Approximate Nearest Neighbors

Low Dimensions

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Overview

1. Introduction
2. ANN with quadtree (bounded spread)
3. Why low-quality approximation helps for unbounded spread
4. Low-quality approximation

Many Applications

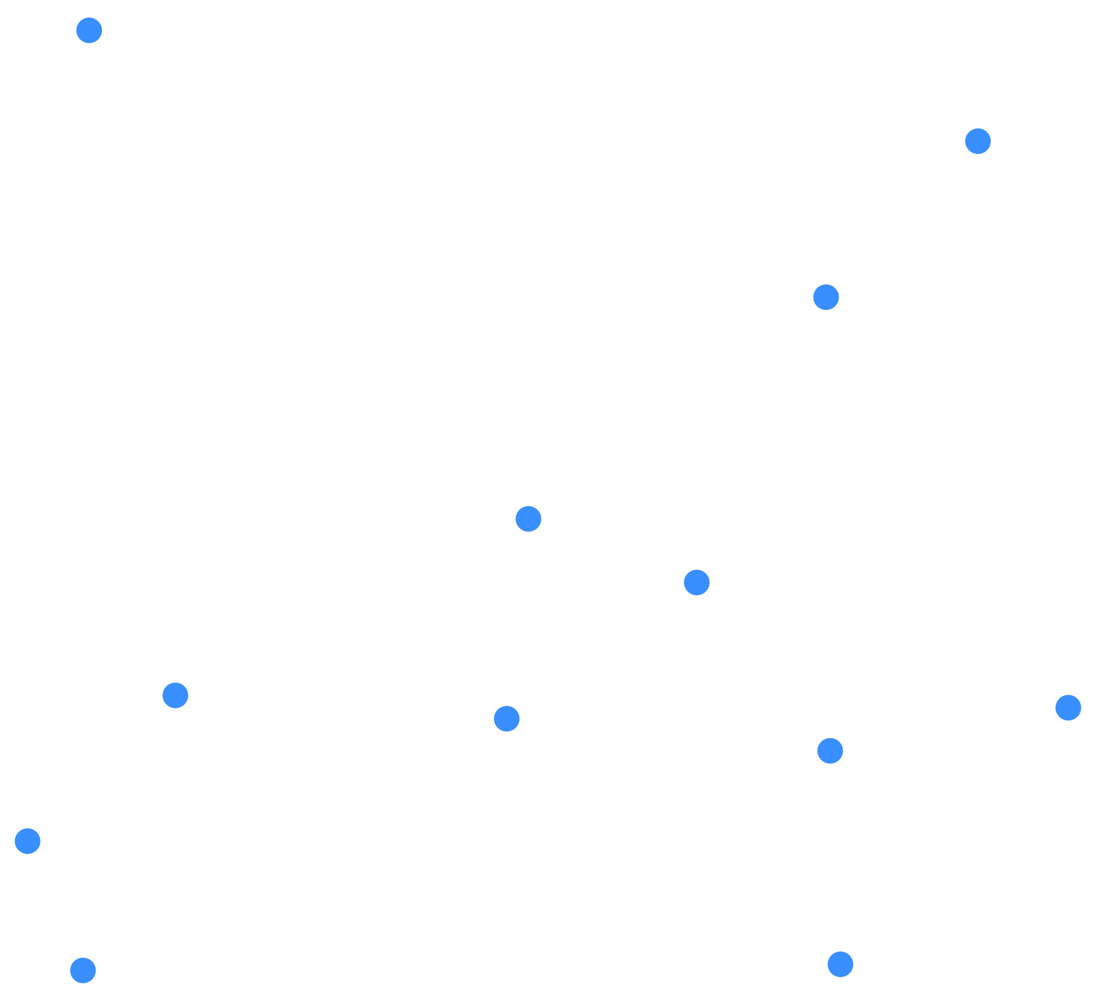
https://en.wikipedia.org/wiki/Nearest_neighbor_search

