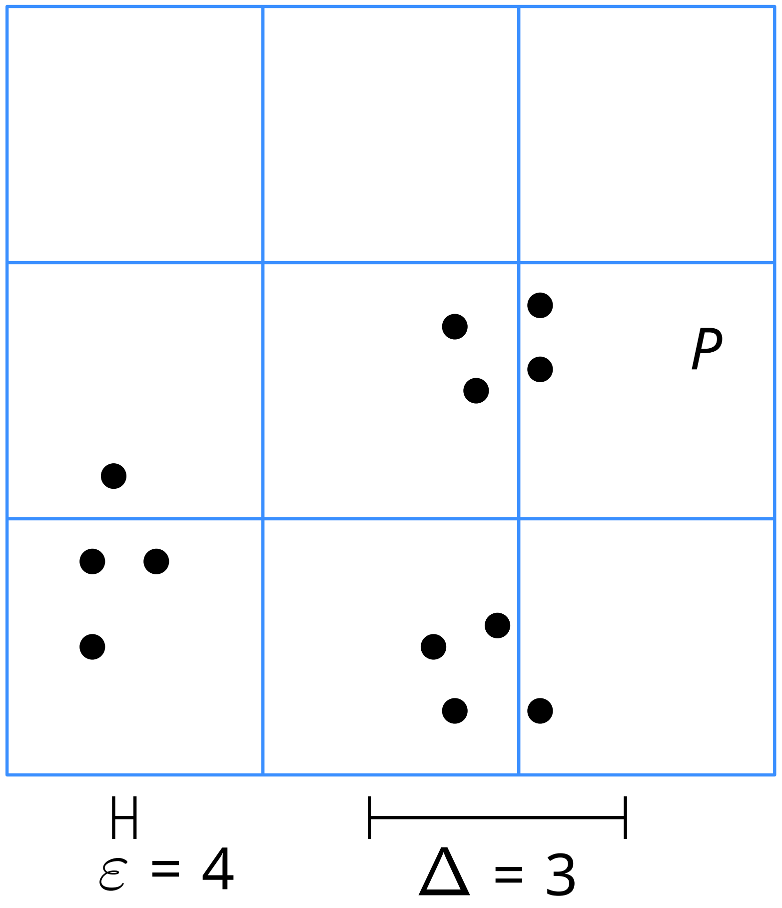
    using slow algorithm



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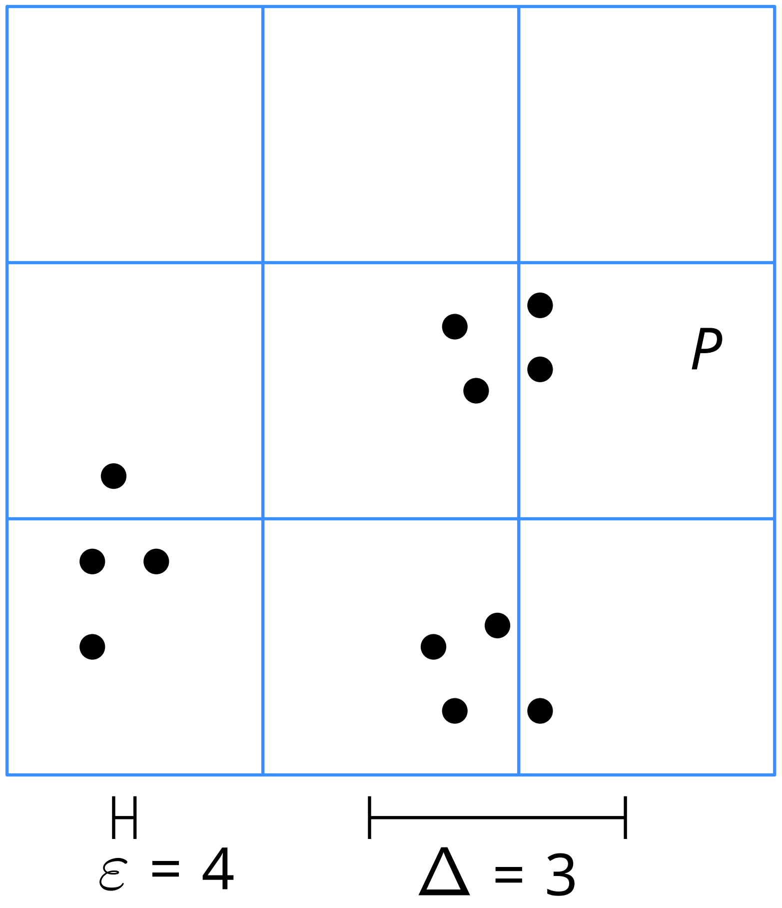

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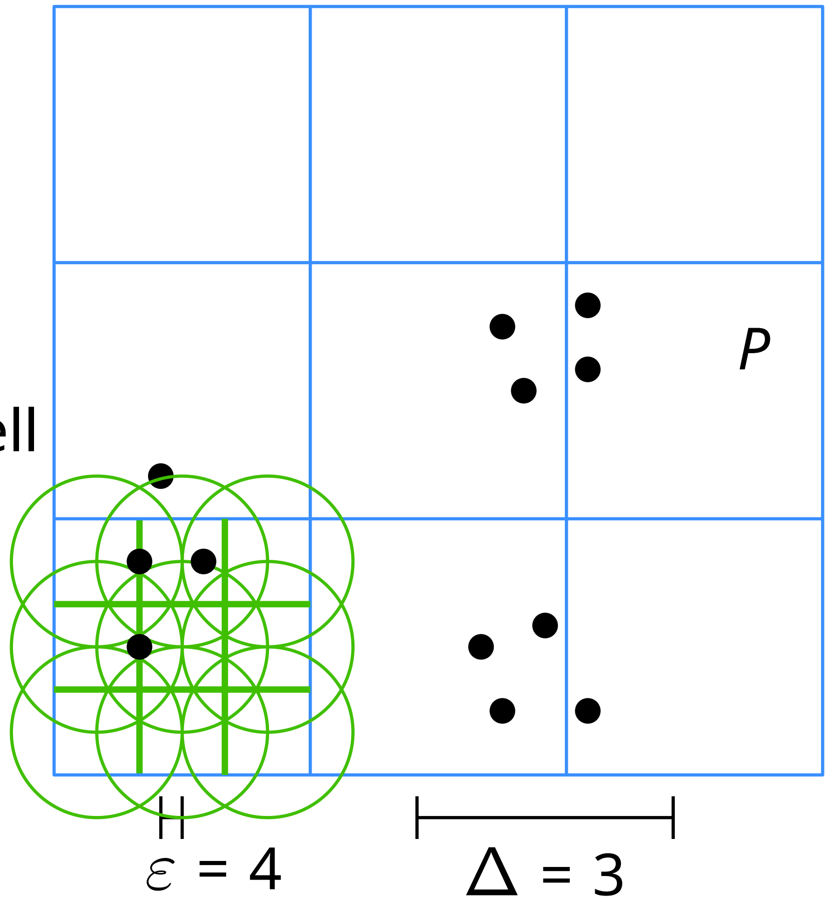

