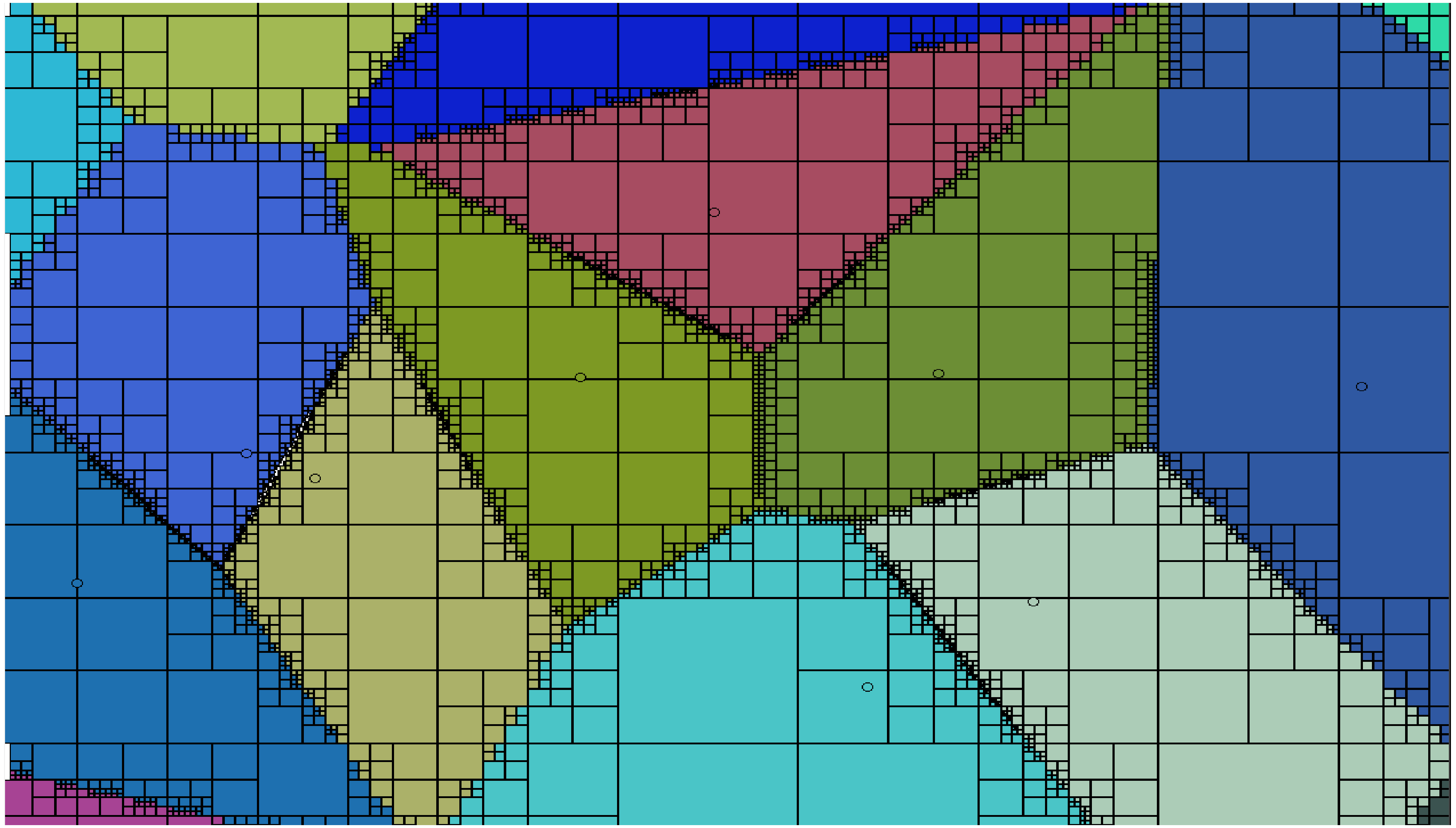
Approximate Voronoi Diagrams



Recap Point Location Among Balls

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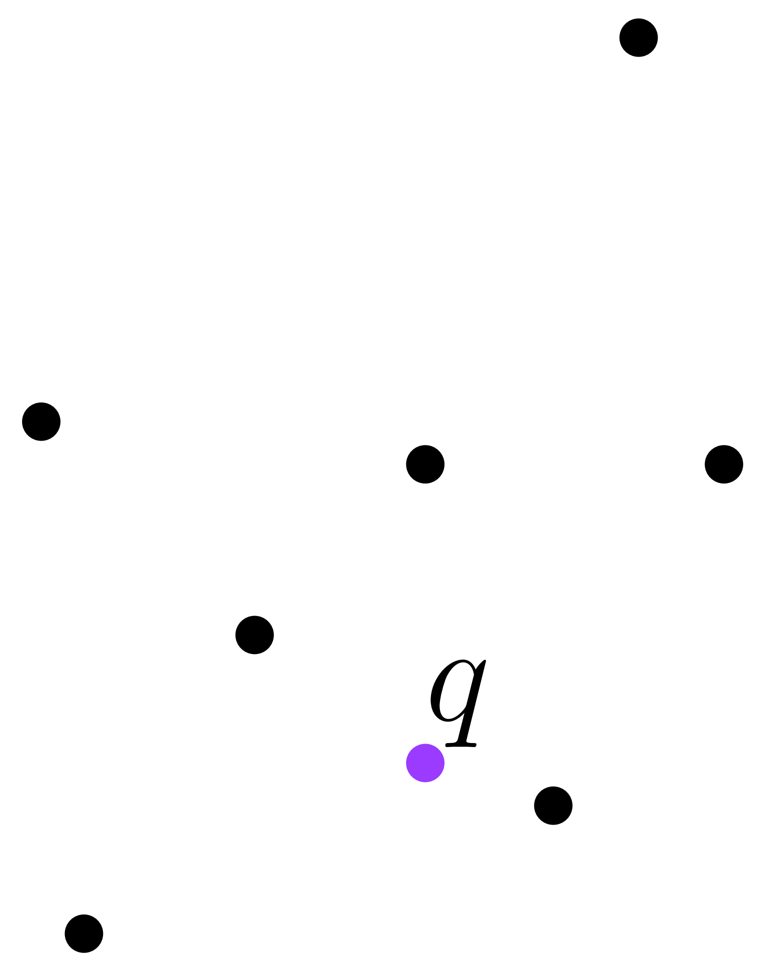

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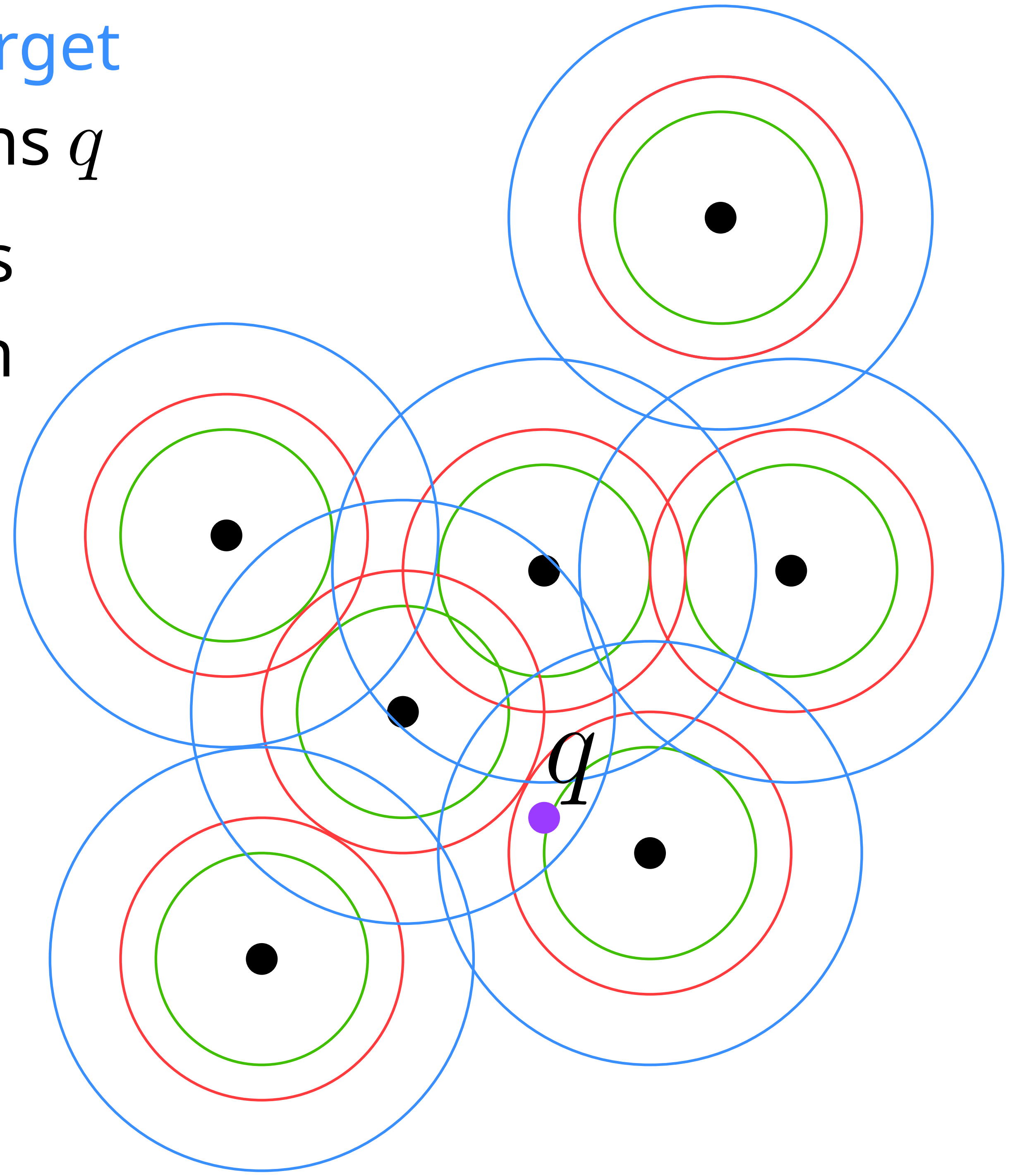

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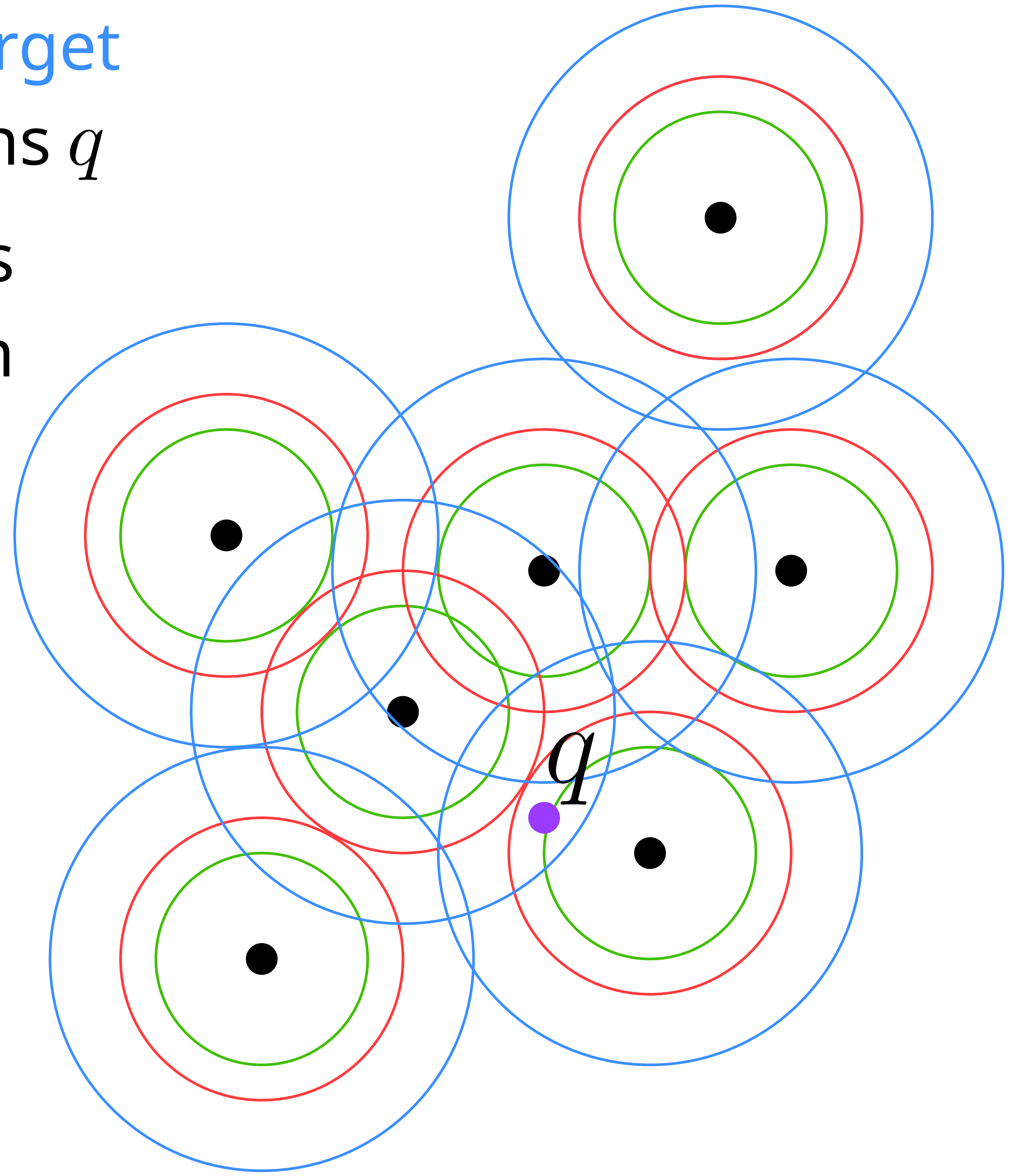

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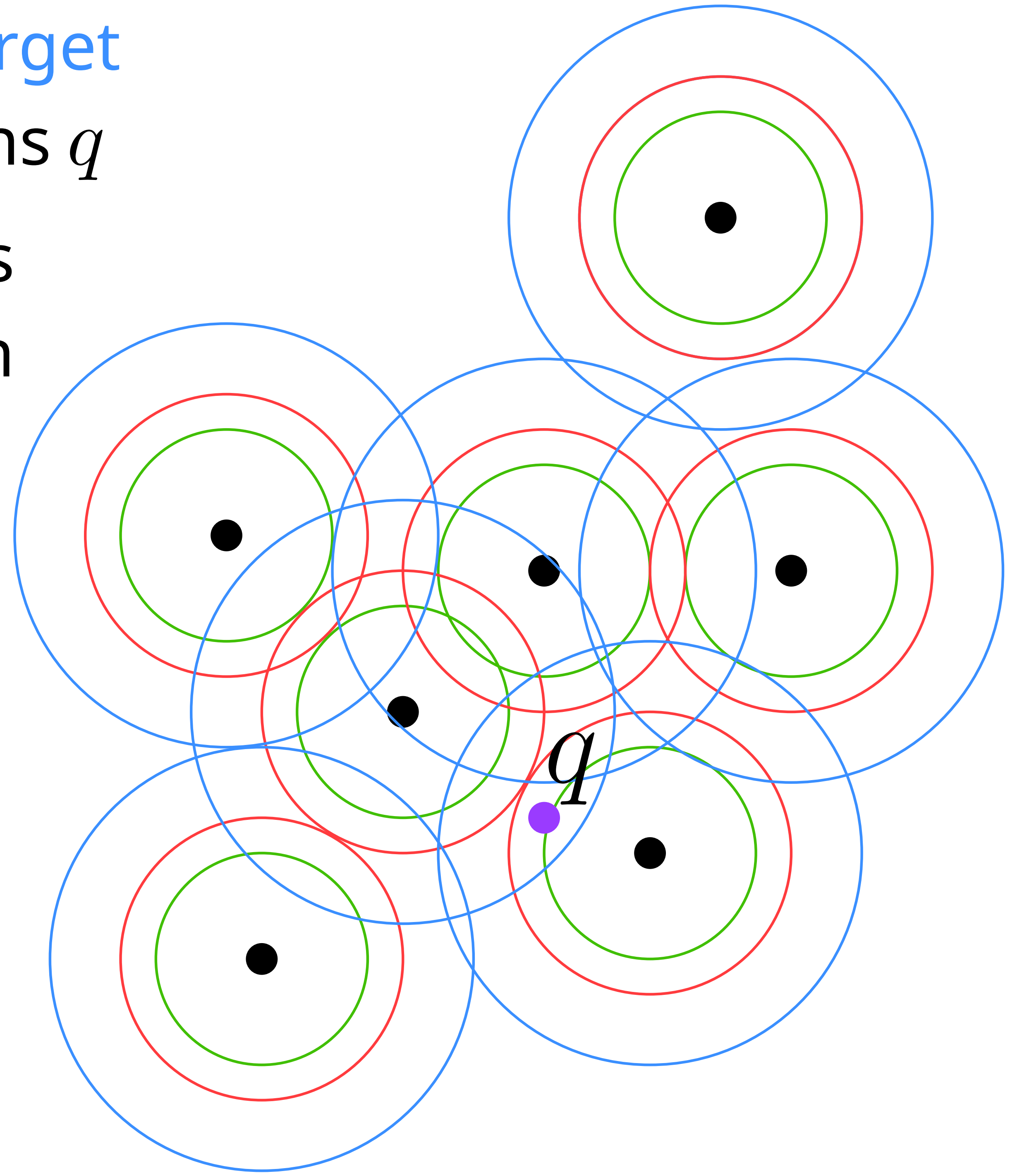

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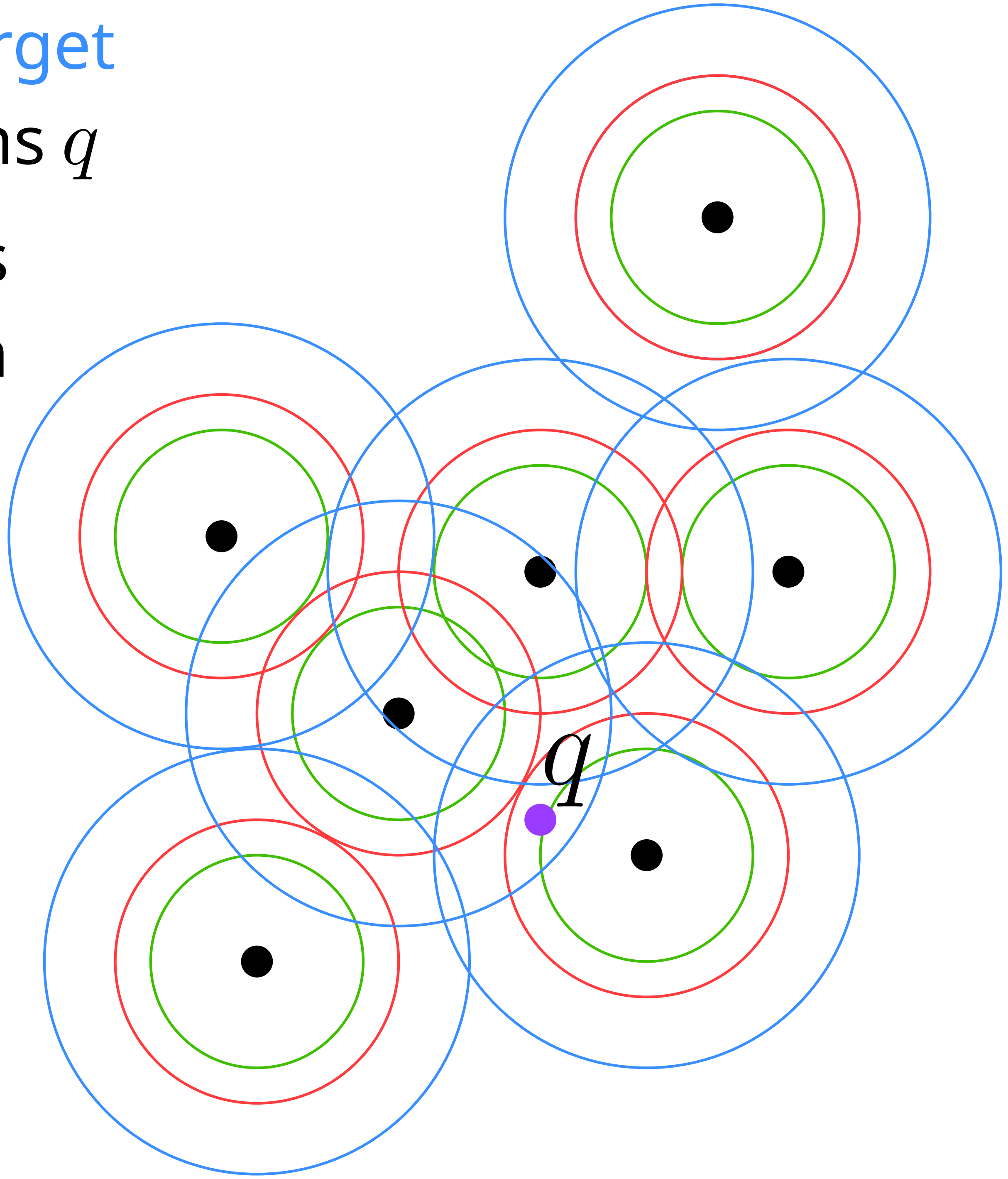