- Pattern recognition – in particular for optical character recognition
- Statistical classification – see k-nearest neighbor algorithm
- Computer vision
- Computational geometry – see Closest pair of points problem
- Databases – e.g. content-based image retrieval
- Coding theory – see maximum likelihood decoding
- Data compression – see MPEG-2 standard
- Robotic sensing[2]
- Recommendation systems, e.g. see Collaborative filtering
- Internet marketing – see contextual advertising and behavioral targeting
- DNA sequencing
- Spell checking – suggesting correct spelling
- Plagiarism detection
- Similarity scores for predicting career paths of professional athletes.
- Cluster analysis – assignment of a set of observations into subsets (called clusters) so that observations in the same cluster are similar in some sense, usually based on Euclidean distance
- Chemical similarity
- Sampling-based motion planning

Exact nearest neighbor

Problem statement

Preprocess set P of n points in $\mathbb{R}^d$ such that
given a query point q, we can find the closest point in P to q quickly.



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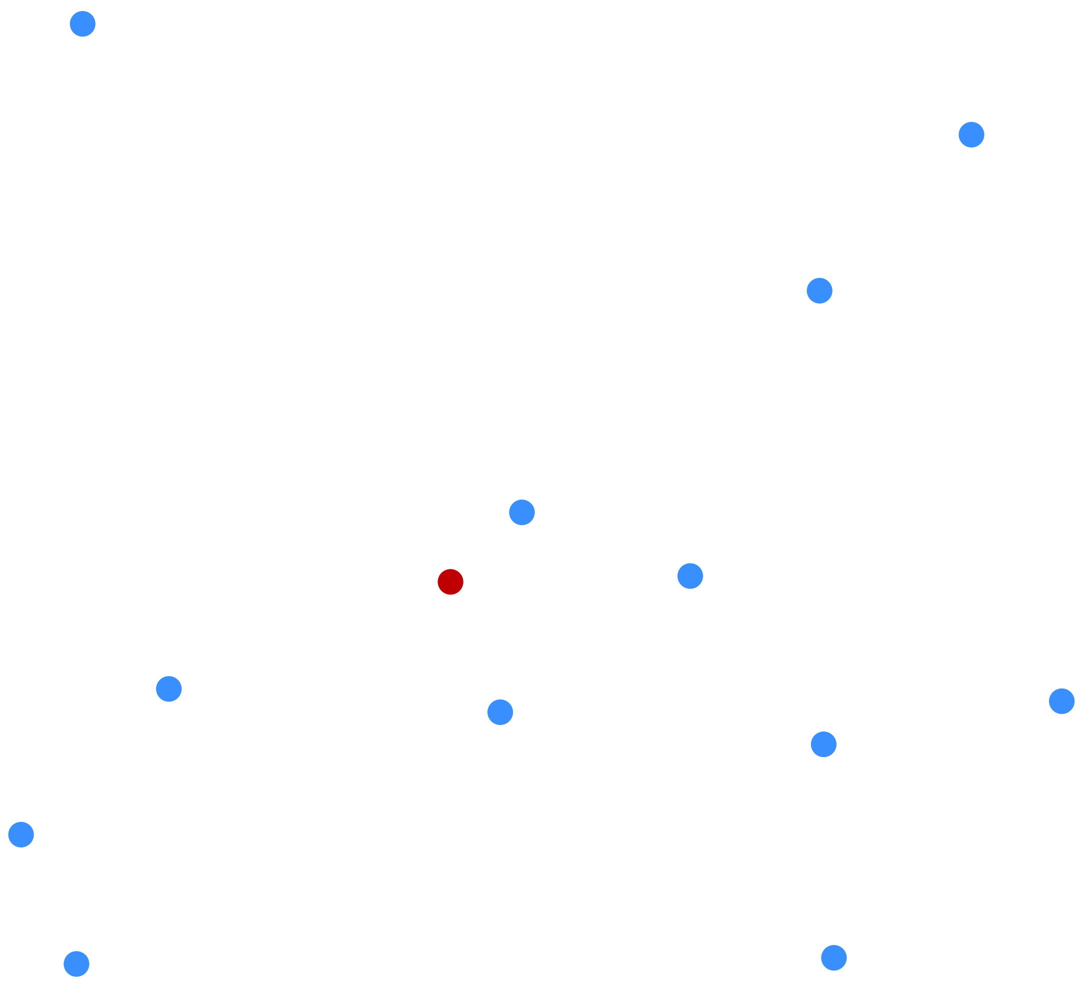