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  using hashing and the fact that each grid cell can be covered by $(\Delta + 1)^2 = O(1/\varepsilon^2)$ many unit disks; hence for at most n cells we can compute this in $O(Mn^{2M+2}) = n^{O(1/\varepsilon^2)}$ time

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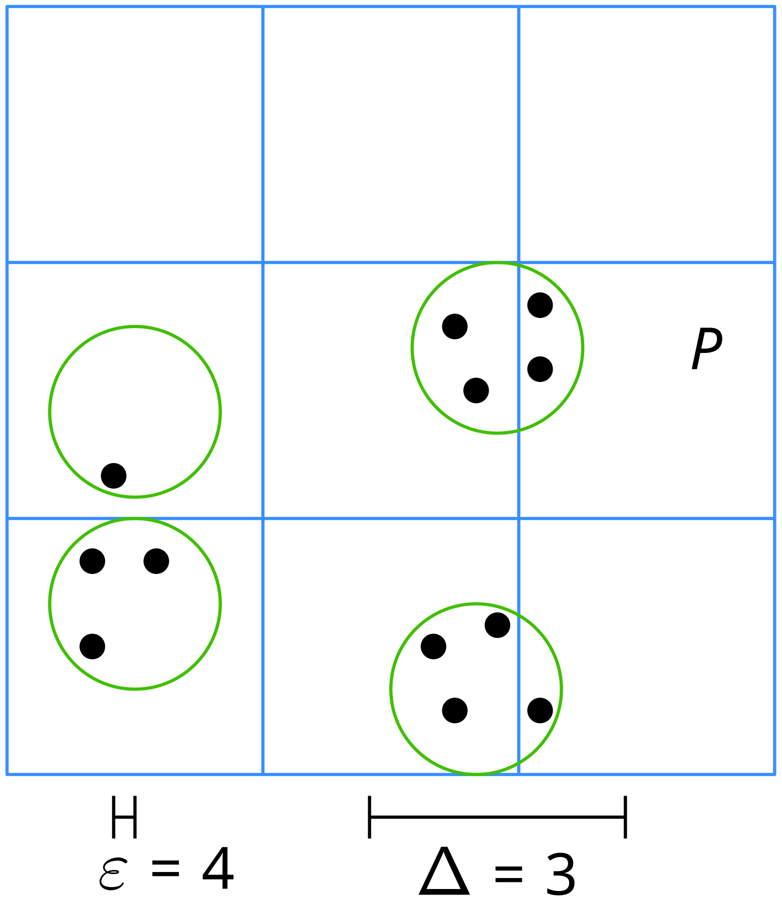

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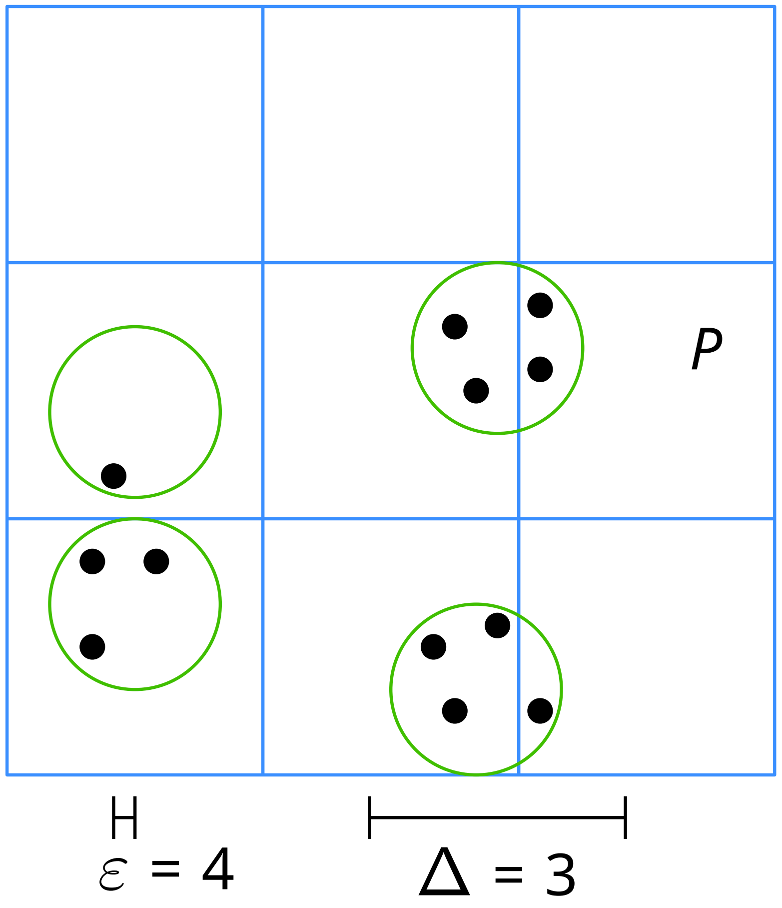