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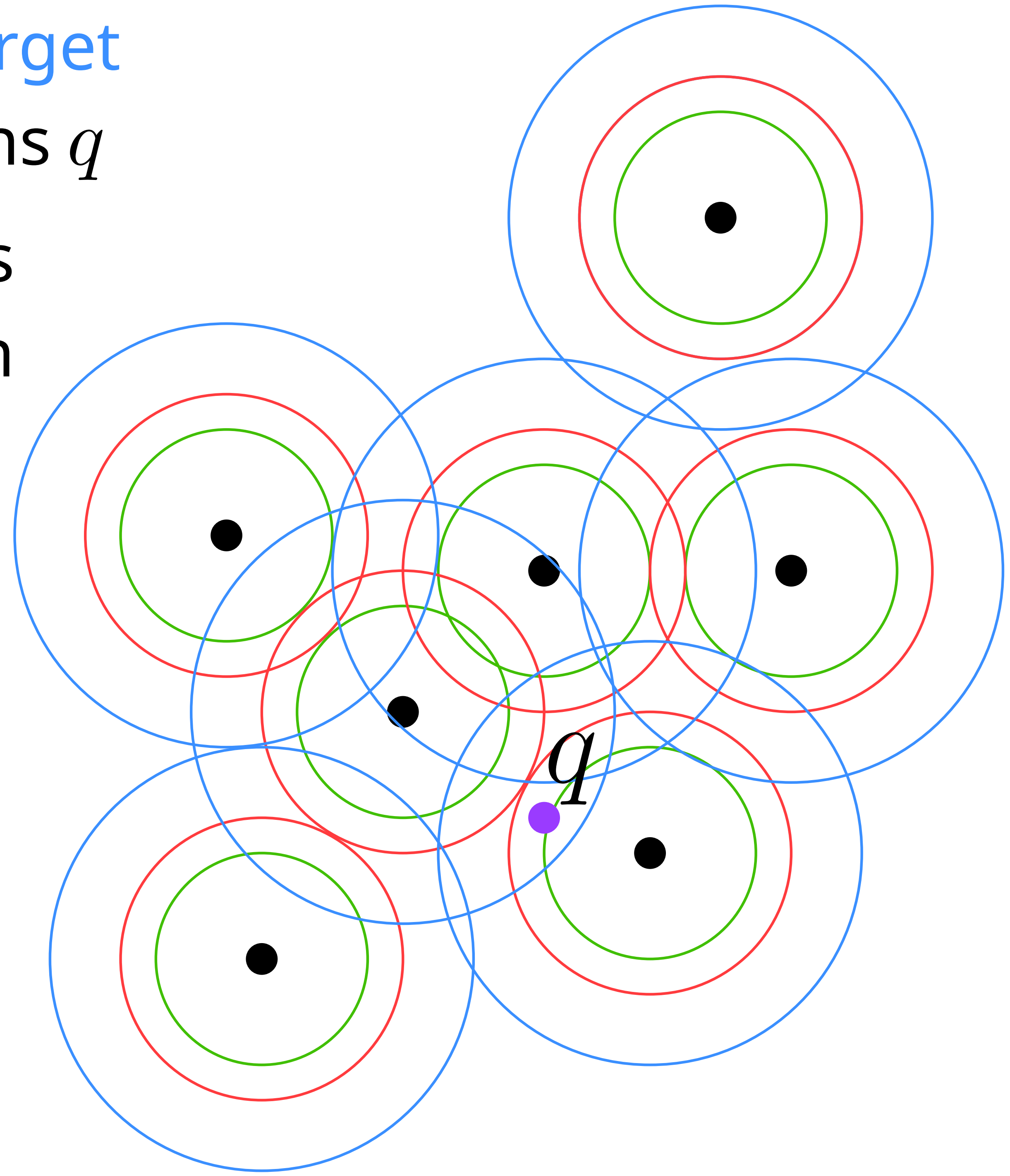

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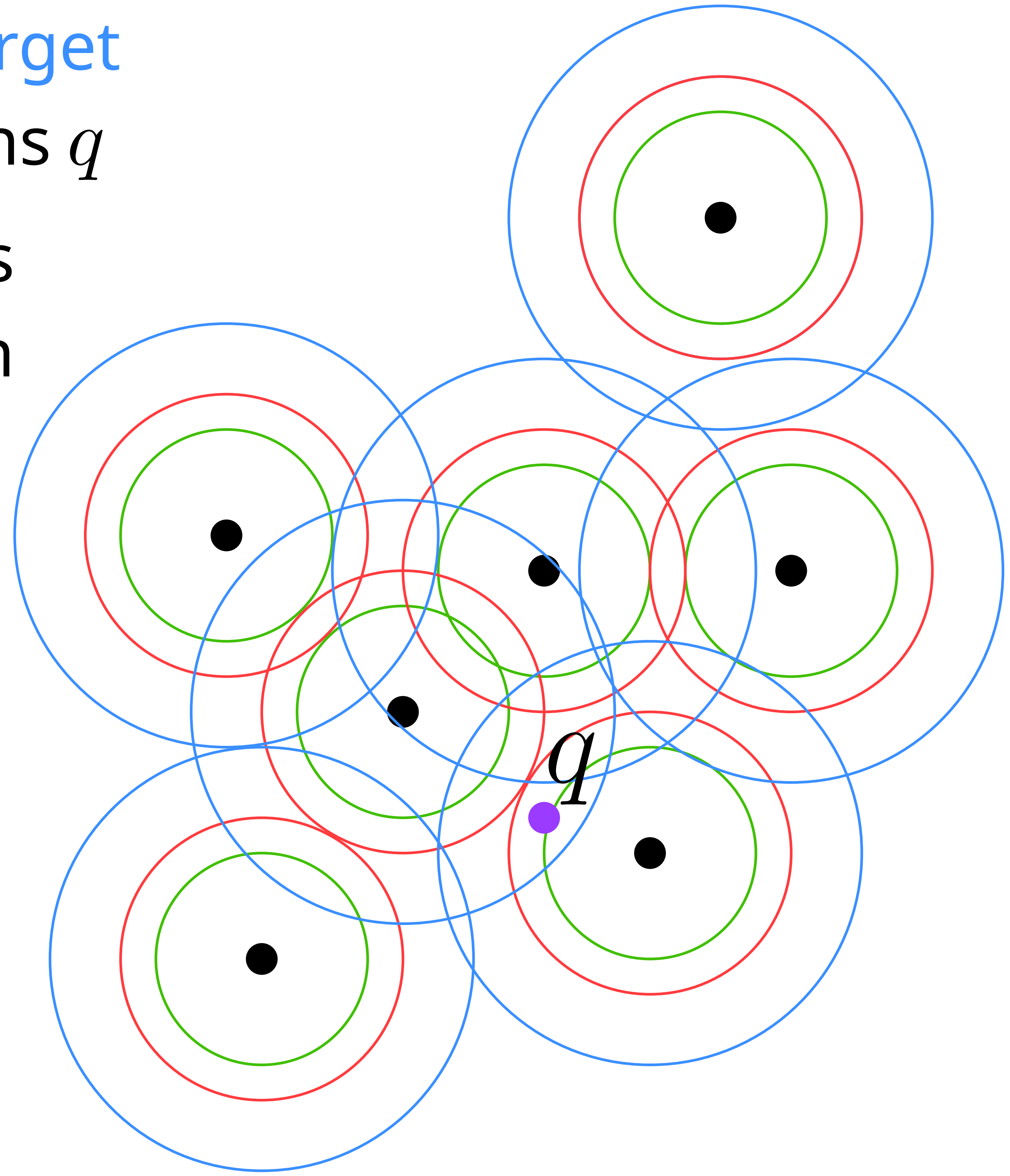
  - Euclidean space: shifted quadtree



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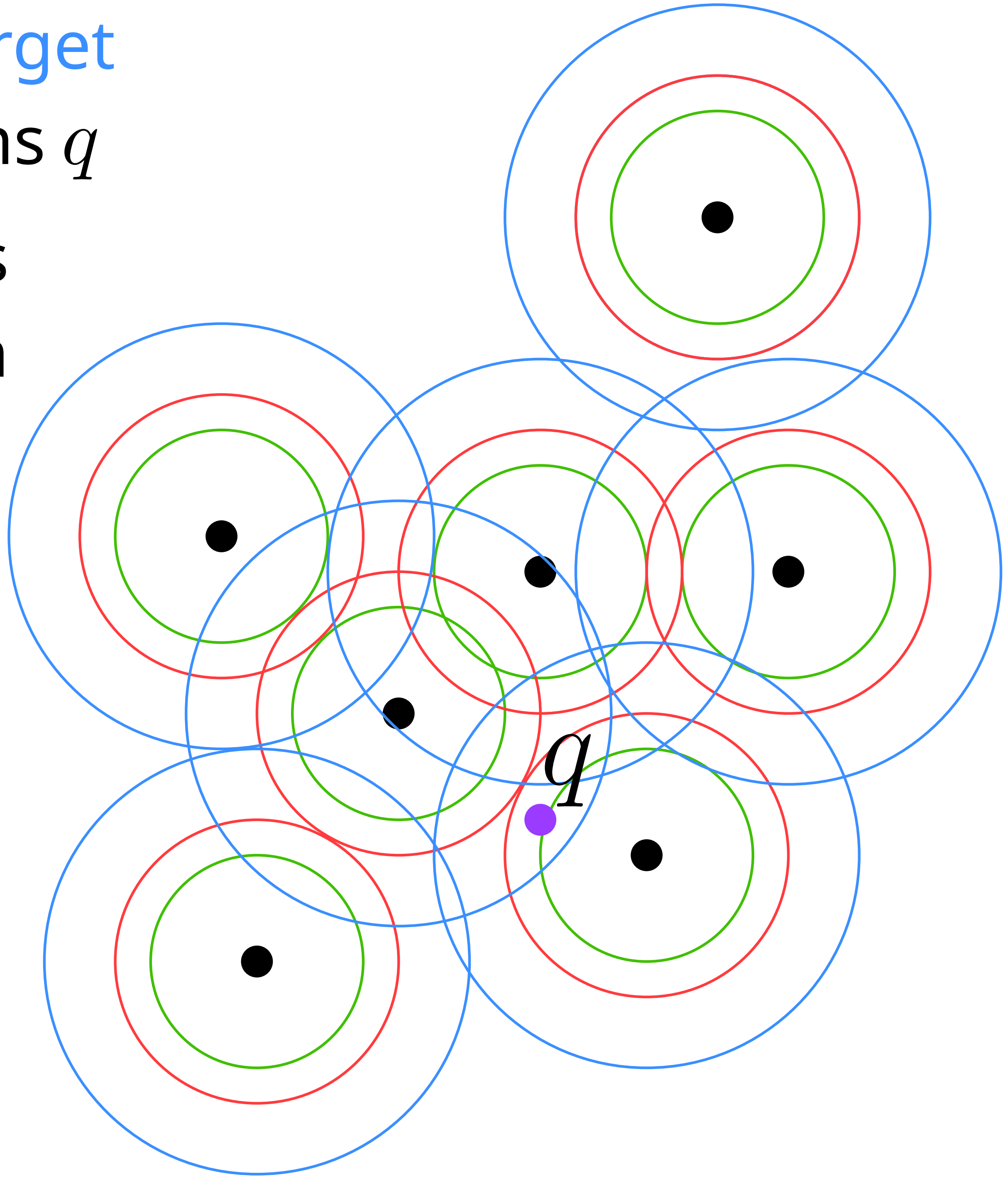

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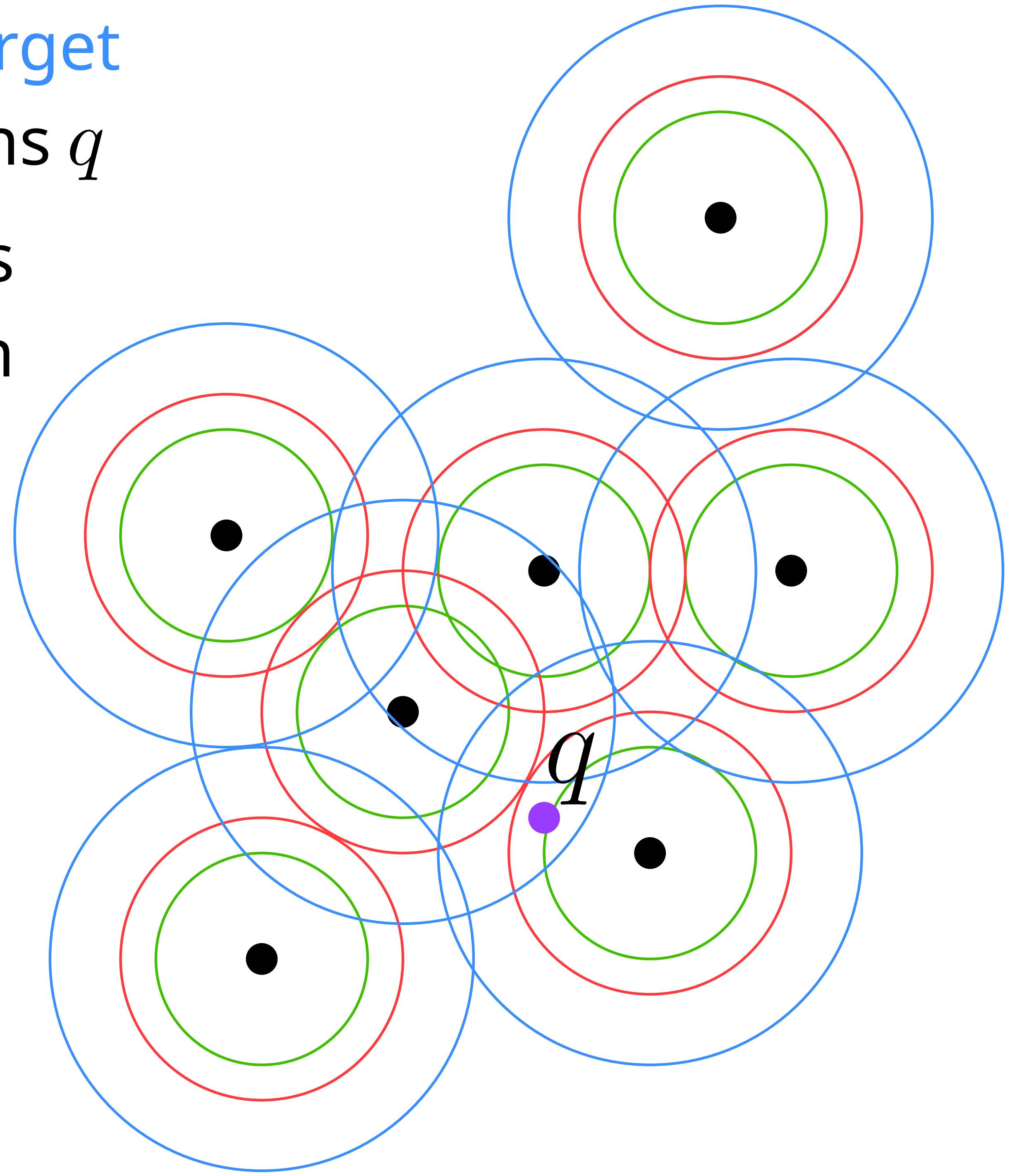

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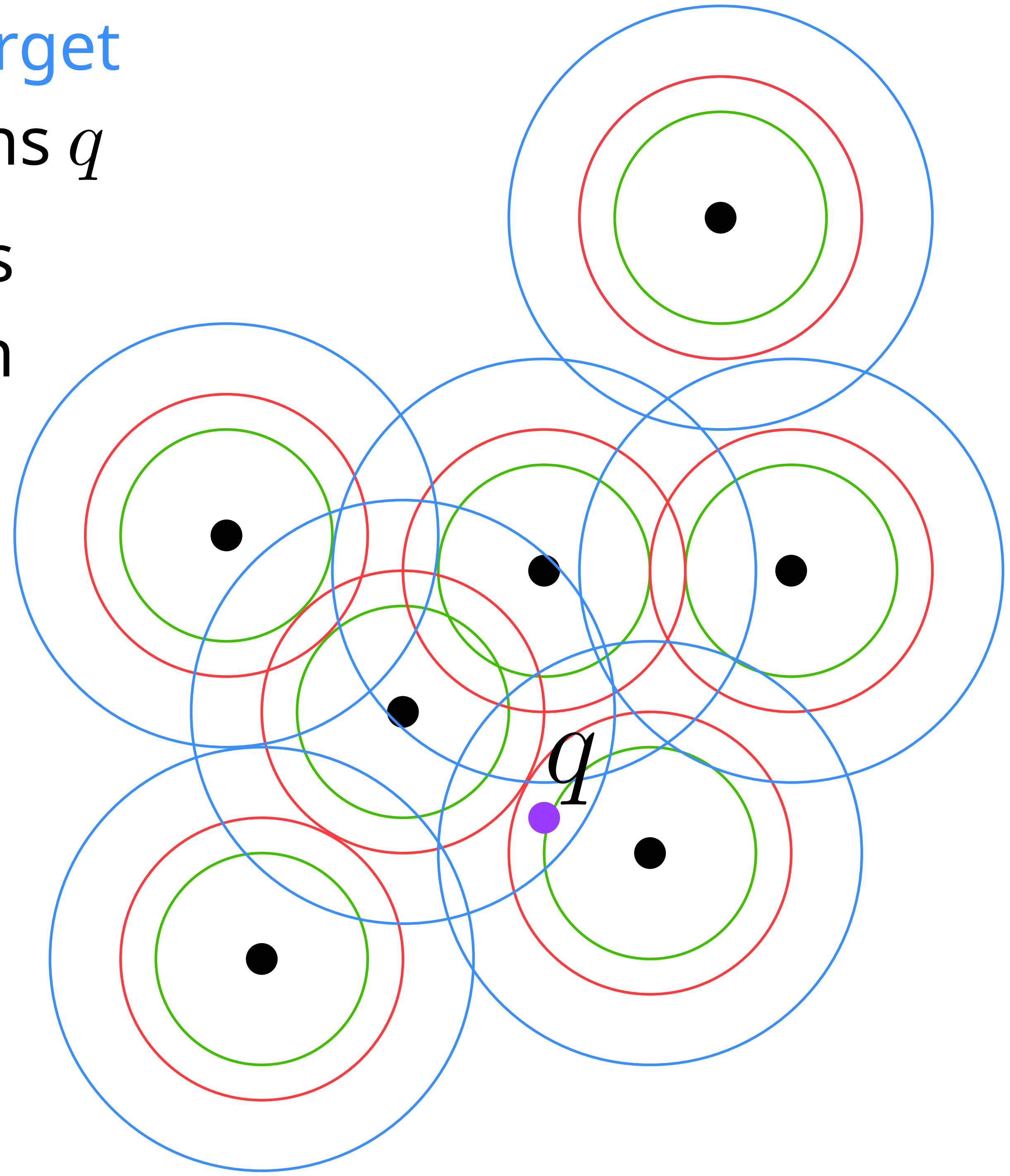
points occur up to $\log n$ times