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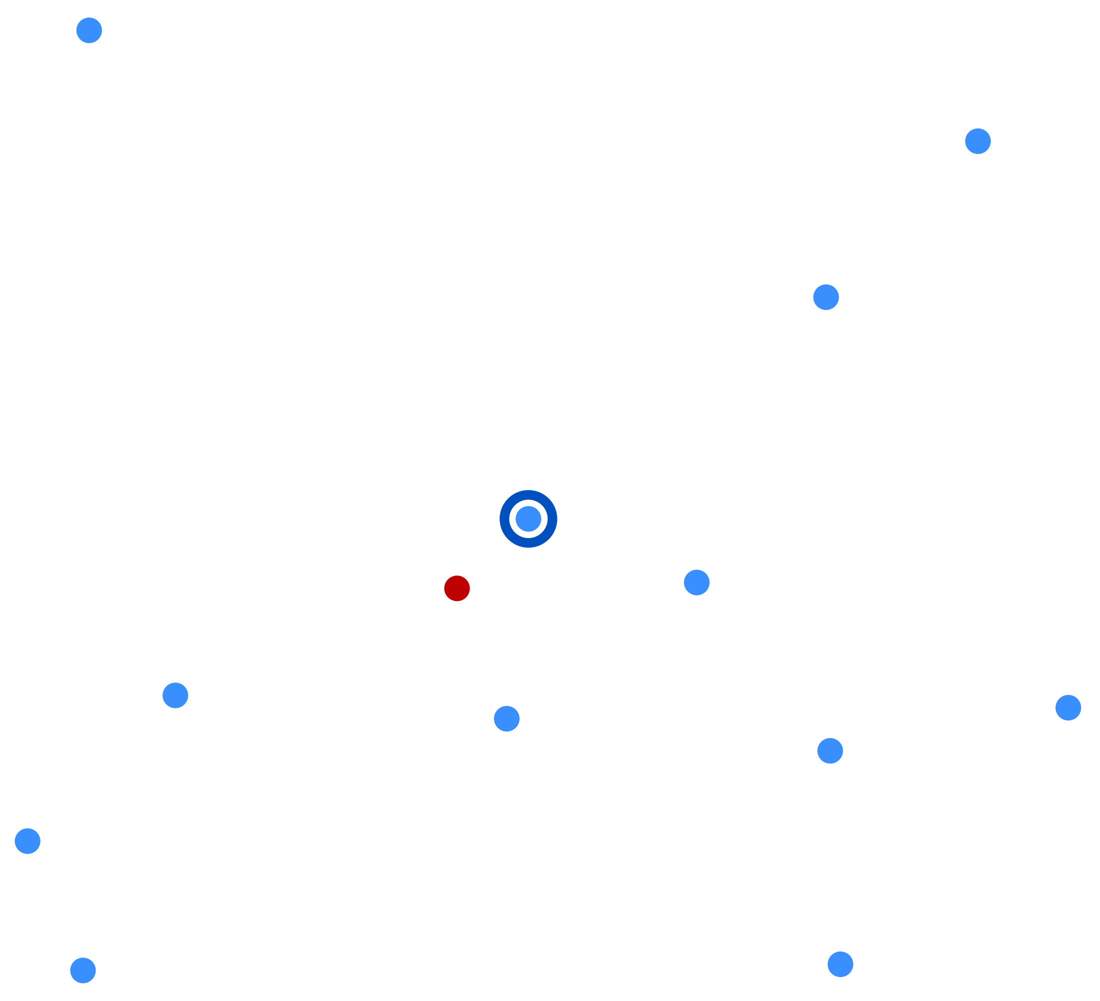