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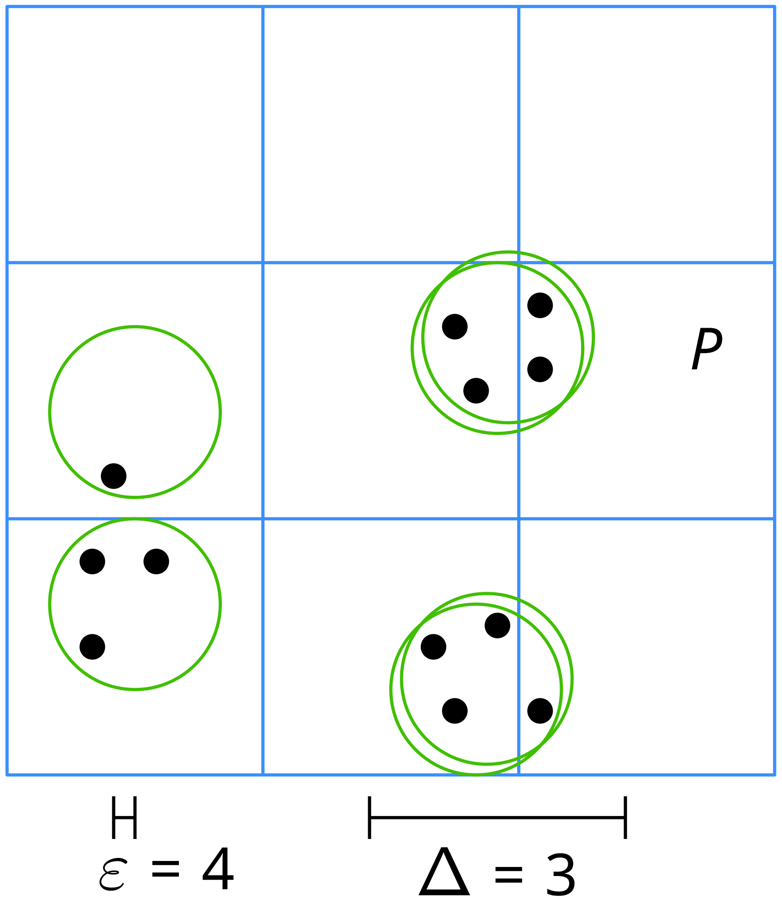

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  For each grid cell C: F_C the disks in F that intersect C. Let $G = \cup_C F_C$ (a multiset).

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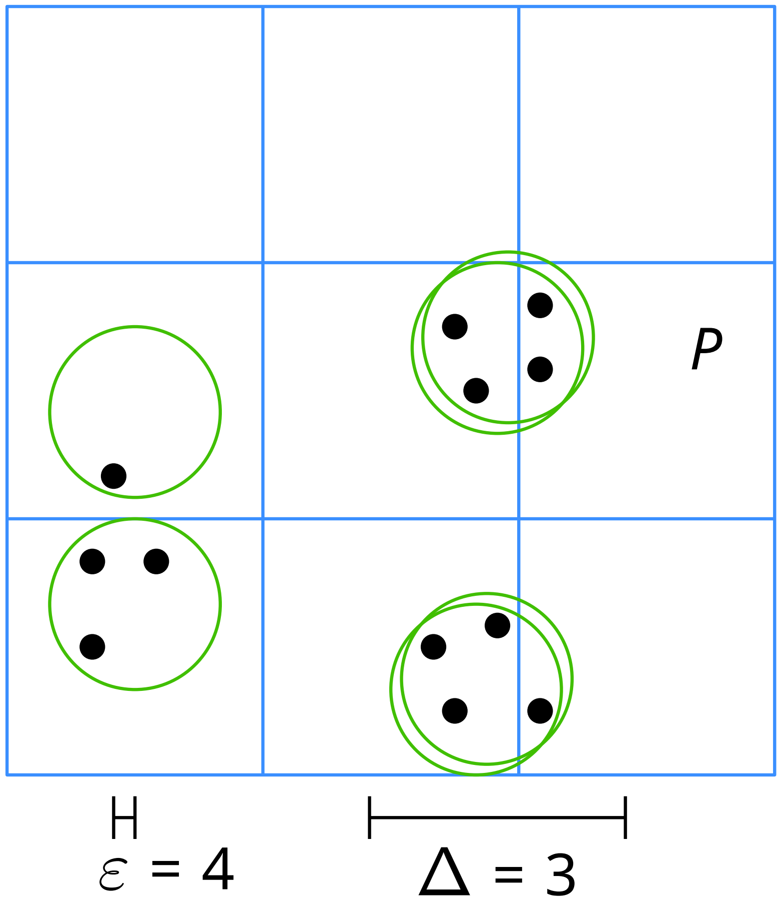

Analysis:

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Faster approximate covering with unit disks