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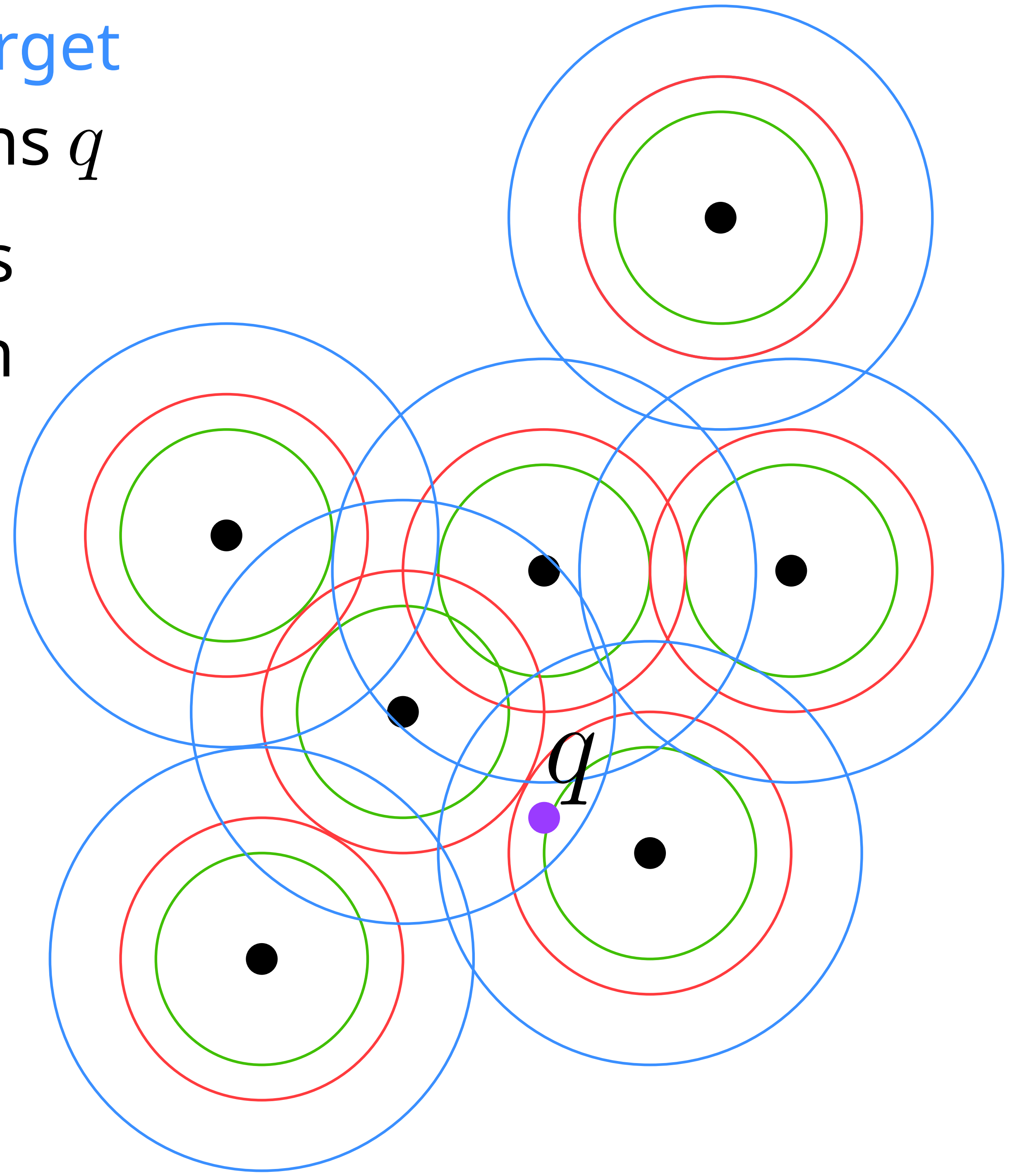

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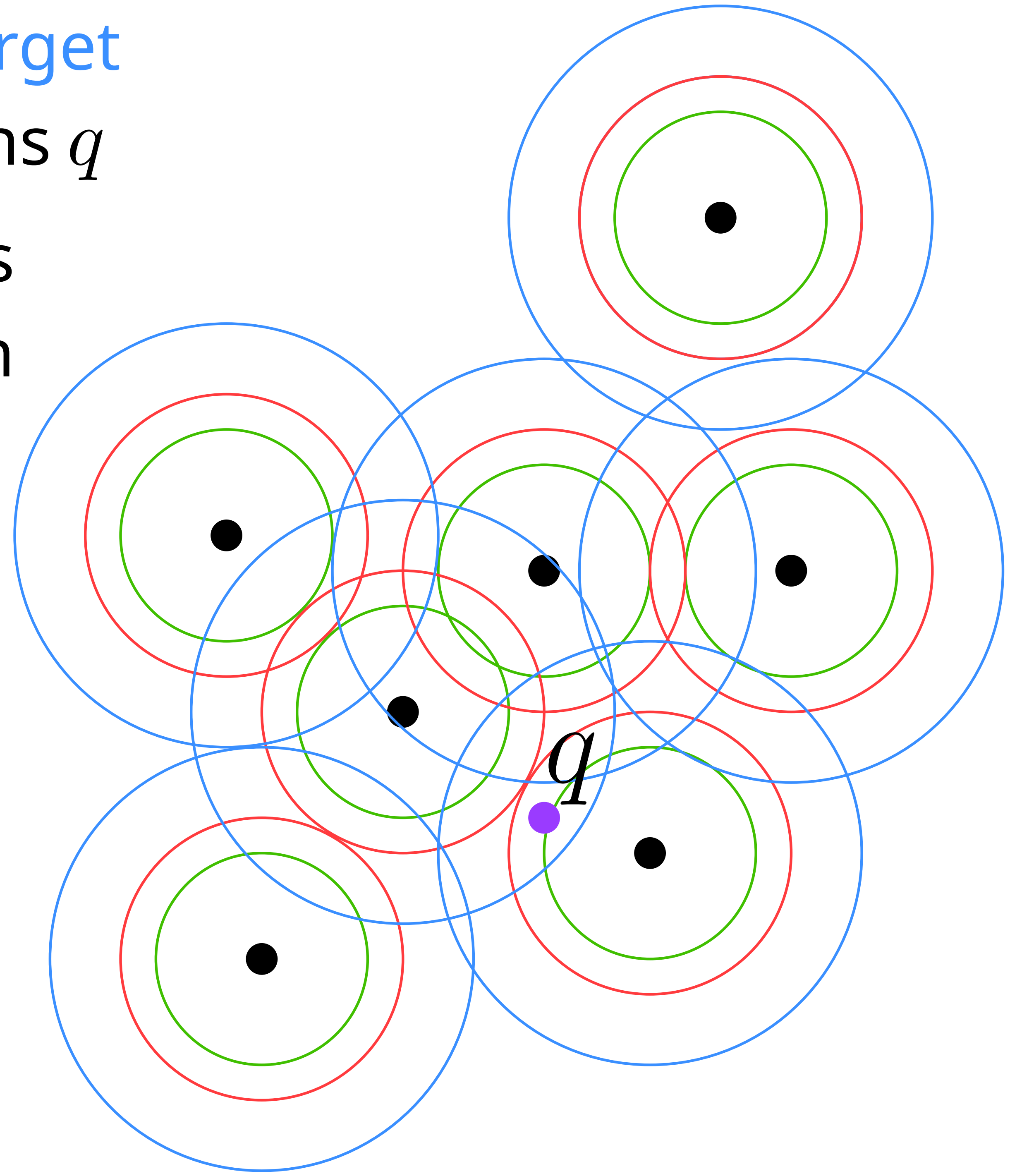

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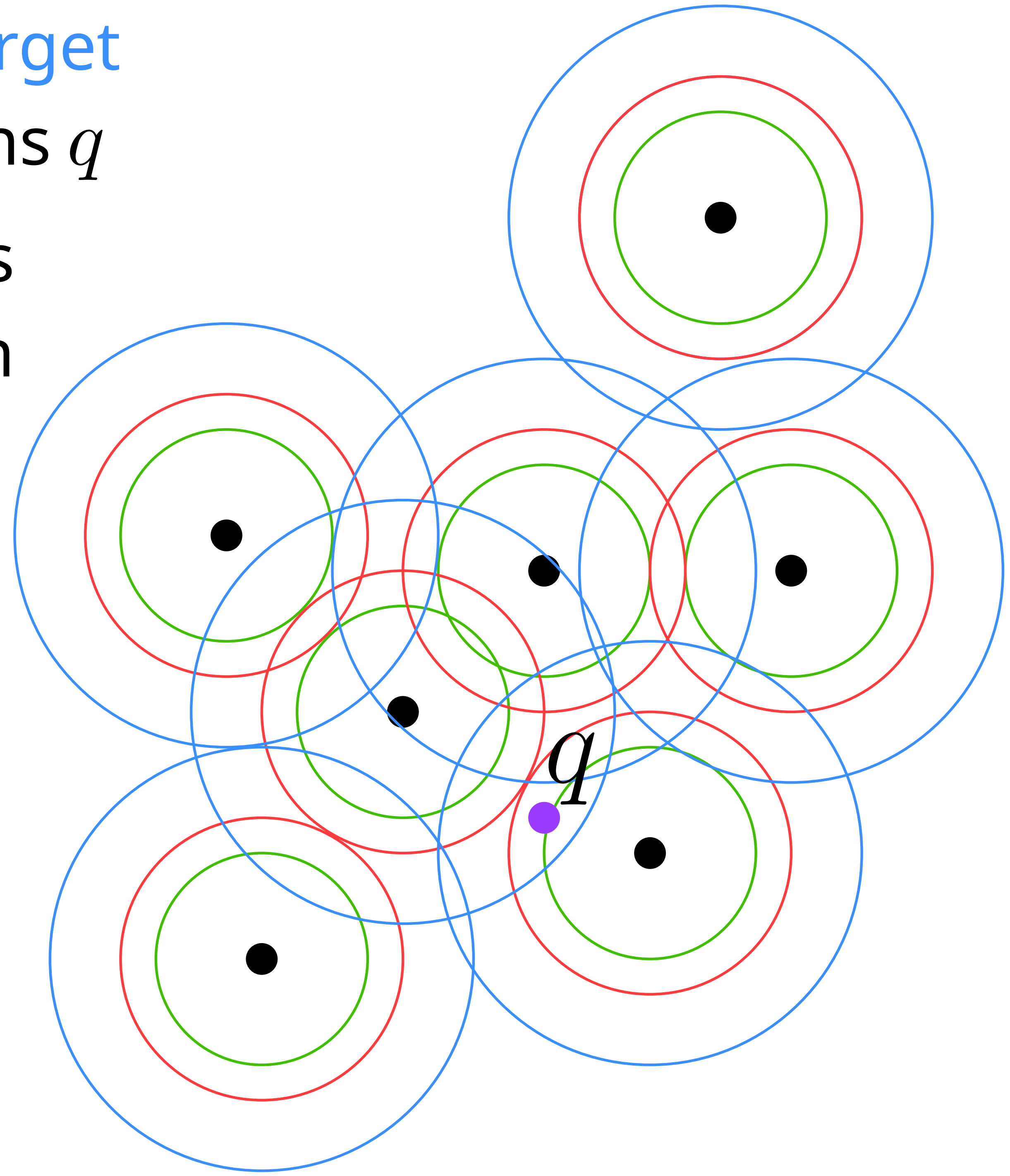

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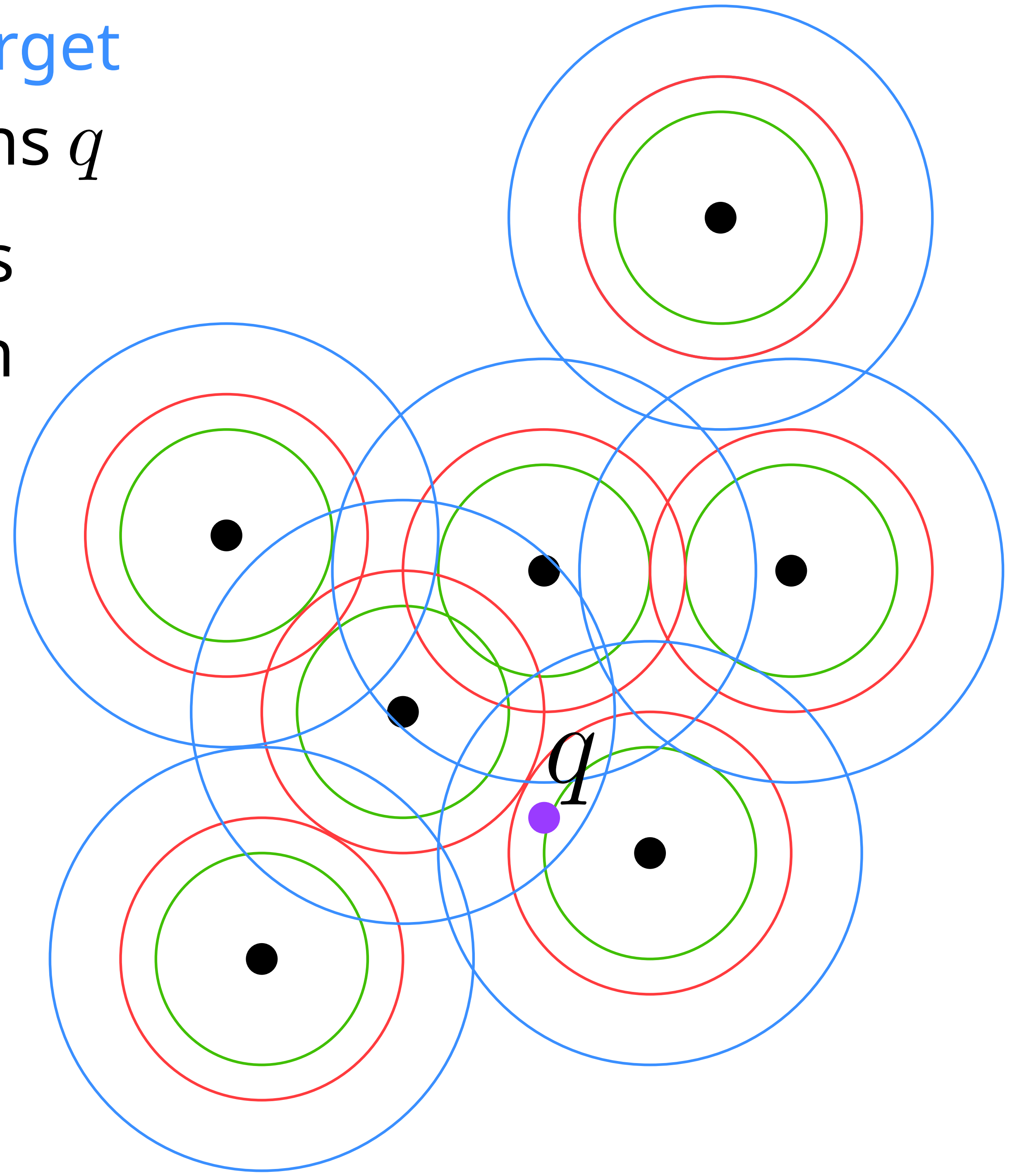

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$O(\log{(n/\varepsilon)})$ queries

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Those queries are also hard . . .

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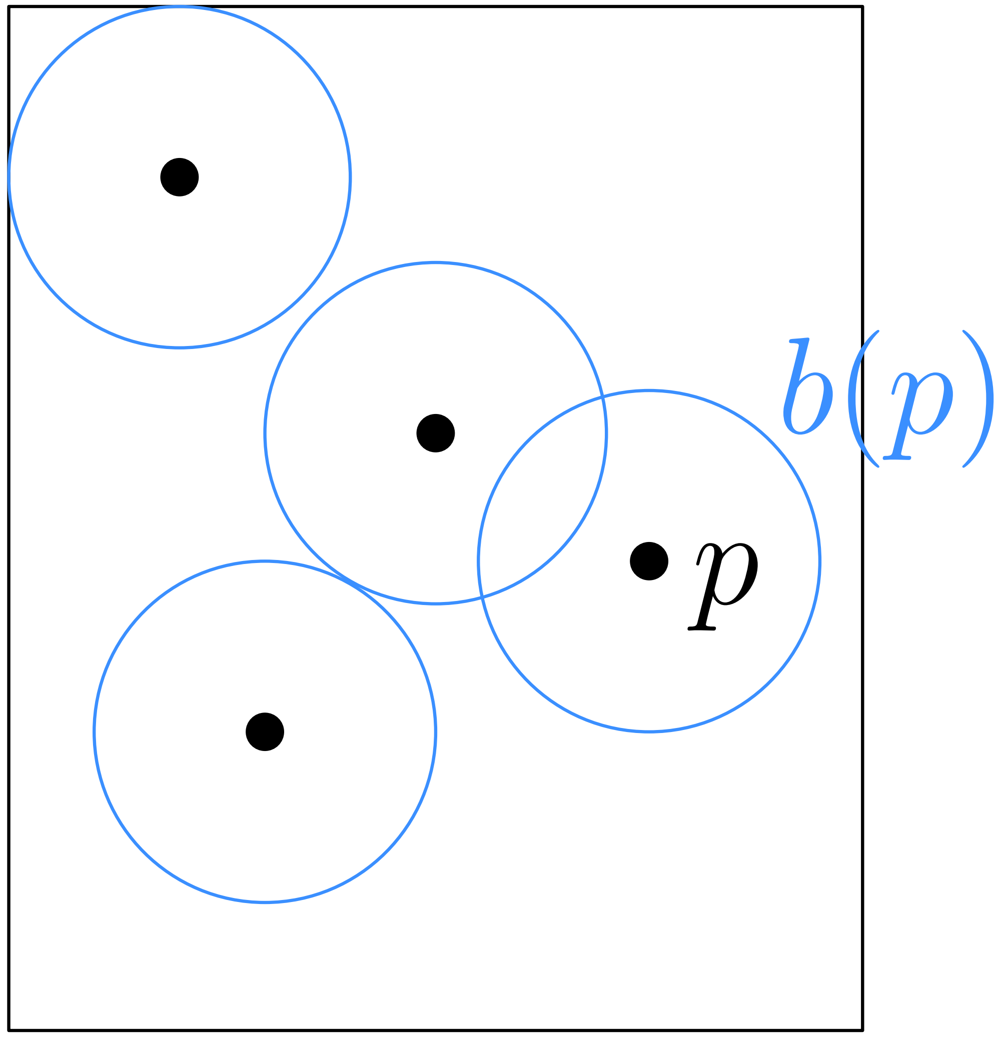
The power of grids!