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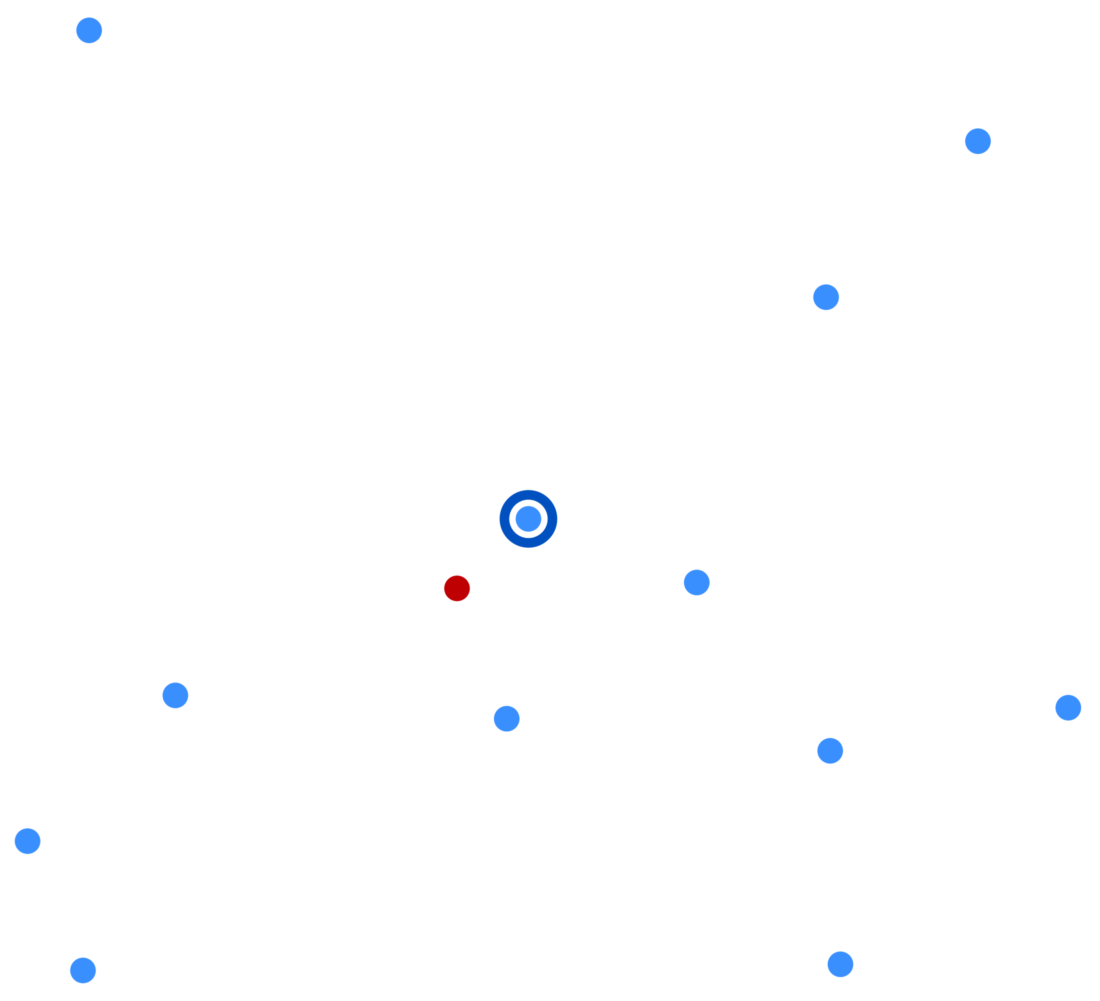