Let $\Delta = 12/\varepsilon$ and consider shifted grid $G^2(b, \Delta)$

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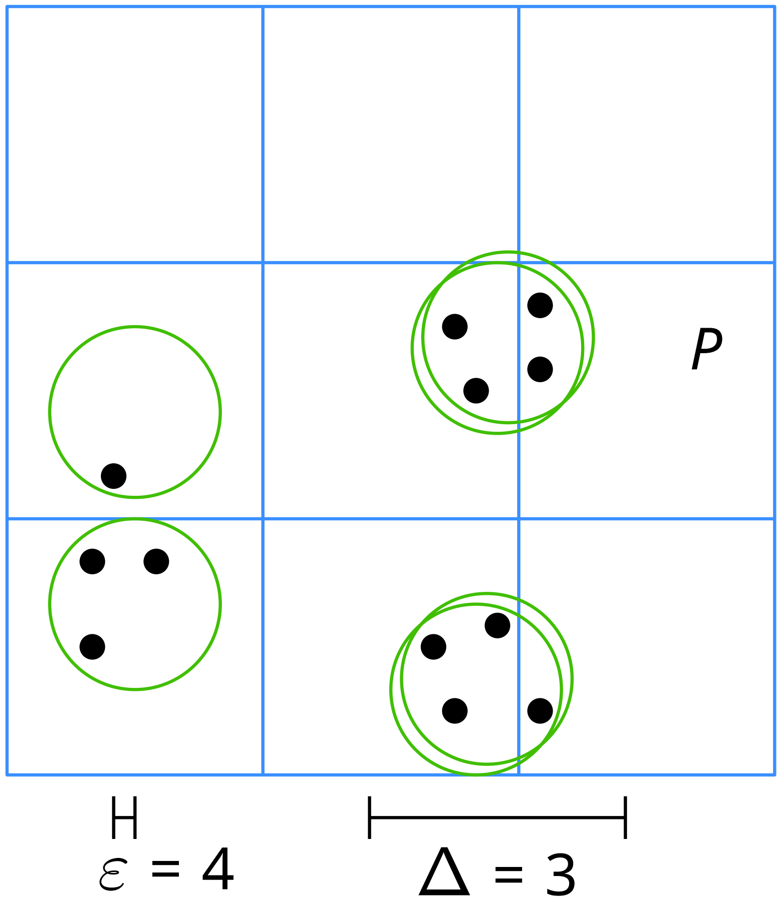

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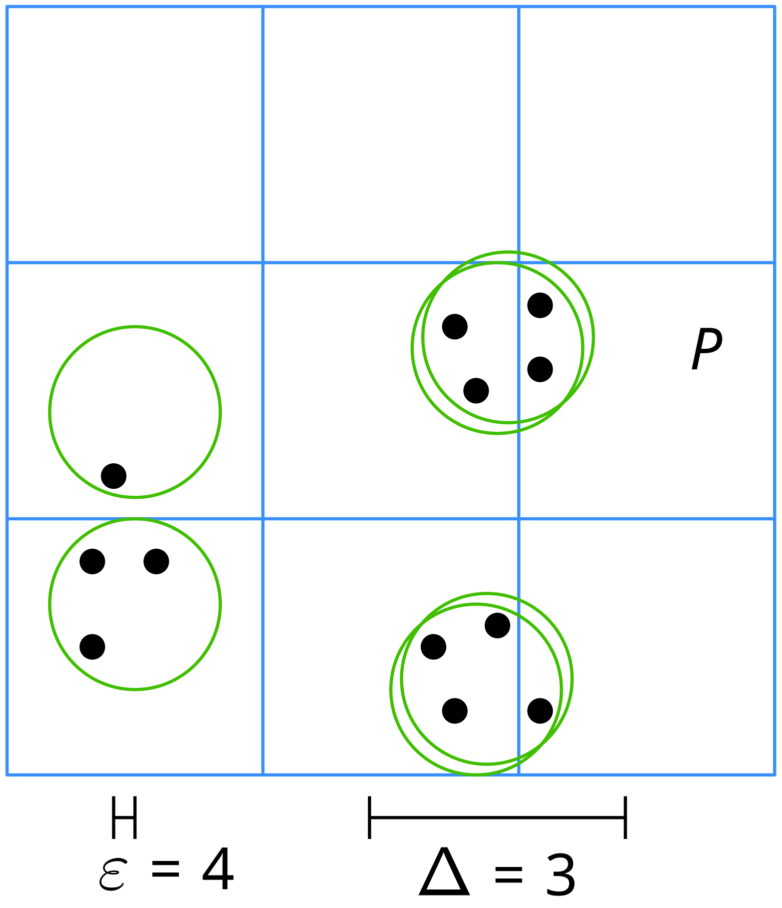

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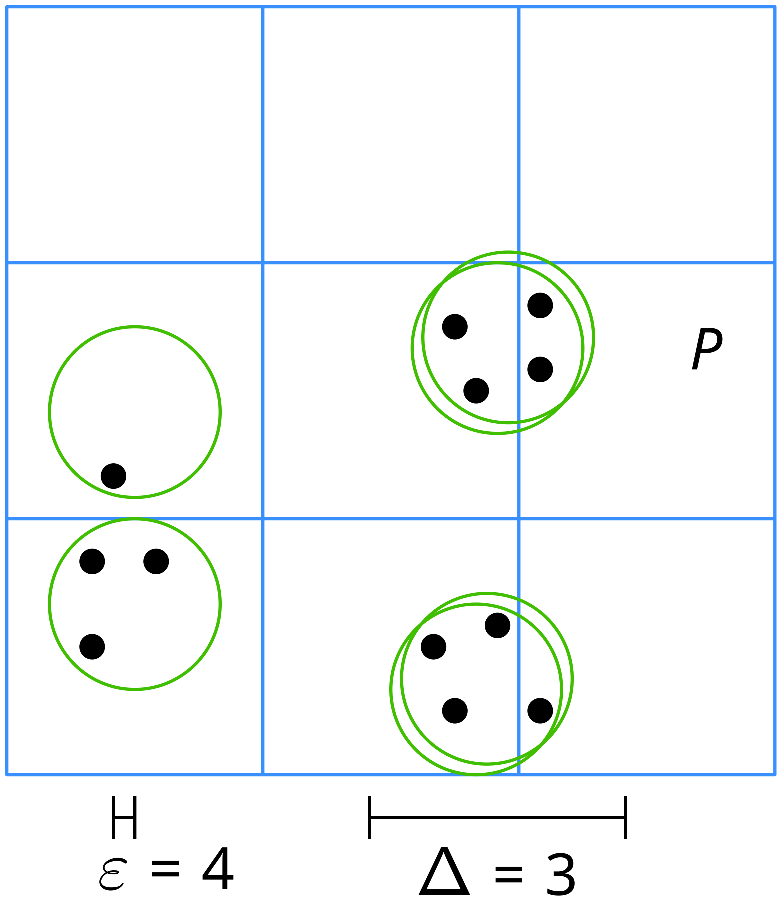

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Shifting Quadtrees in 1 dimension