Approximating the ball



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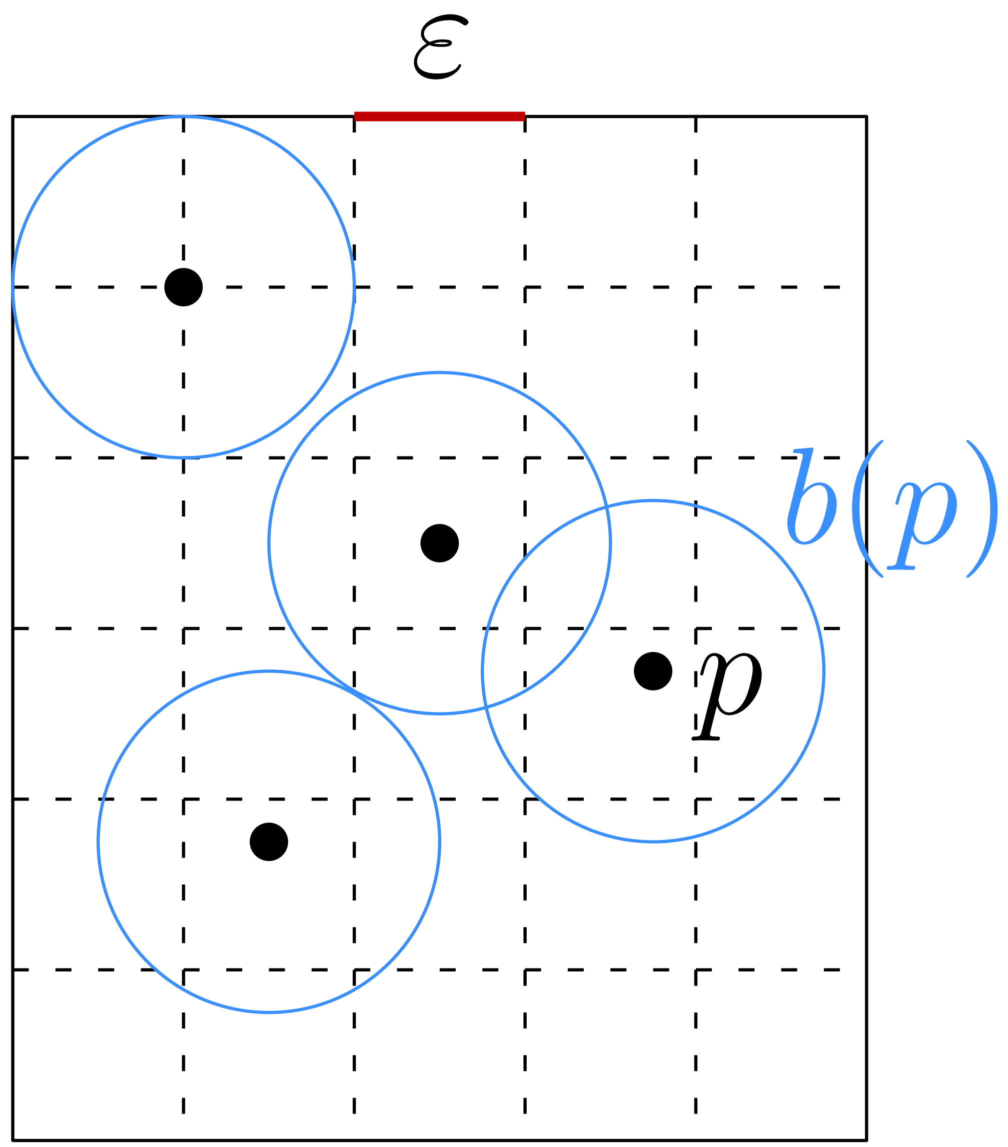

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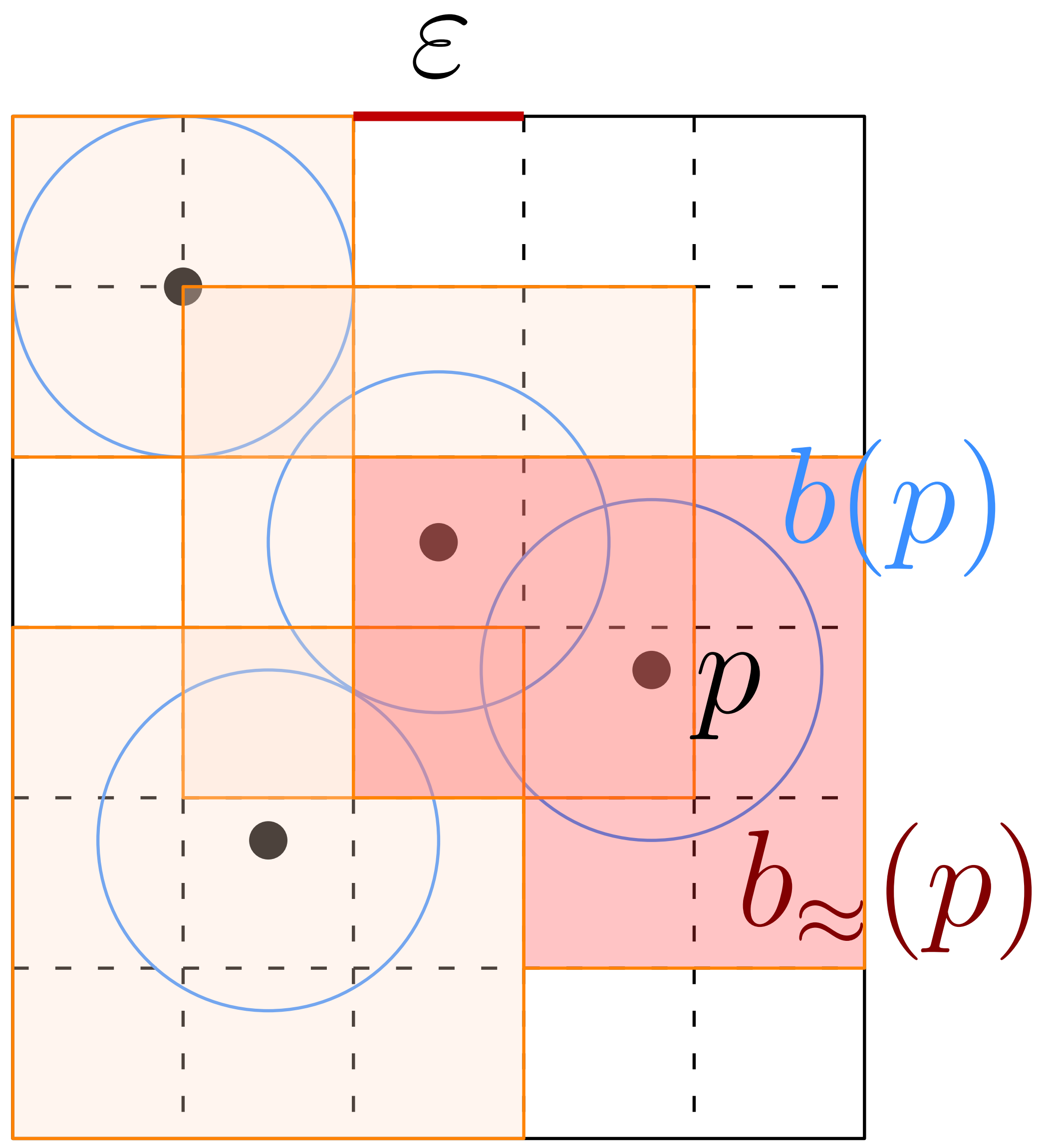

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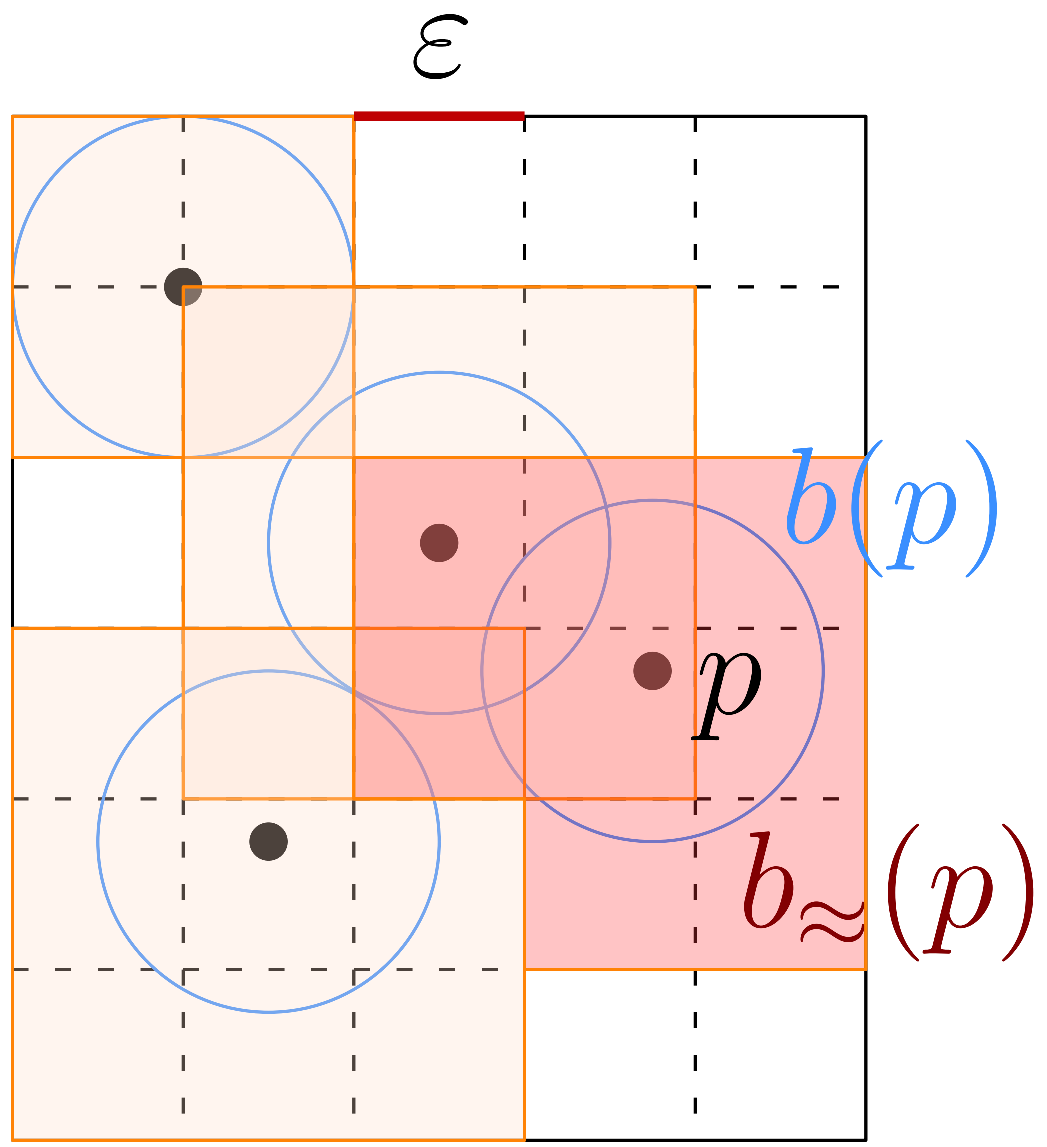

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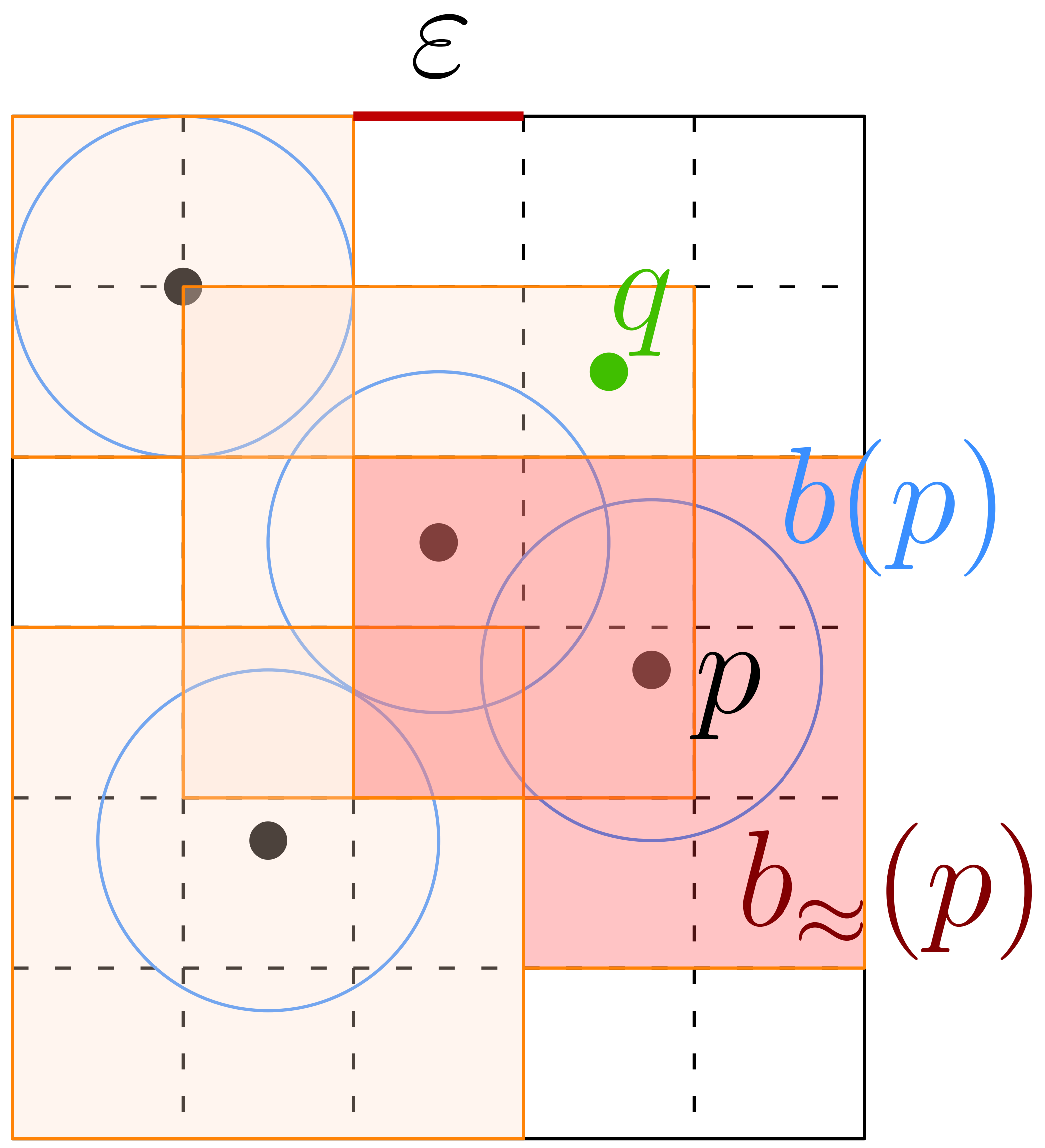

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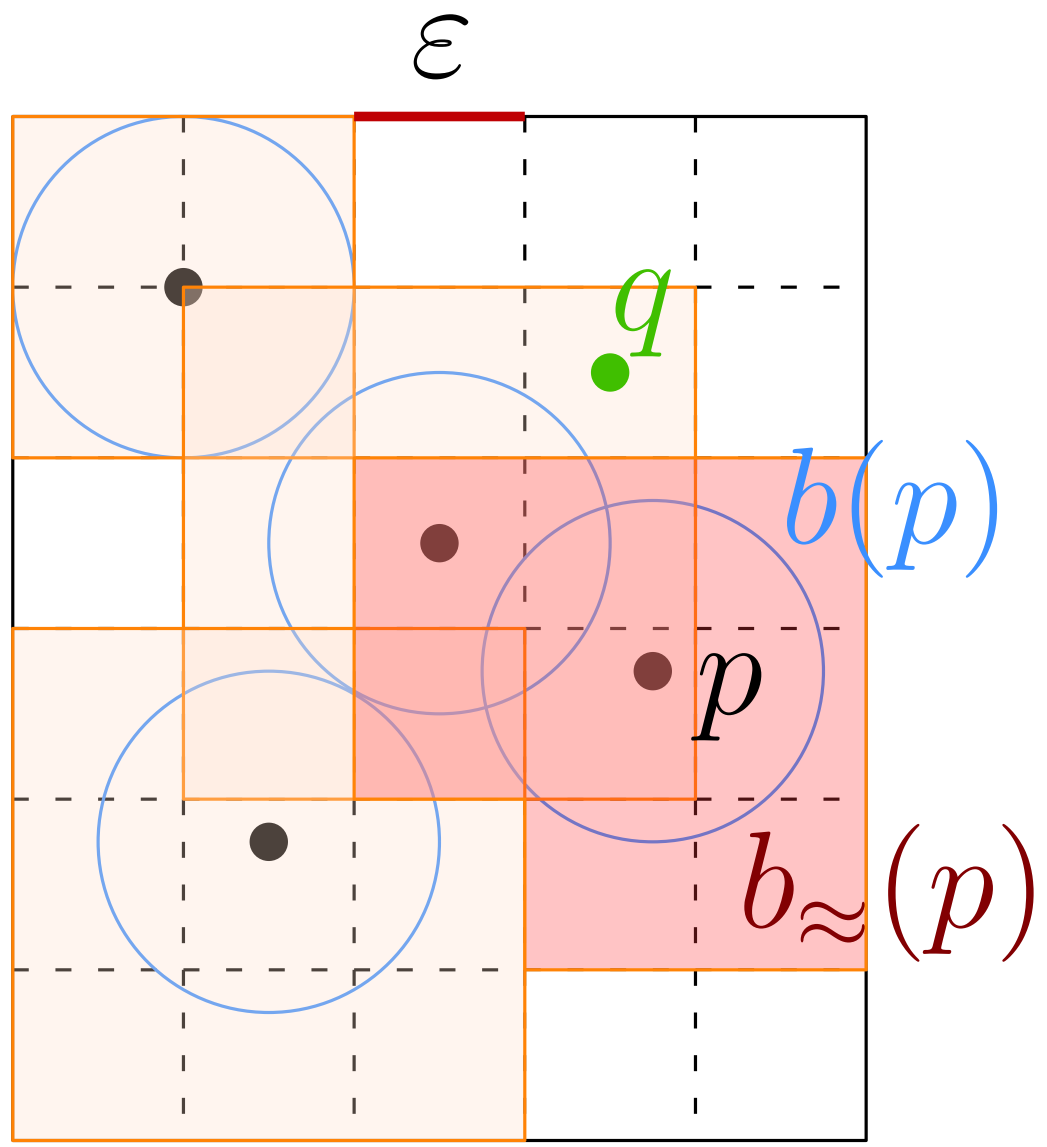

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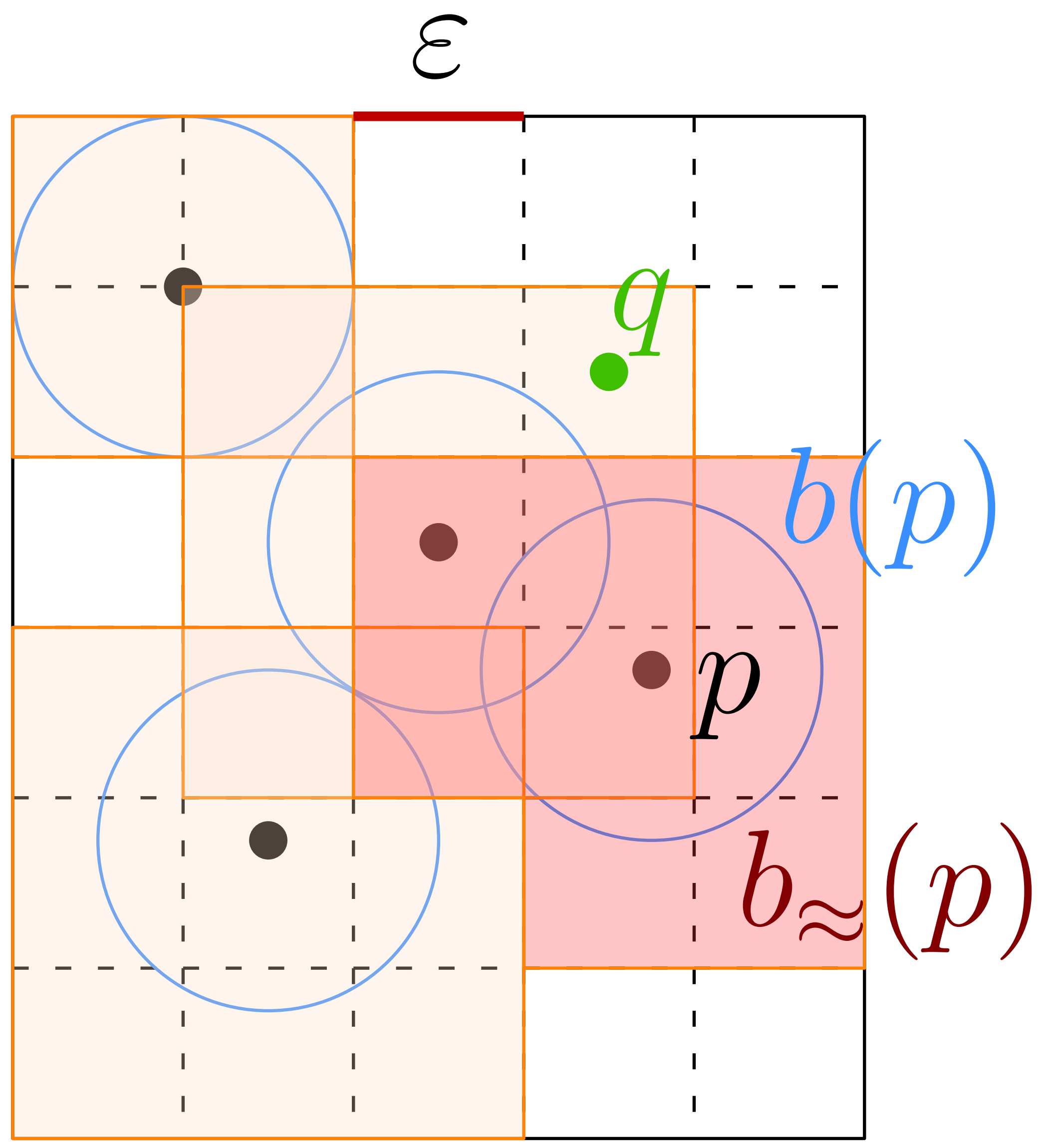

But we don't have constant ball sizes...

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Approximation from using balls

Approximation from approximating the balls

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Substitute

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$$\boxed{1+\frac{2\varepsilon}{16}+\frac{\varepsilon^2}{16^2} = 1+\frac{\varepsilon}{8}+\frac{\varepsilon}{16} = 1+\frac{3\varepsilon}{16} = O(1+\frac{4}{\varepsilon})}$$

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- s.t. answering $(1+\varepsilon)$-ANN queries on P can be answered by doing a single target query on $\mathcal{B}$
- Furthermore, if we $(1+\varepsilon/16)$-approximate each ball the target query becomes easier.

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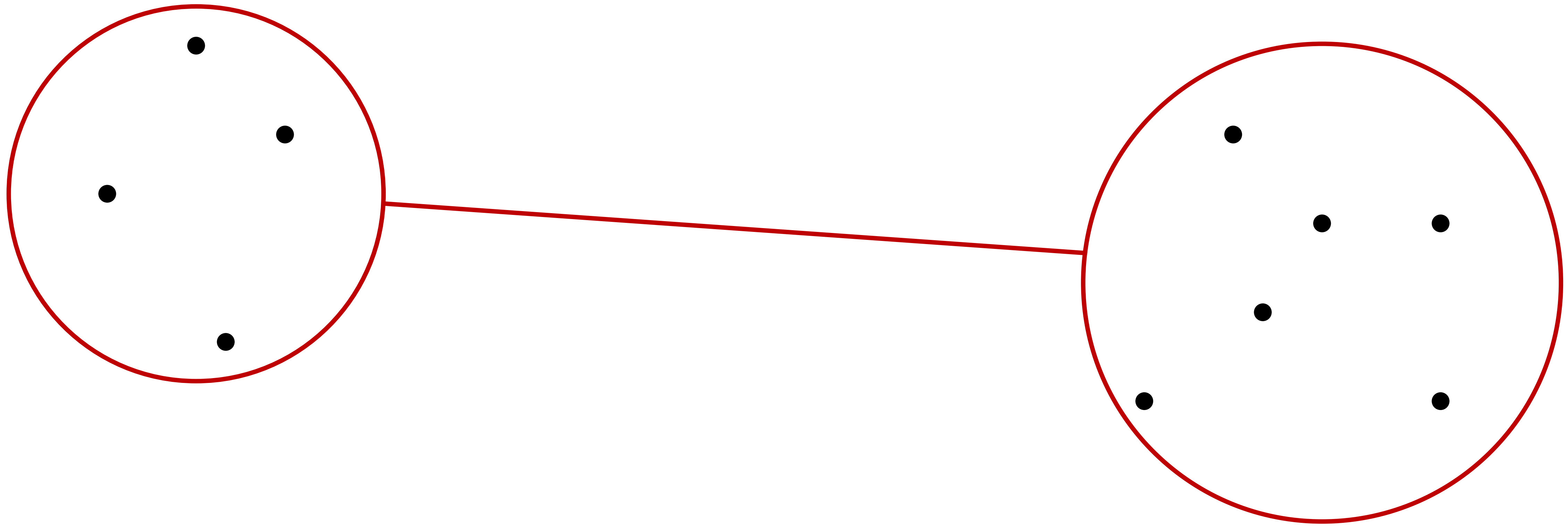
- Well Separated Pair Decomposition!



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Improvements in low dimensions

- Construct a (c/ε)-WSPD $\mathcal{W}$ of P, where c is sufficiently large
- The number of pairs in a WSPD is $O(\frac{n}{\varepsilon^d})$
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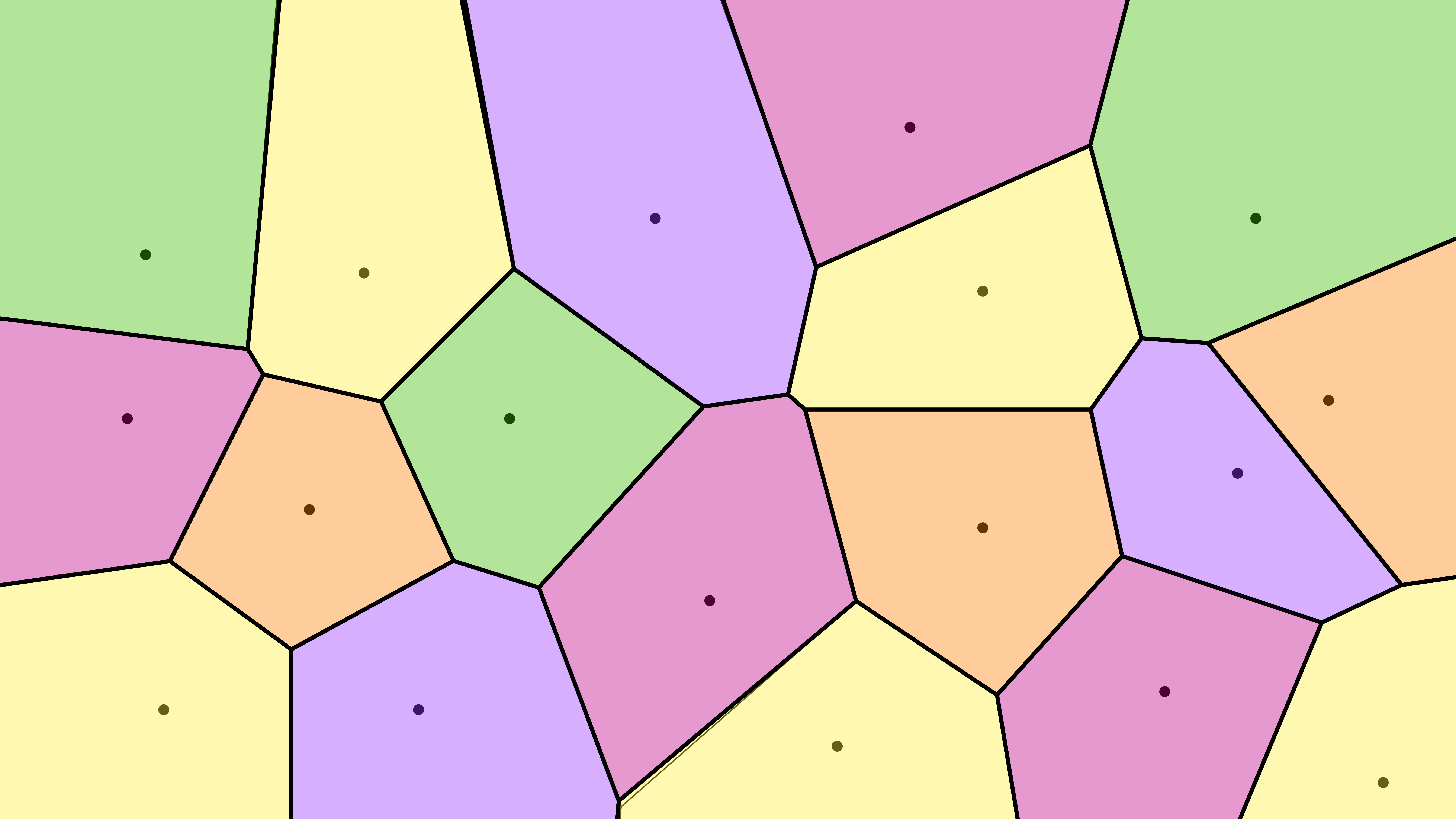
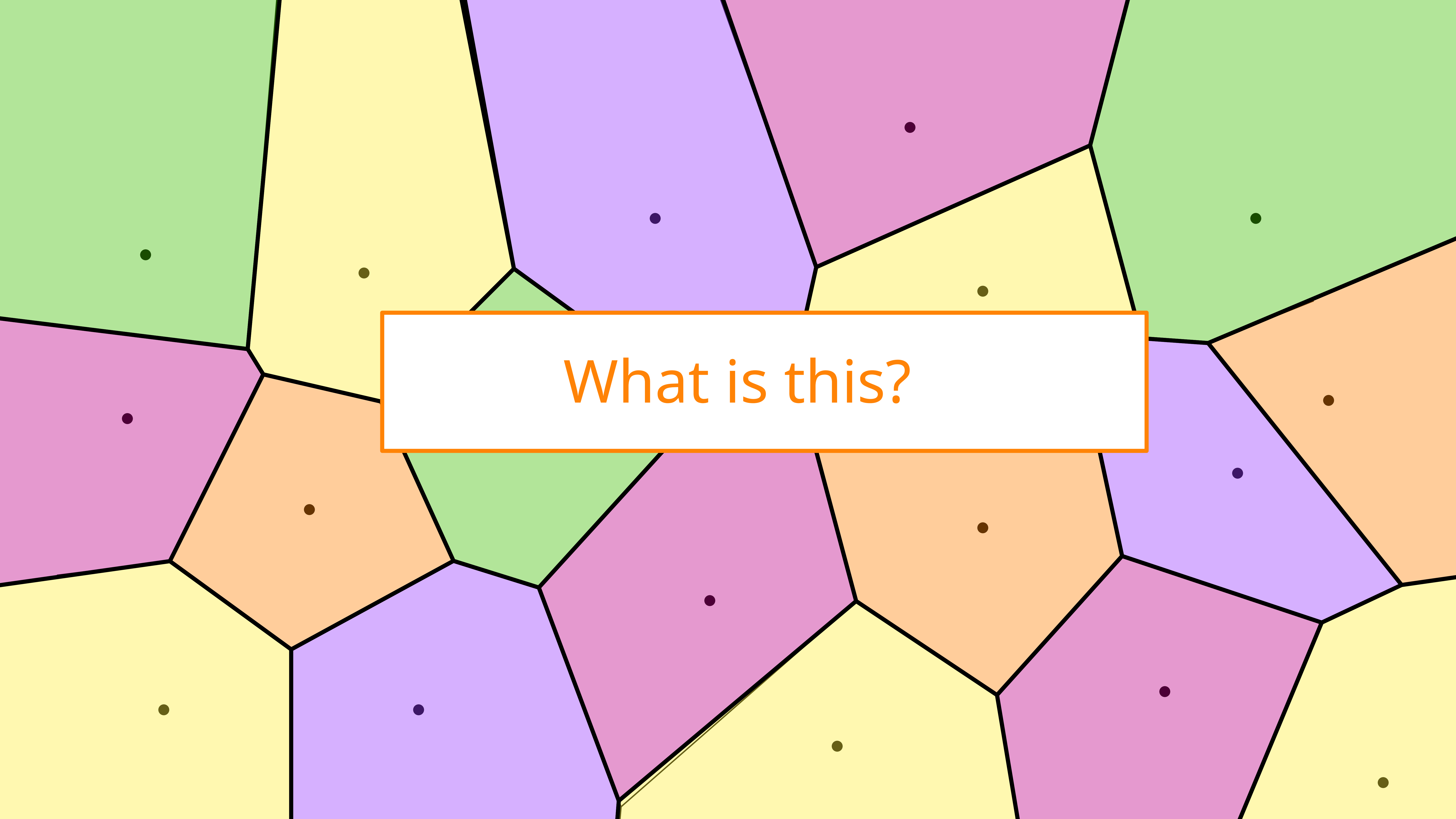
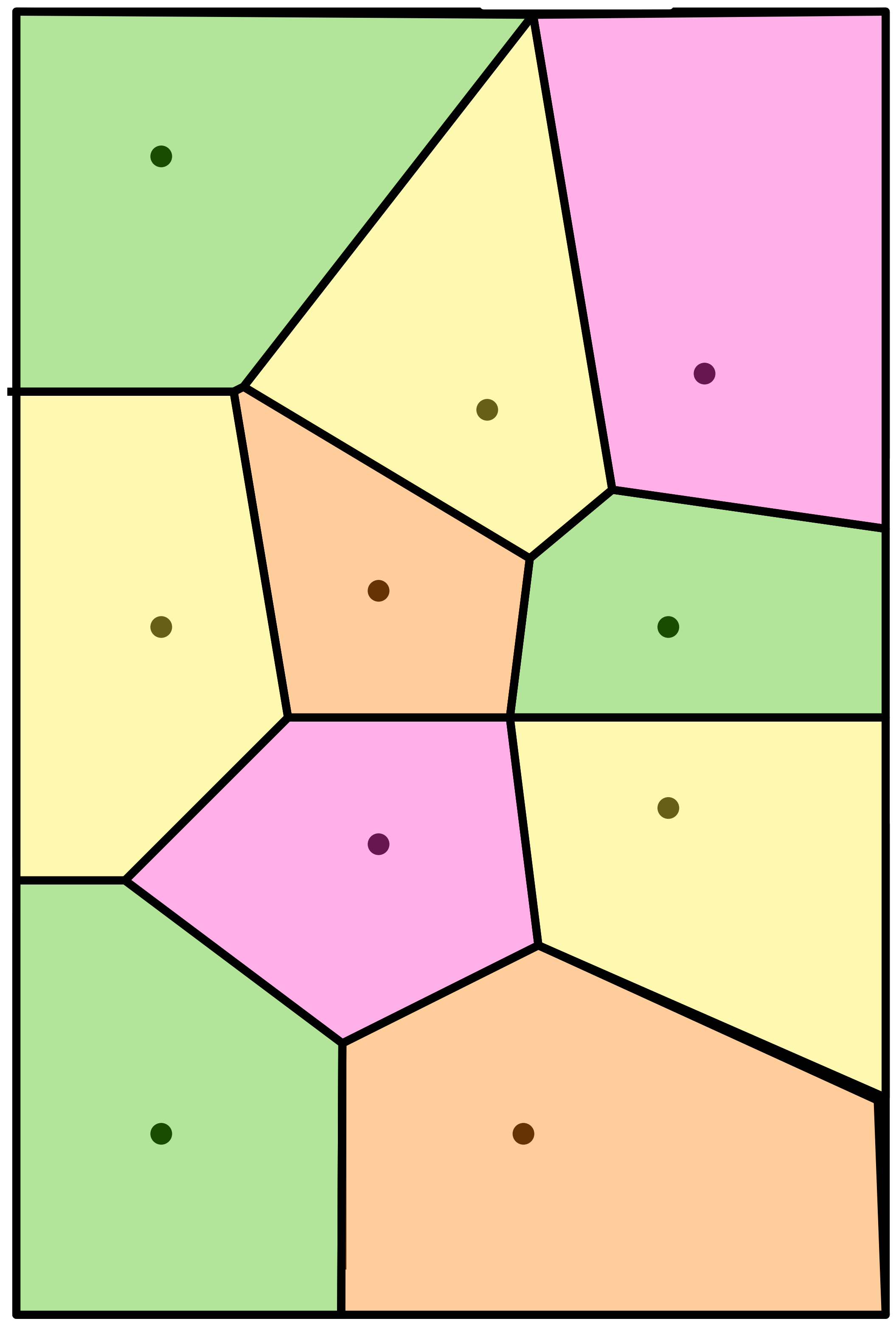
Correctness proof: as exercise




What is this?