Given point set P of n points in $\left[\frac{1}{2}, \frac{3}{4}\right]$. Draw $b \in \left[0, \frac{1}{2}\right]$ uniformly at random.
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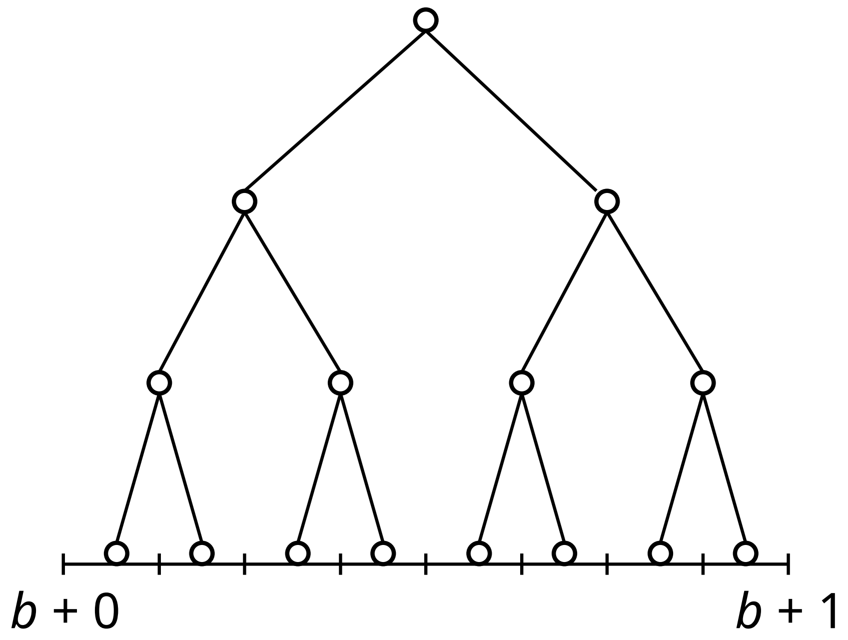

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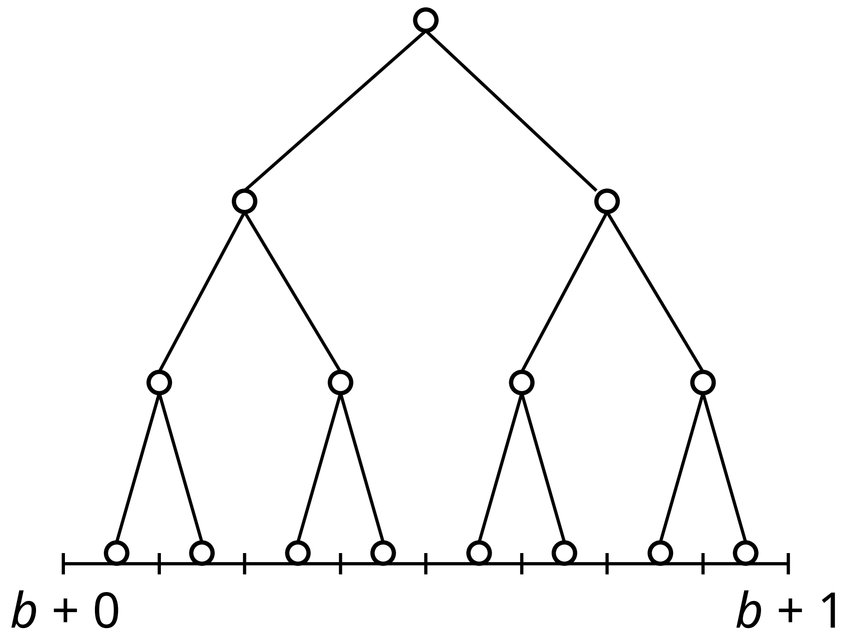

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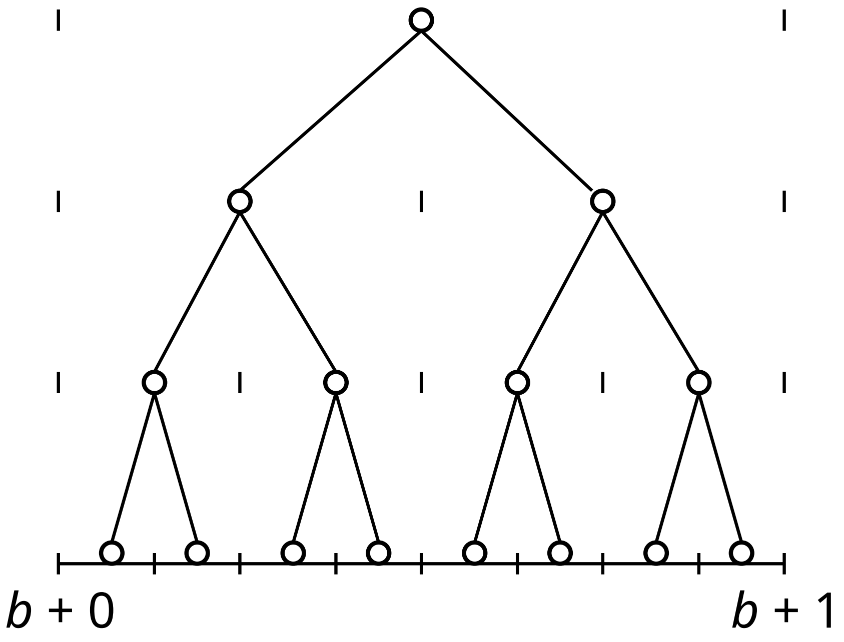