Motivation



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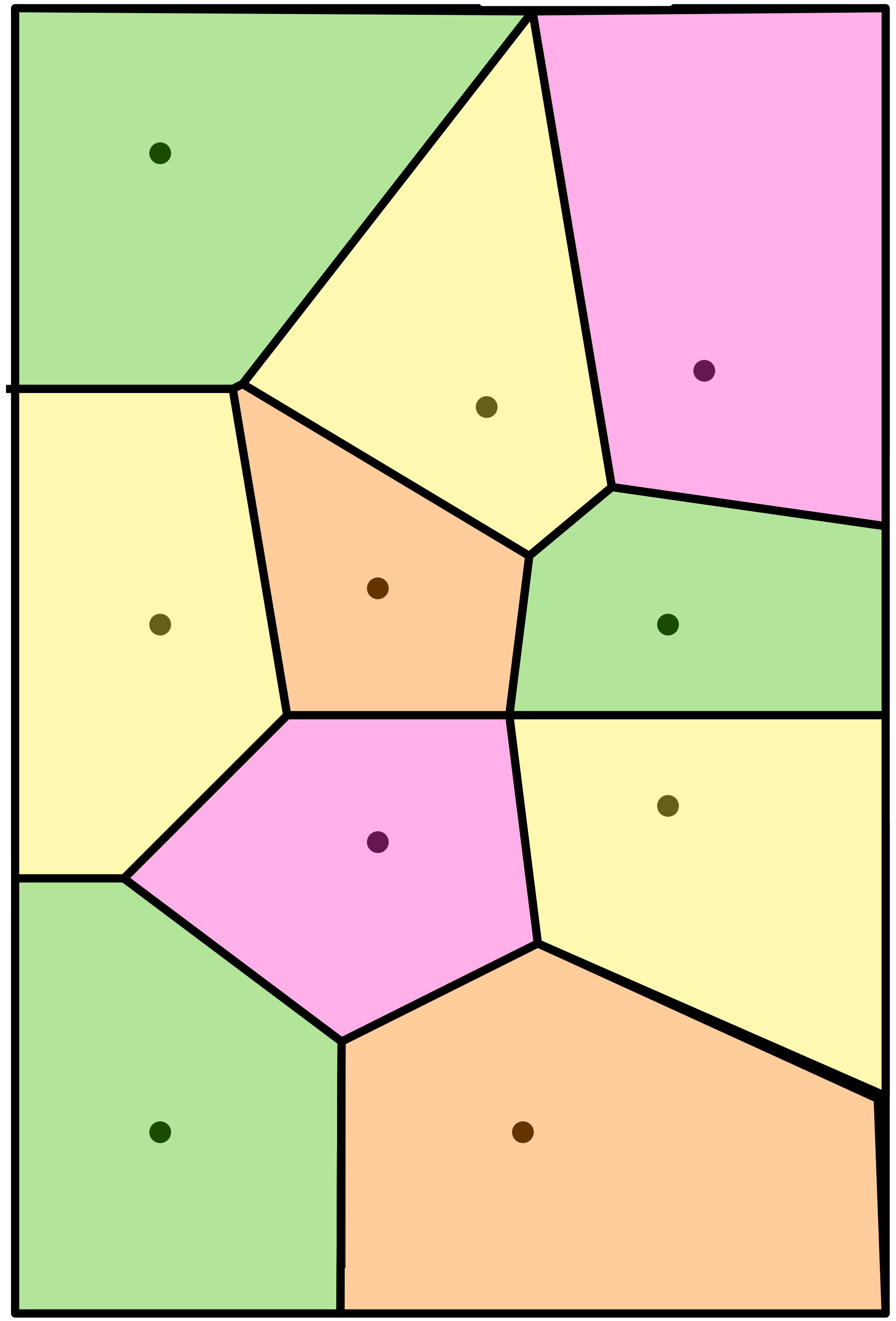
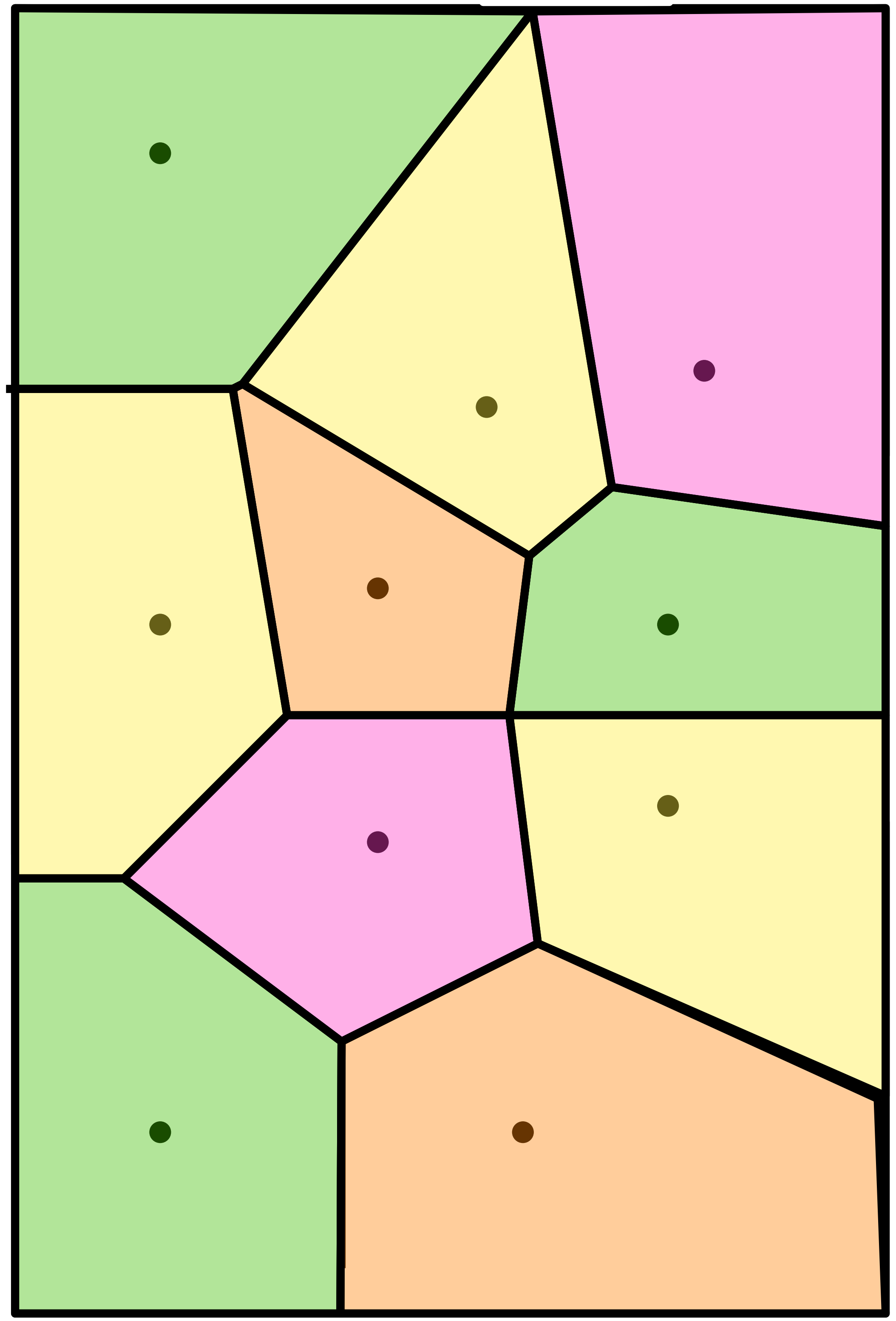
Voronoi diagrams have a multitude of uses:



Motivation

Voronoi diagrams have a multitude of uses:

- *Biology* Model biological structures like cells
- *Hydrology* Calculate the rainfall in an area based on point measurements
- *Aviation* Find the nearest safe landing zone in case of failure



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A Voronoi diagram V of a point set $P \subseteq \mathbb{R}^d$ is a partition of space into regions such that a cell of point $p \in P$ is:

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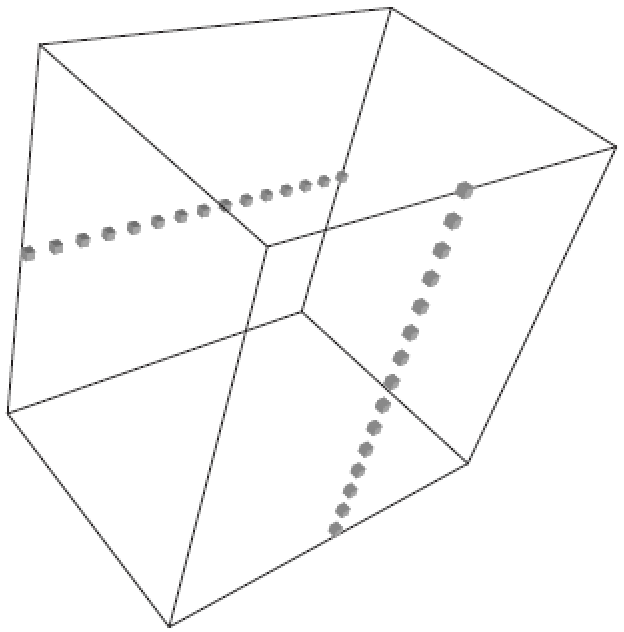 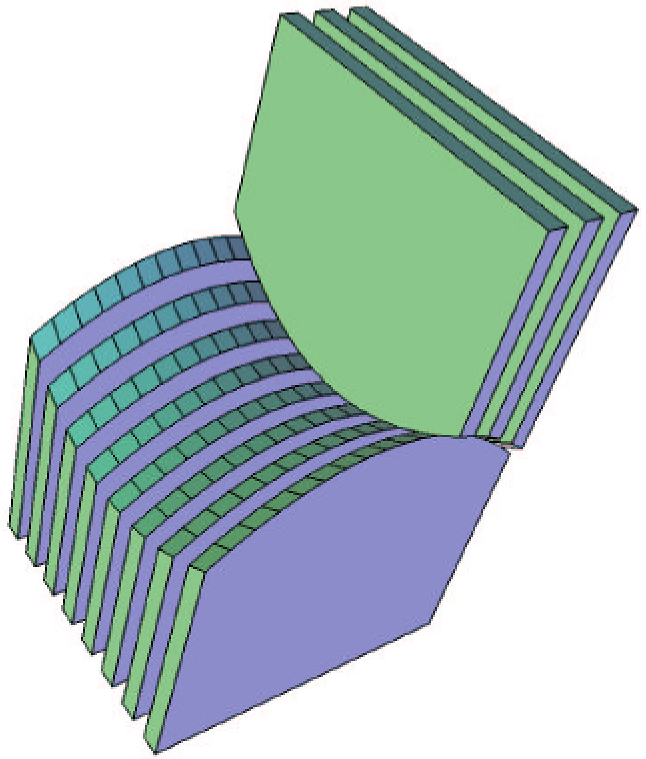 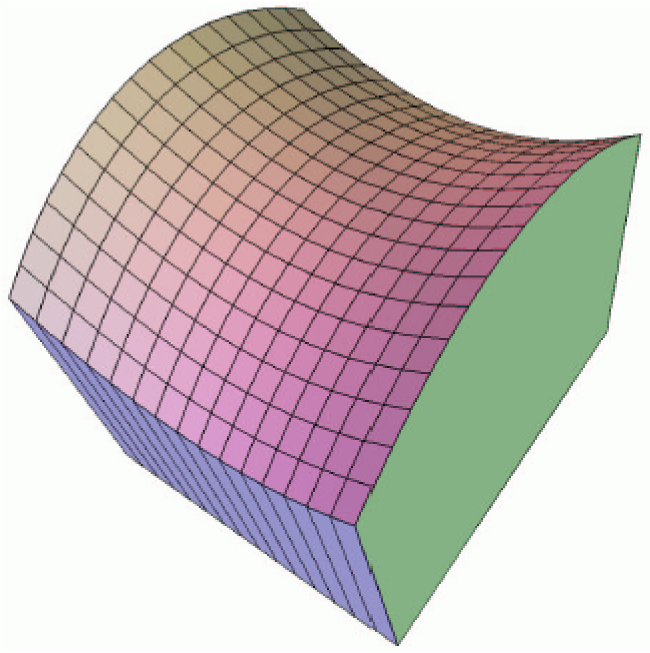

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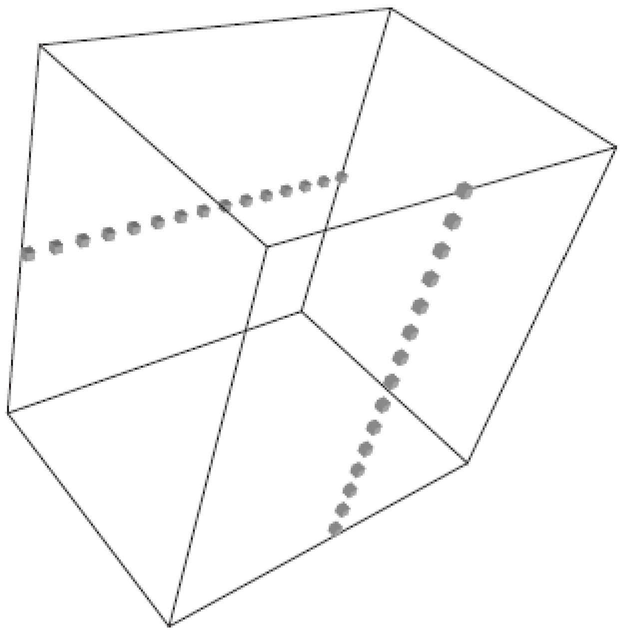
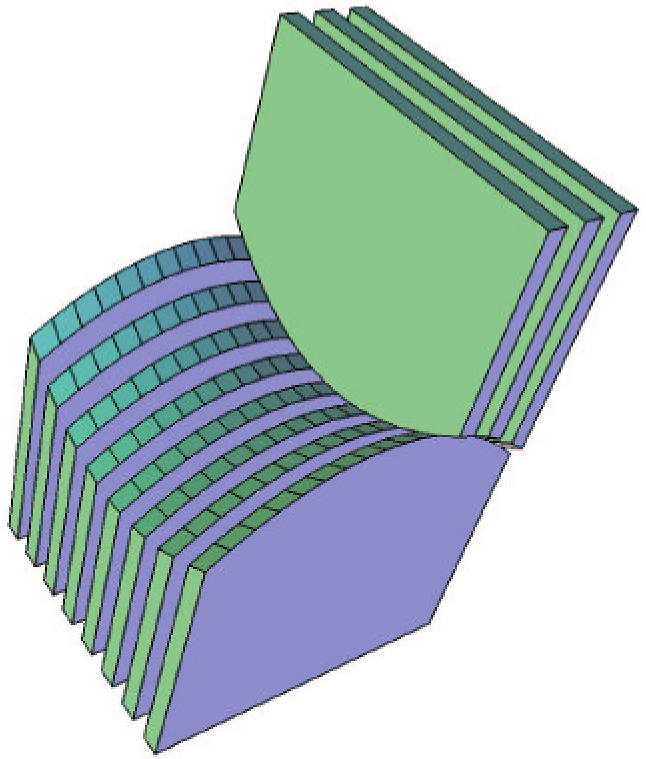
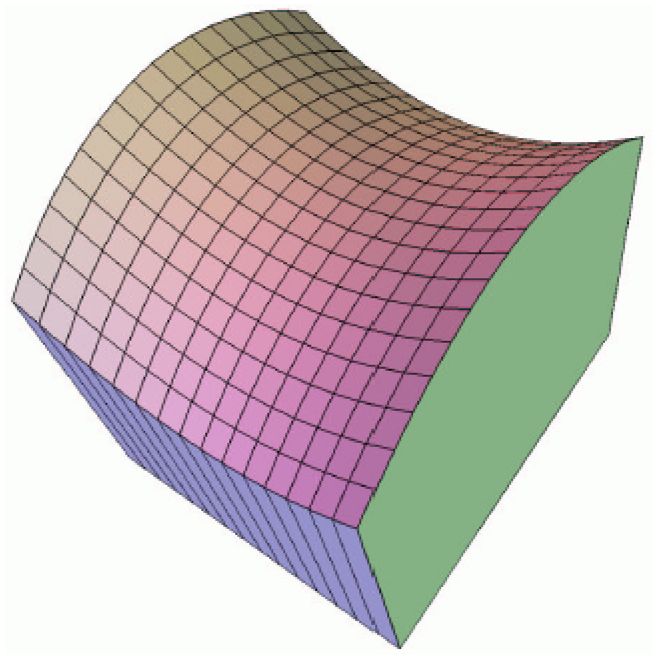
Can we do better?







Approximate Voronoi diagrams

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Given a set P of n points in $\mathbb{R}^d$ and parameter $\varepsilon > 0$, a $(1+\varepsilon)$-*Approximated Voronoi Diagram(AVS)* of P is a partition $\mathcal{V}$ of $\mathbb{R}^d$ into regions φ, s.t. for any region $\varphi \in \mathcal{V}$ we have that rep_φ is a $(1+\varepsilon)$-ANN for x, that is:

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Approximate Nearest Neighbors in $\mathbb{R}^d$

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(now fast, using approximate Voronoi diagrams)

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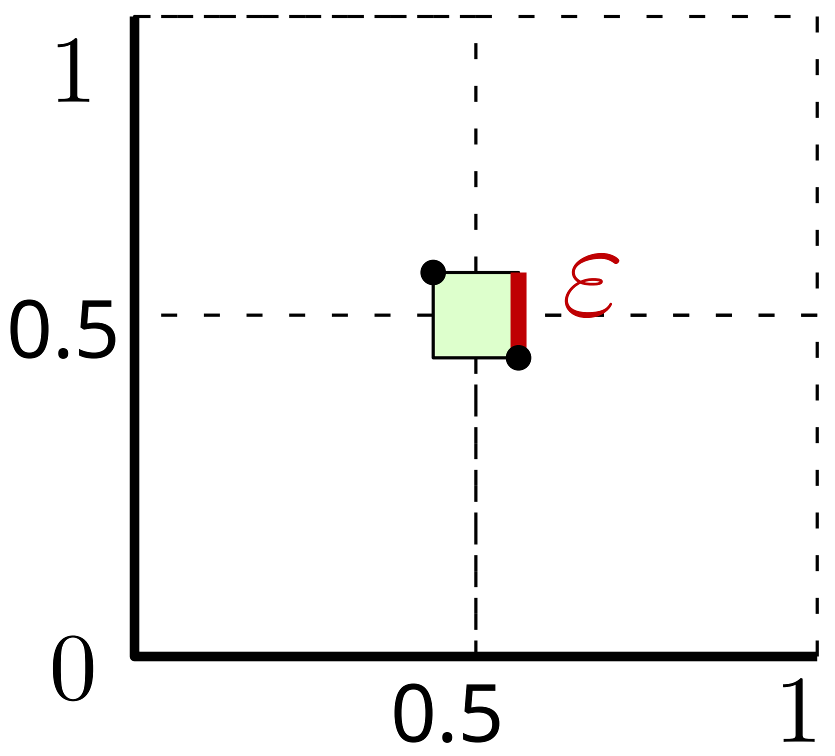

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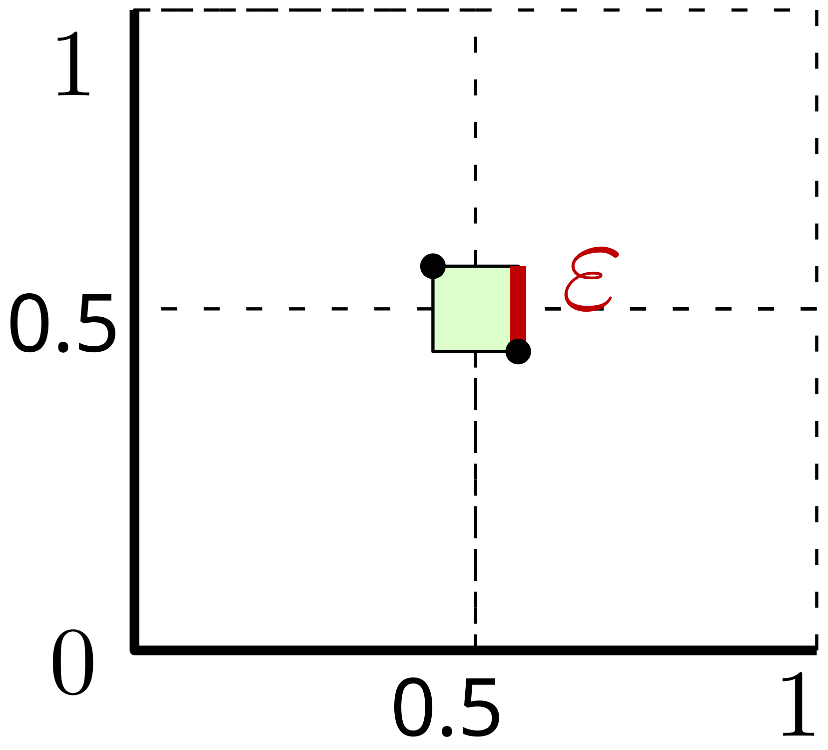

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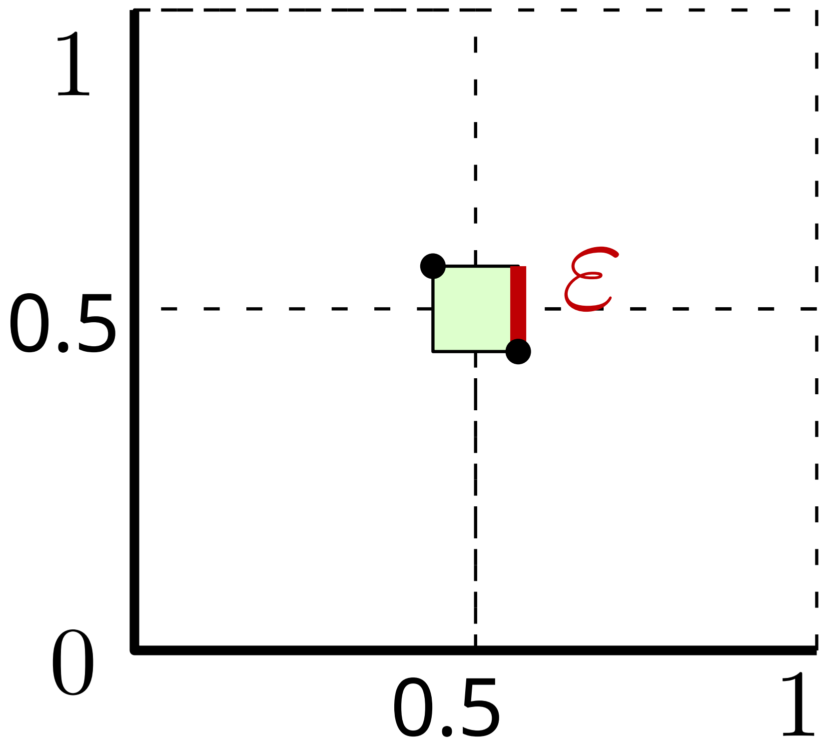

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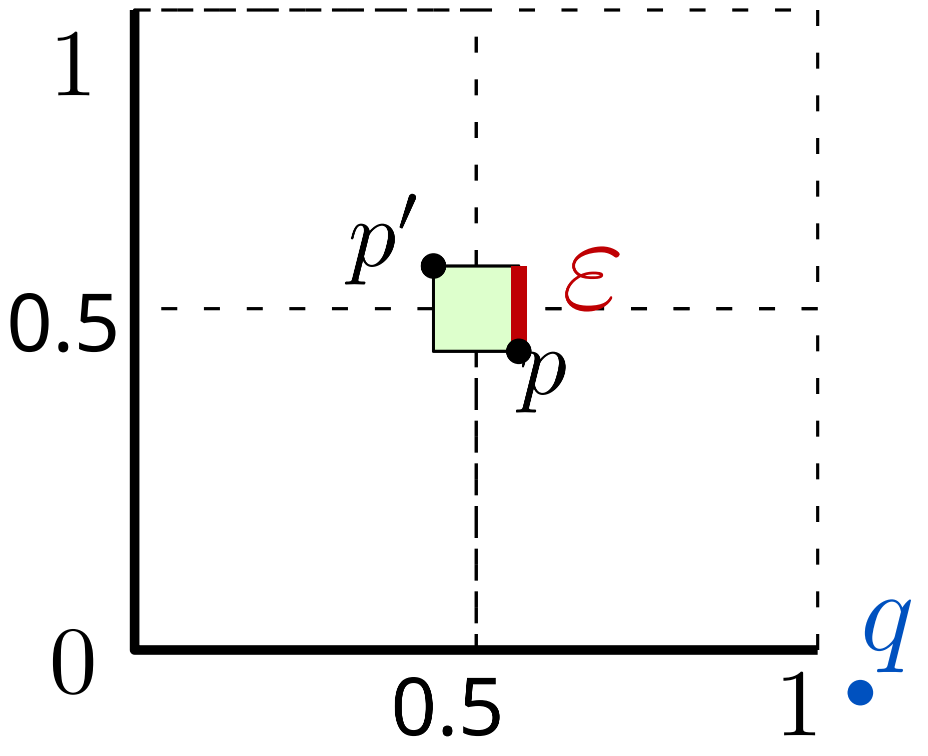

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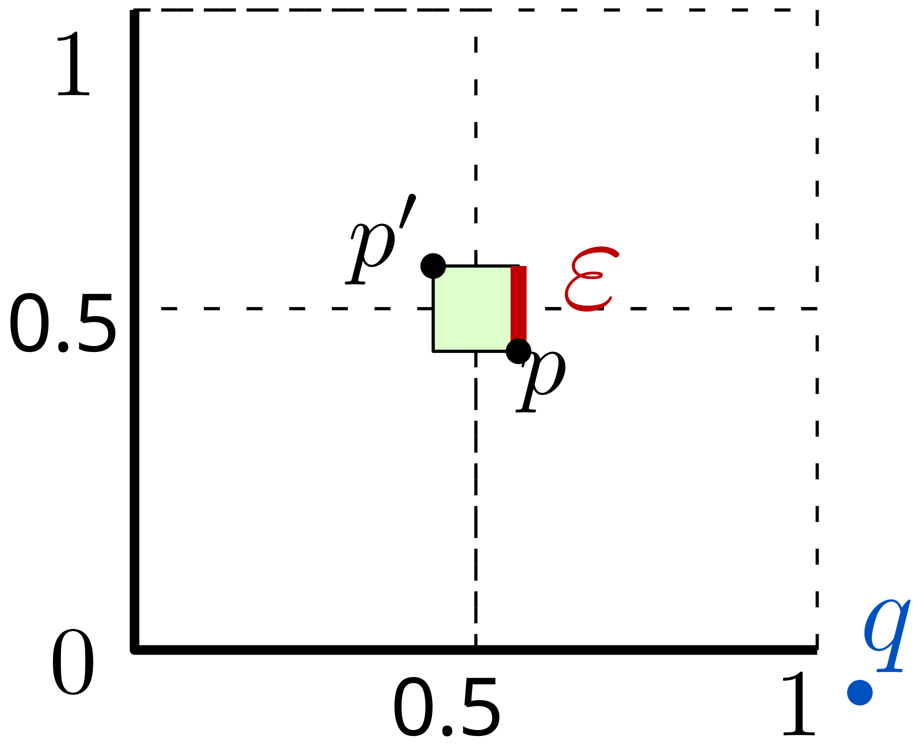
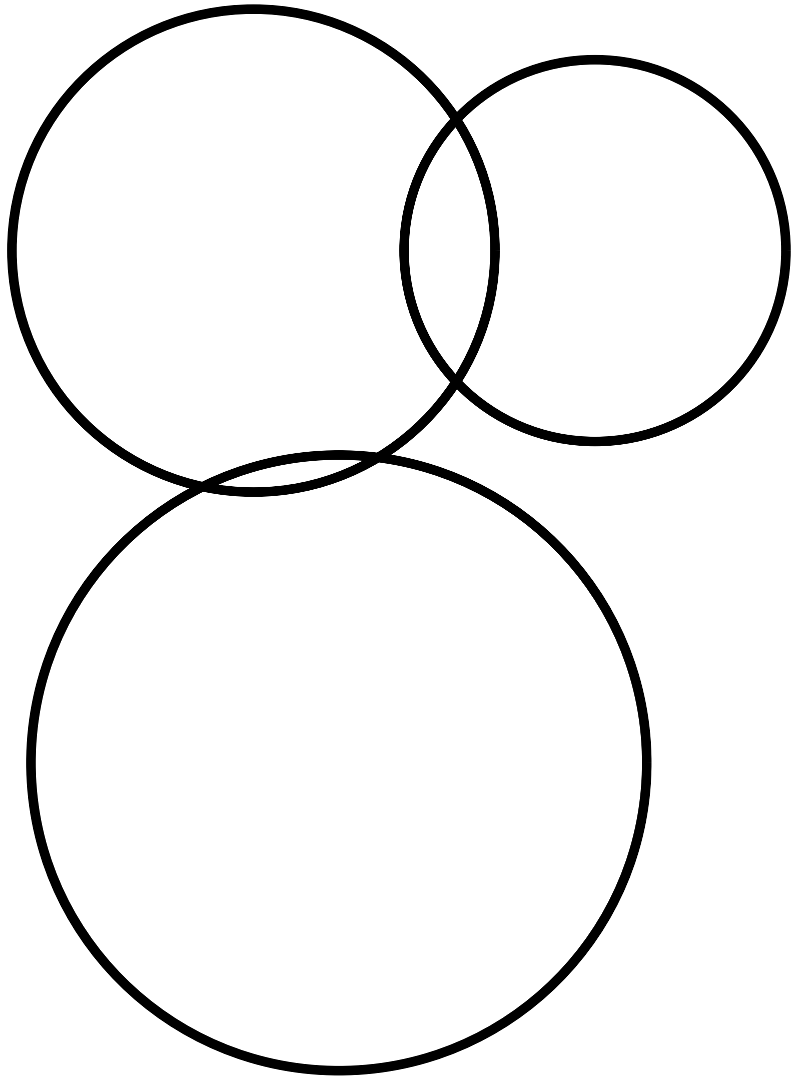


Thus only consider ANN for points inside $[0,1]^d$

Creating the AVD

- Remember we can compute a set $\mathcal{B}$ of $O(\frac{n}{\varepsilon^{d+1}} \log \frac{1}{\varepsilon})$ balls



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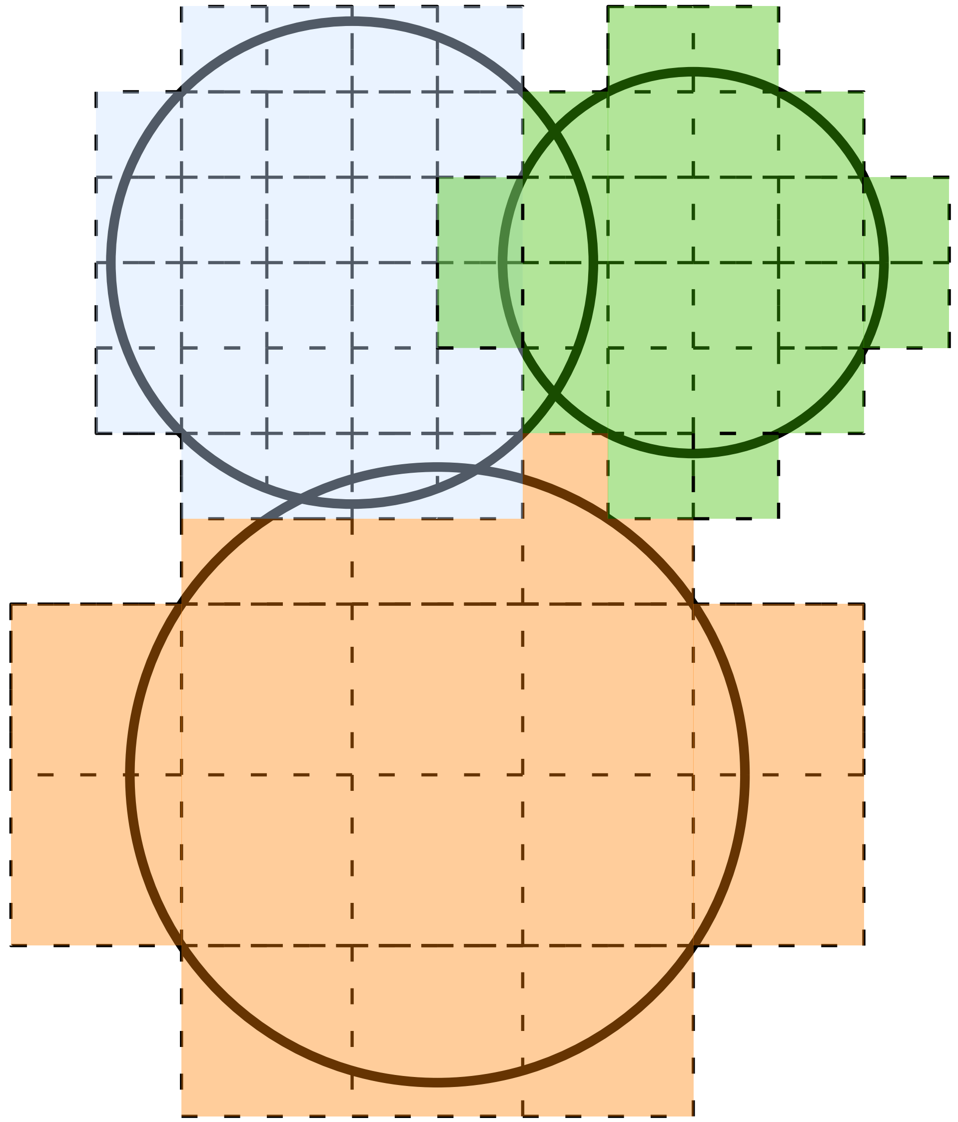

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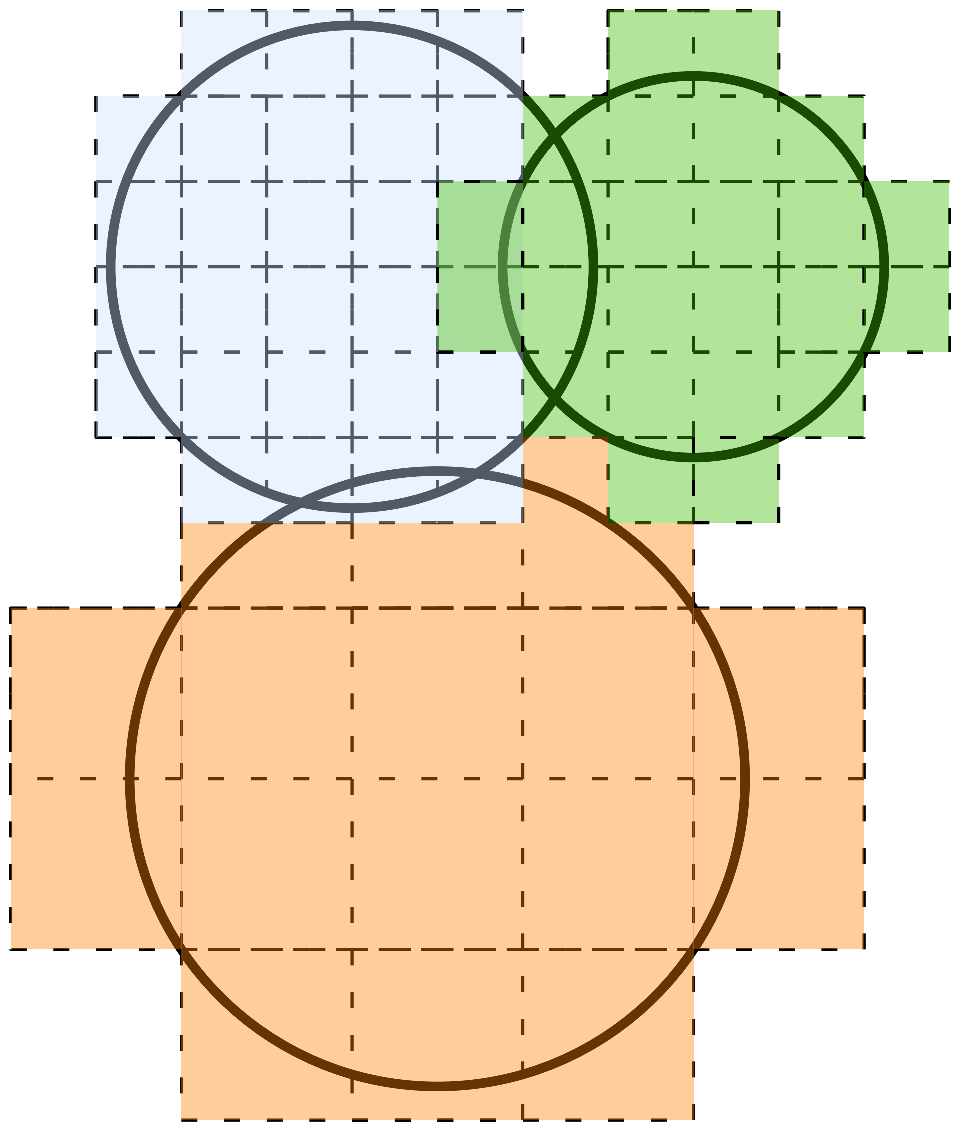

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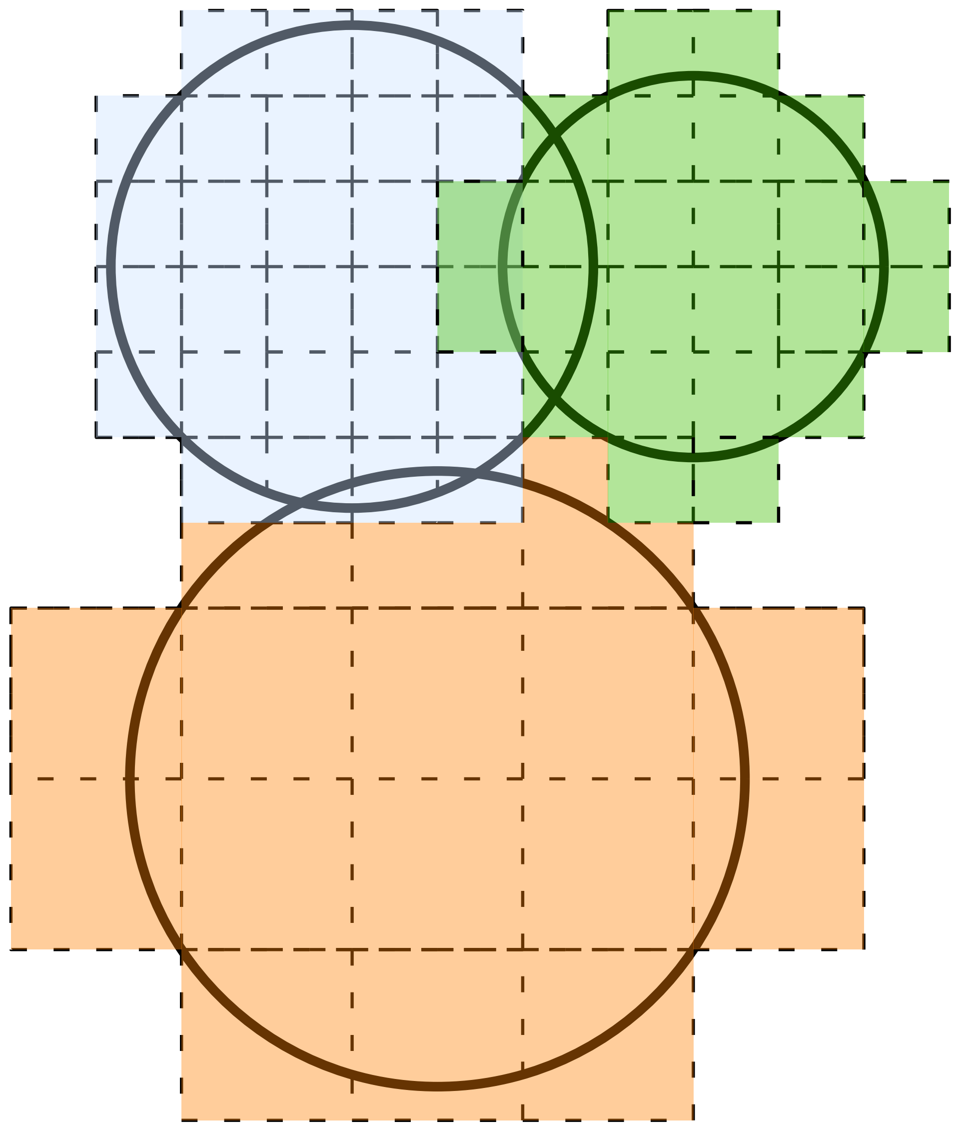