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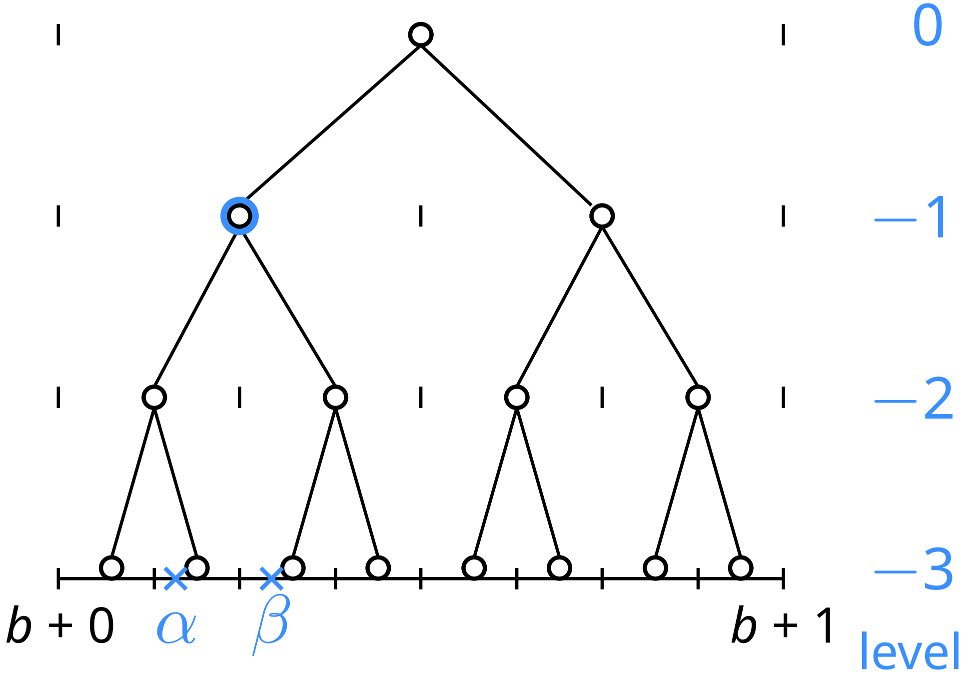

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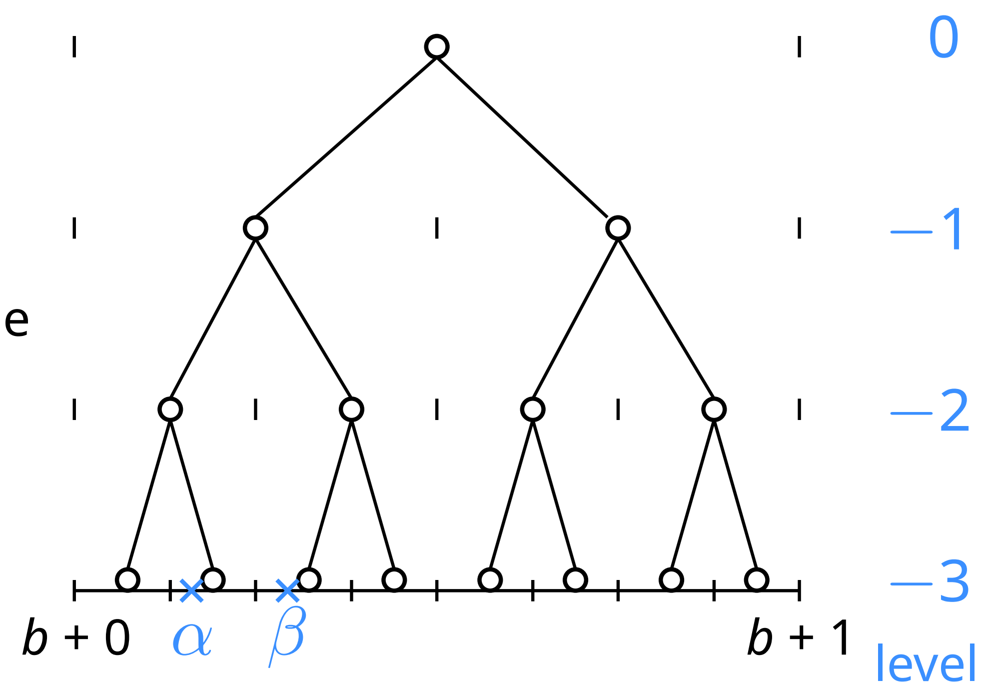

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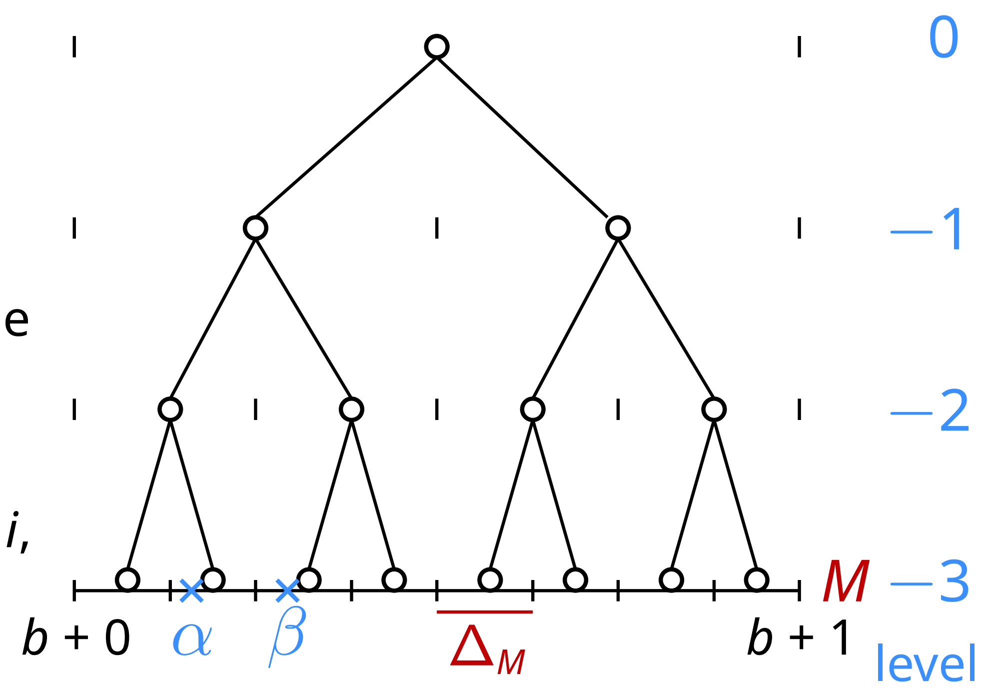

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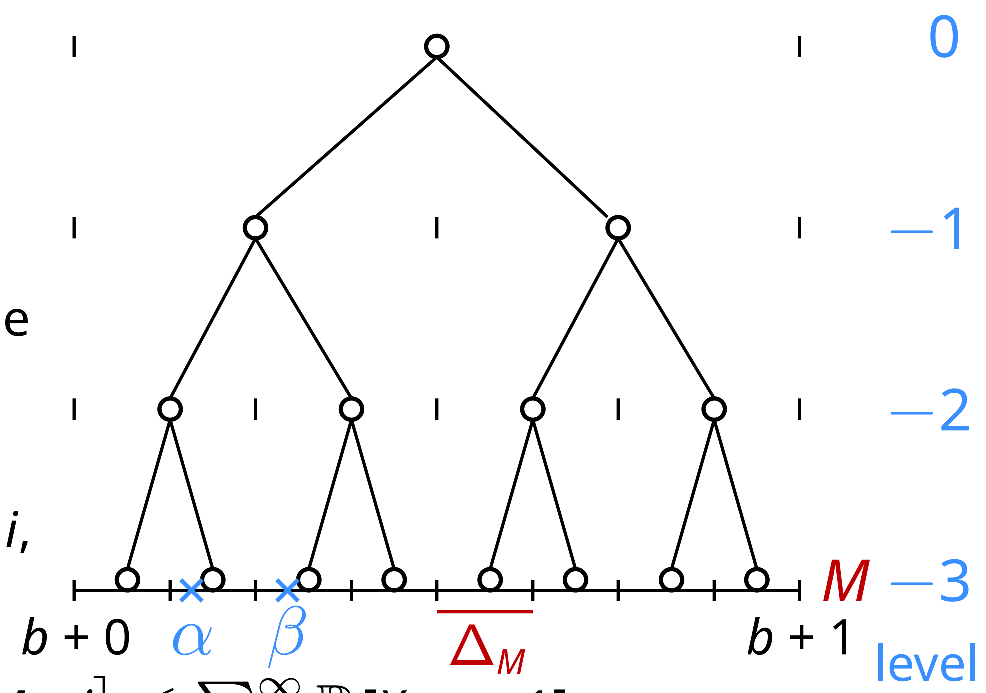

Hence $\mathbb{P}\left[\mathbb{L}_b(\alpha, \beta) > \log_2 |\alpha - \beta| + t\right] \leq \sum_{i=1+t}^{\infty} \mathbb{P}\left[\mathbb{L}_b(\alpha, \beta) = M + i\right] \leq \sum_{i=t}^{\infty} \mathbb{P}[X_{M+i} = 1]$
$\leq \sum_{i=t}^{\infty} \frac{|\alpha - \beta|}{\Delta_{M+i}} \leq \sum_{i=t}^{\infty} 2^{1-i} \leq 2^{2-t}$

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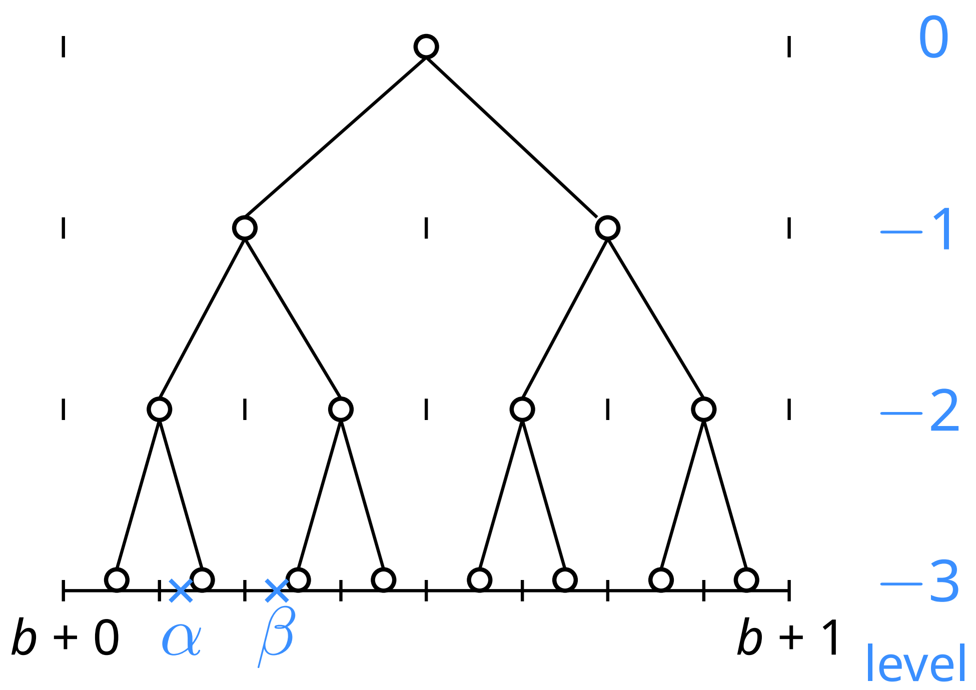

Corollary: Let $\alpha, \beta \in \left[\frac{1}{2}, \frac{3}{4}\right]$ and $b \in \left[0, \frac{1}{2}\right]$.
For $c > 1$ holds $\mathbb{P}\left[\mathbb{L}_b(\alpha, \beta) > \log_2|\alpha - \beta| + c \log n\right] \leq \frac{4}{n^c}$ where $|P| = n$.

Shifting Quadtrees in higher dimensions

Now let P be a set of n points in $\left[\frac{1}{2}, \frac{3}{4}\right]^d$ and b in $\left[0, \frac{1}{2}\right]^d$.

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Note that T is the combination of 1dim Quadtrees $T_1, \dots, T_d$ in each coordinate.

Hence $\mathbb{L}_b(p, q) = \max_{i=1}^{d} \mathbb{L}_{b_i}(p_i, q_i)$ and is again independent of all other points in P.