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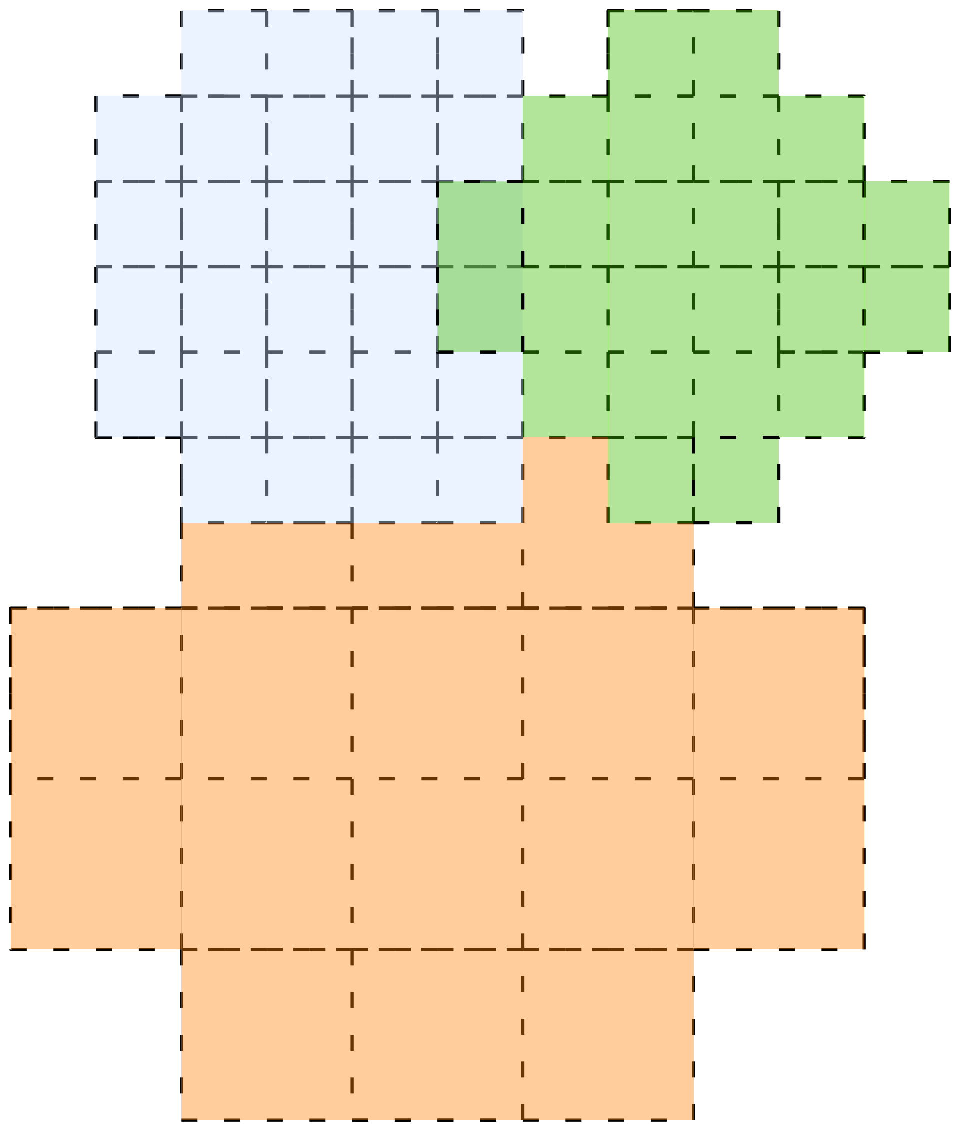

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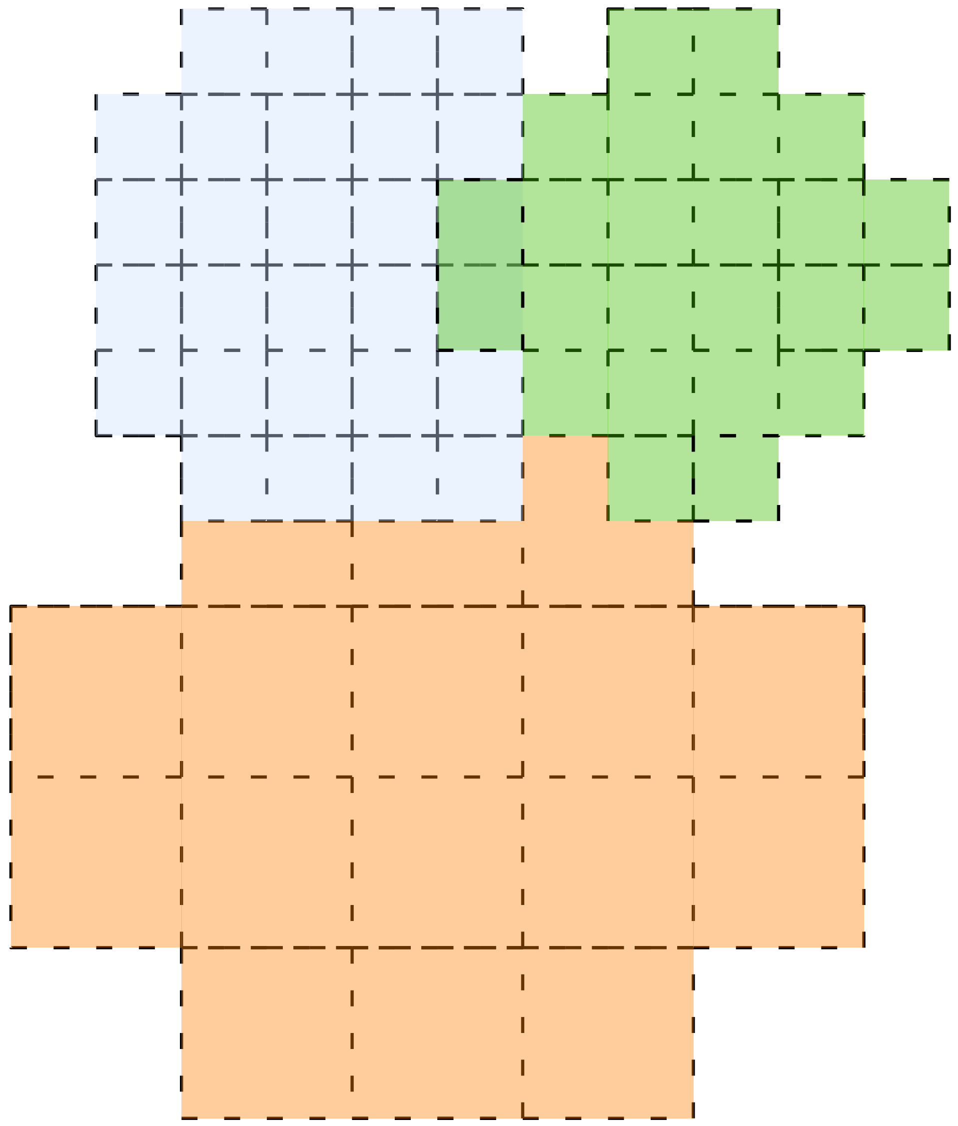
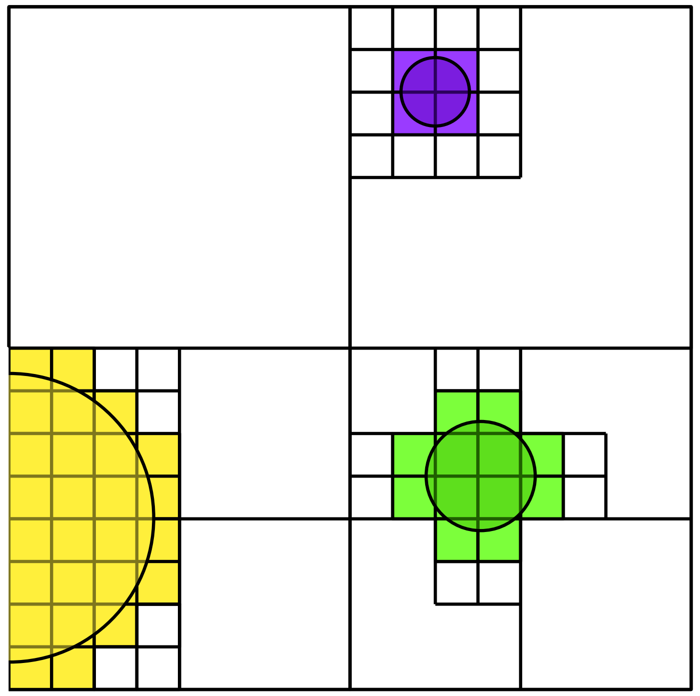


Point location on the grids



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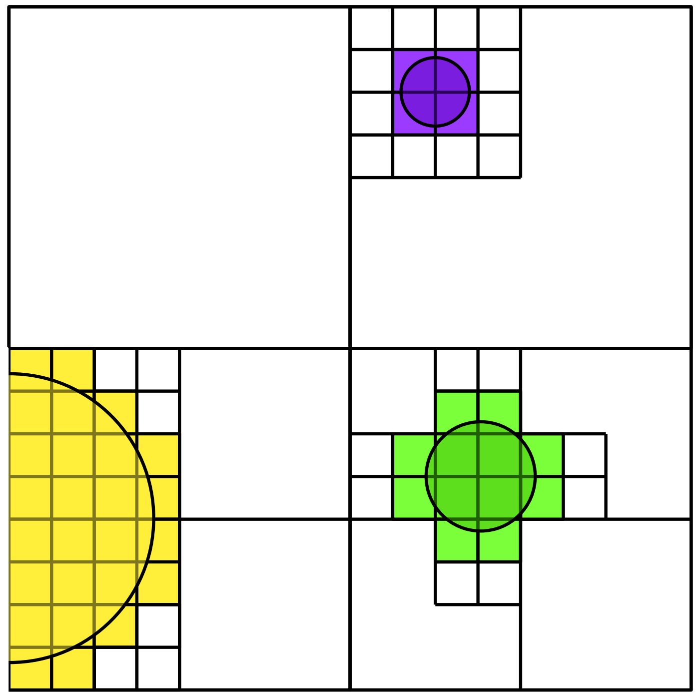

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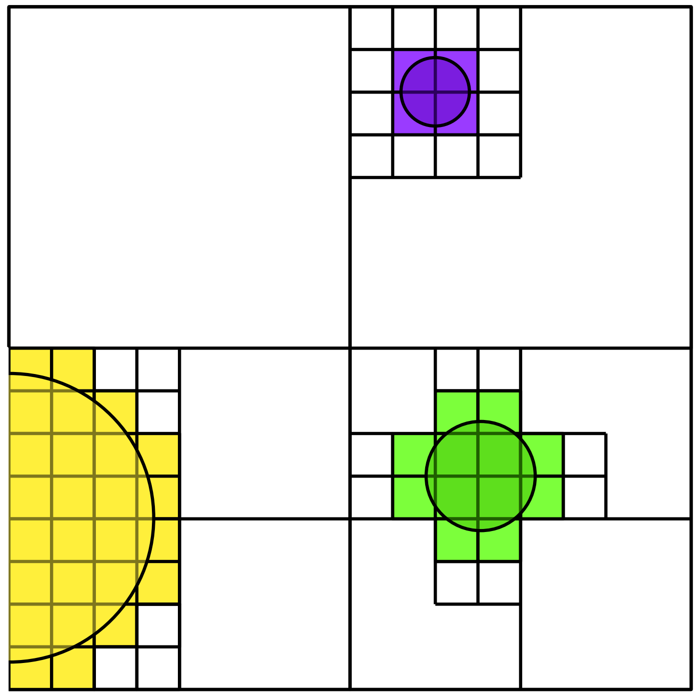

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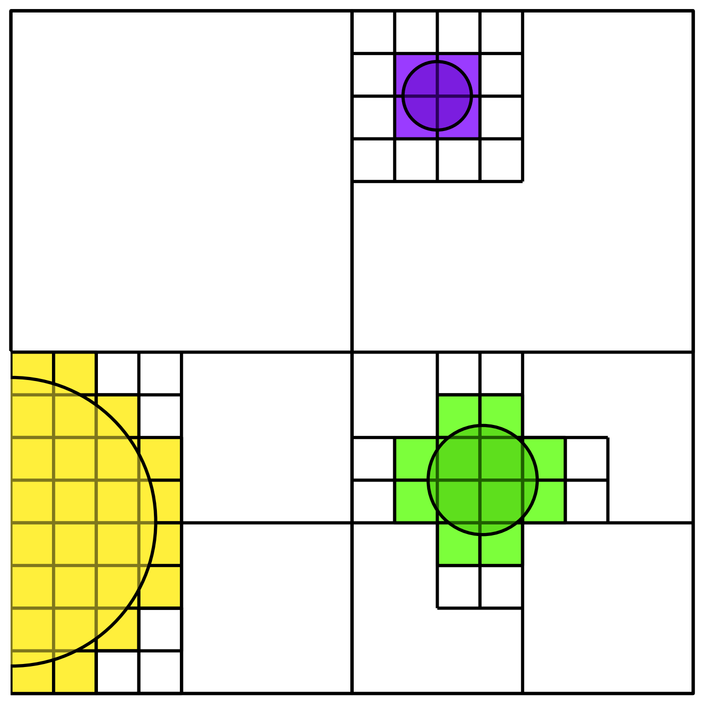

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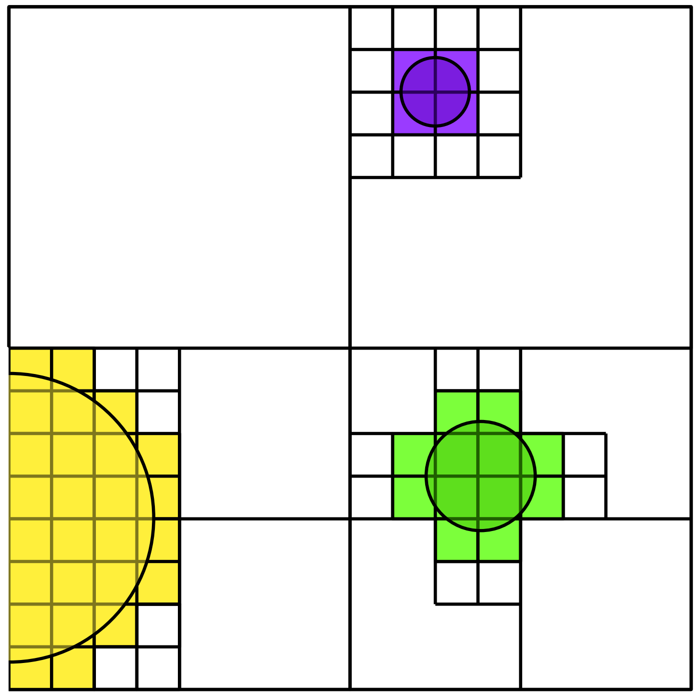

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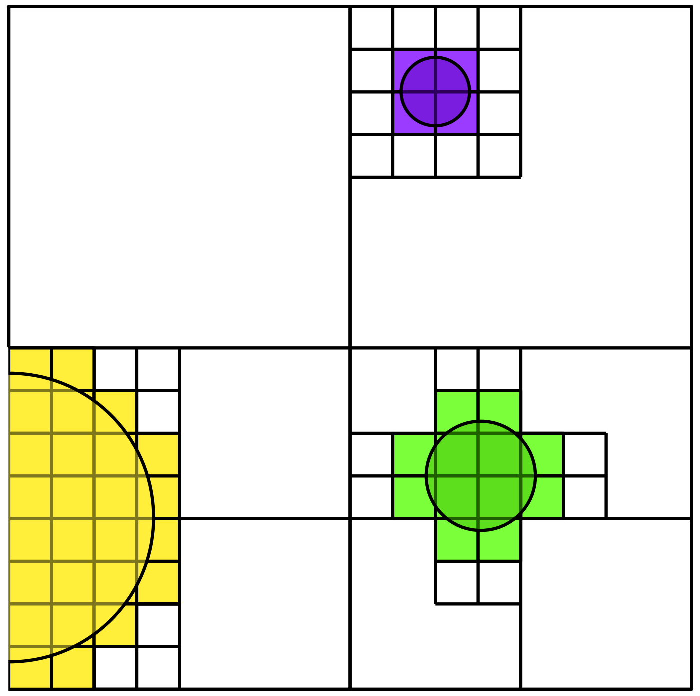

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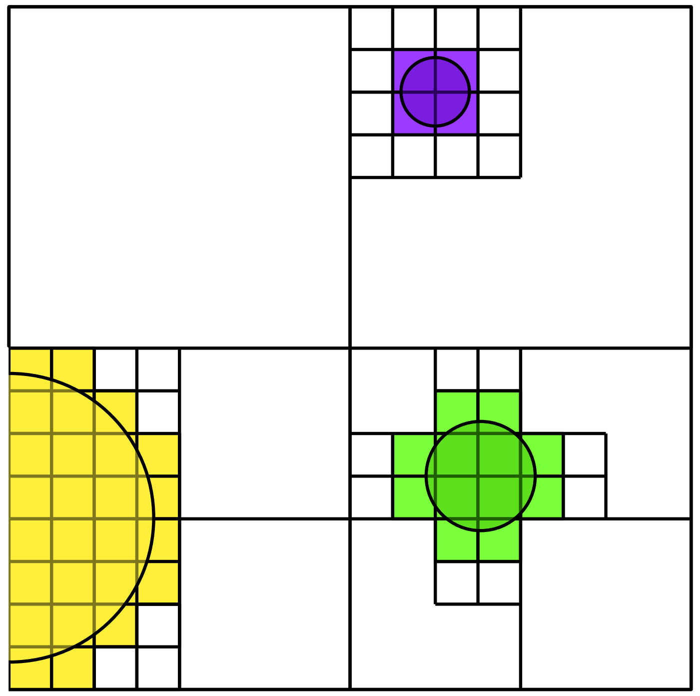

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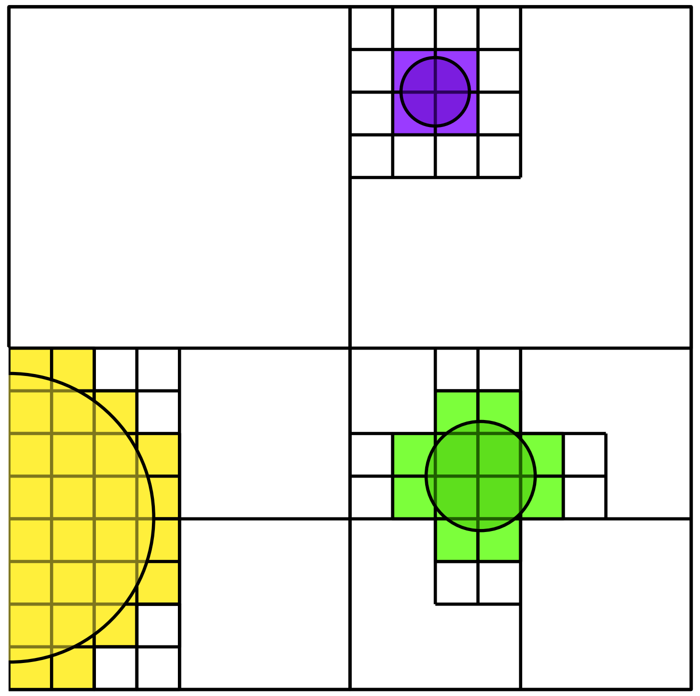

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- $|C|$ is naively bound by $N = O(\frac{|\mathcal{B}|}{\varepsilon^d})$
- $|C|$ can also be computed in that time
- $|\mathcal{B}| = O(\frac{n}{\varepsilon^{d+1}} \log \frac{1}{\varepsilon})$
- $N = O(\frac{n}{\varepsilon^{2d+1}} \log \frac{1}{\varepsilon})$
- $\log N = \log \frac{n}{\varepsilon^{2d+1}} \boxed{\log \frac{1}{\varepsilon}} \leq \log \frac{n}{\varepsilon^{2d+2}}$

$$= \frac{1}{2d+2} \log \frac{\boxed{n^{1/(2d+2)}}}{\varepsilon}$$

$$\boxed{\log \frac{1}{\varepsilon} = O(\frac{1}{\varepsilon})}$$

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