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We consider $\mathbb{L}_b(p, q)$ as random variable and use
Lemma
For $t > 0$ holds $\mathbb{P}\left[\mathbb{L}_b(p, q) > \log_2 ||p - q|| + t\right] \leq \frac{4d}{2^t}$.

Low quality ANN-Search

Now we want to use shifted quadtrees to quickly answer ANN-queries in $\mathbb{R}^d$.
That is, we want to preprocess a set P of n points in $\mathbb{R}^d$, so that for query point q we can quickly find $p \in P$, s.t. $||q - p|| \leq \tau\, d(q, P)$ where $d(q, P) = min_{p \in P} ||q - p||$.

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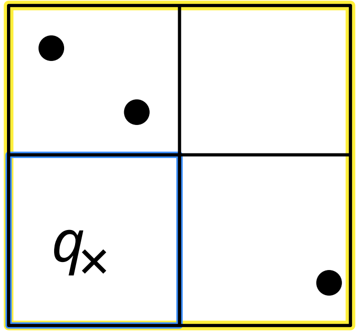

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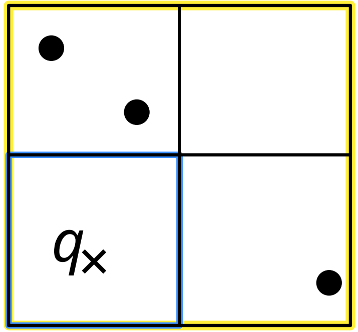

In 1. and 3. $||q - p|| \leq diam(v)$ and in 2. $||q - p|| \leq 2diam(v)$

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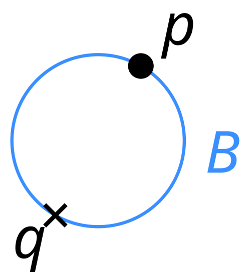

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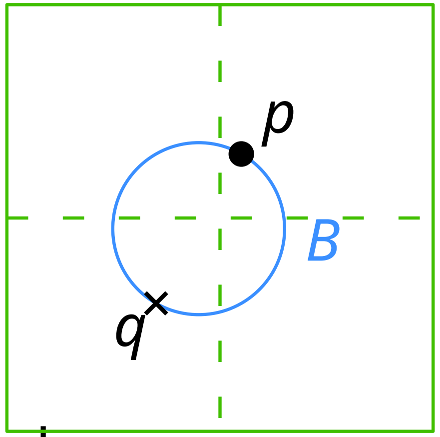

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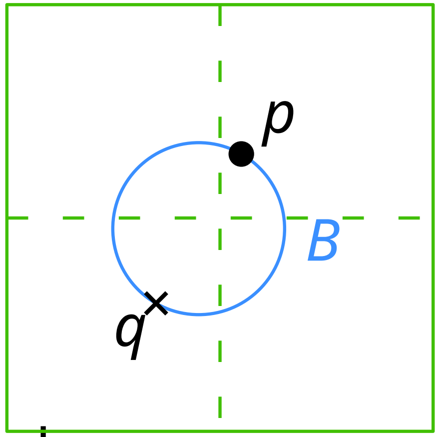

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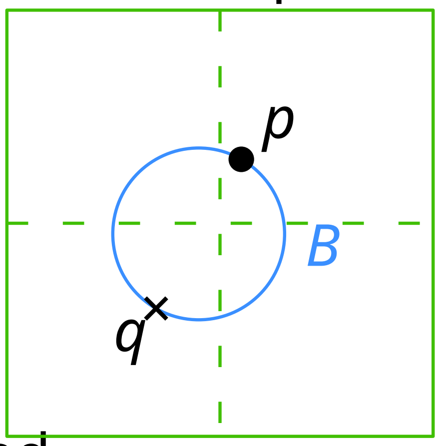

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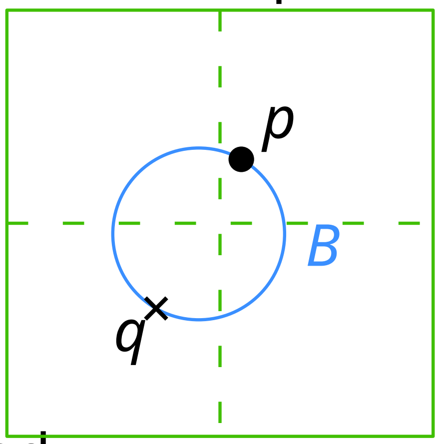
If B lies completely in a cell, let $\square$ be this cell.

The ANN returned has distance $\leq 2diam(\square) \leq 2\sqrt{d}2^i$ hence quality $2\sqrt{d}2^i/\ell$

Low quality ANN-Search

Lemma: For $\tau > 1$ and query point q, a τ-approximate NN is returned with probability at least $(1 - 4d^{3/2})/\tau$.



Proof:

Let p be NN of q in P; consider ball B defined by p, q.

Let u be the lowest node in T that fully contains B.

The query returns either u or one of its descendants.

Hence an ANN at distance at most $2diam(v) \leq 2diam(u)$ is returned.

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If B lies completely in a cell, let $\square$ be this cell.

The ANN returned has distance $\leq 2diam(\square) \leq 2\sqrt{d}2^i$ hence quality $2\sqrt{d}2^i/\ell$

And it holds $2\sqrt{d}2^i/\ell \leq \tau \Leftrightarrow i \leq \log_2 \frac{\ell\tau}{2\sqrt{d}}$

Set $i := \lfloor \log_2(\frac{\ell\tau}{2\sqrt{d}}) \rfloor$ then it follows with $(\star)$ that an τ-ANN is returned with probability at least $1 - \frac{d\ell}{2^i} \geq 1 - \frac{4d^{3/2}}{\tau}$

Summary

shifting grids $\rightarrow$ approximate disk cover

shifting quadtrees $\rightarrow$ approximate nearest neighbor query

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With high probability p and q in same cell at level $\log_2 ||p - q|| + c \log n$