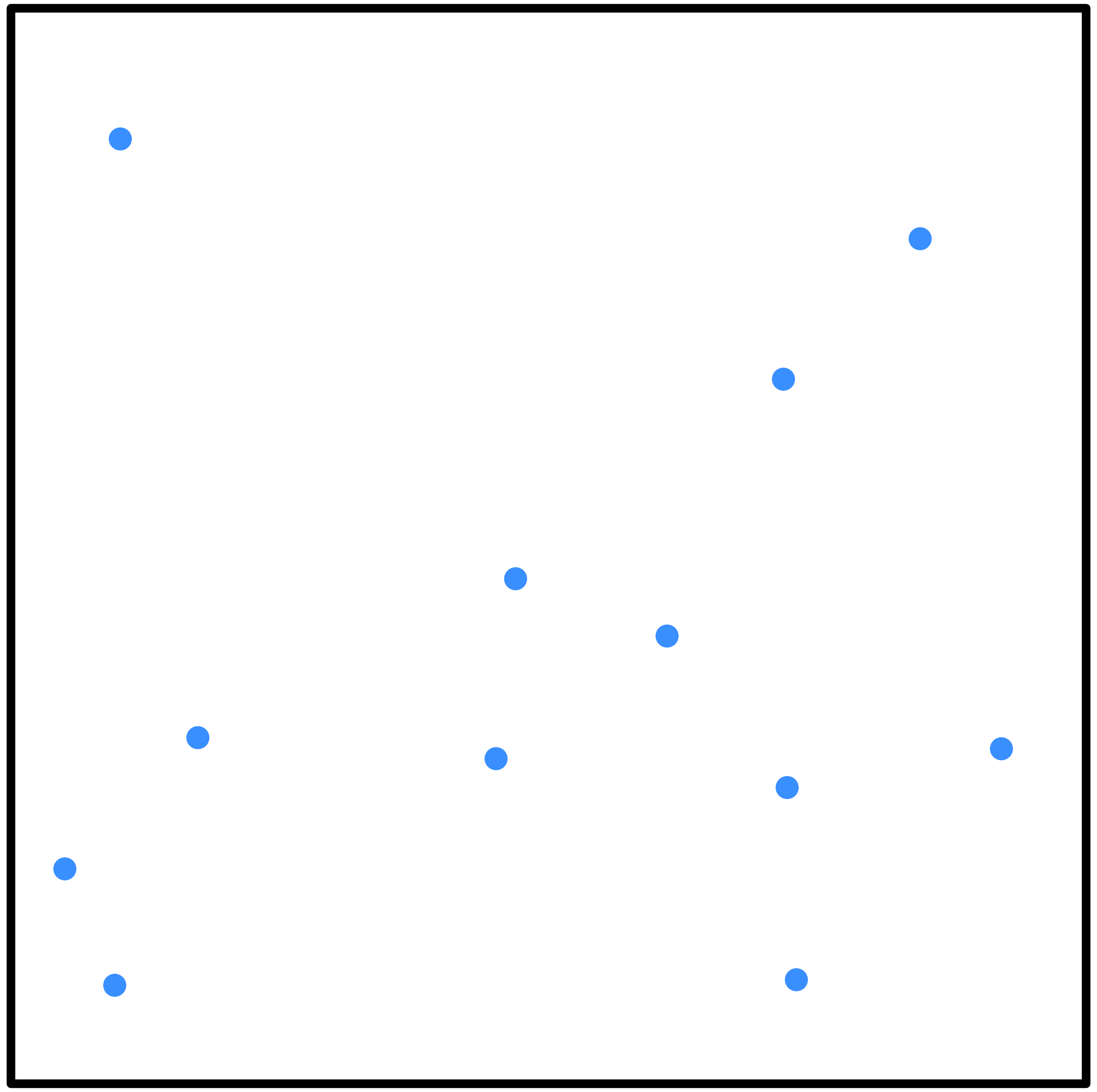


notation:

Nearest neighbor of q: $nn(q) = nn(q, P)$

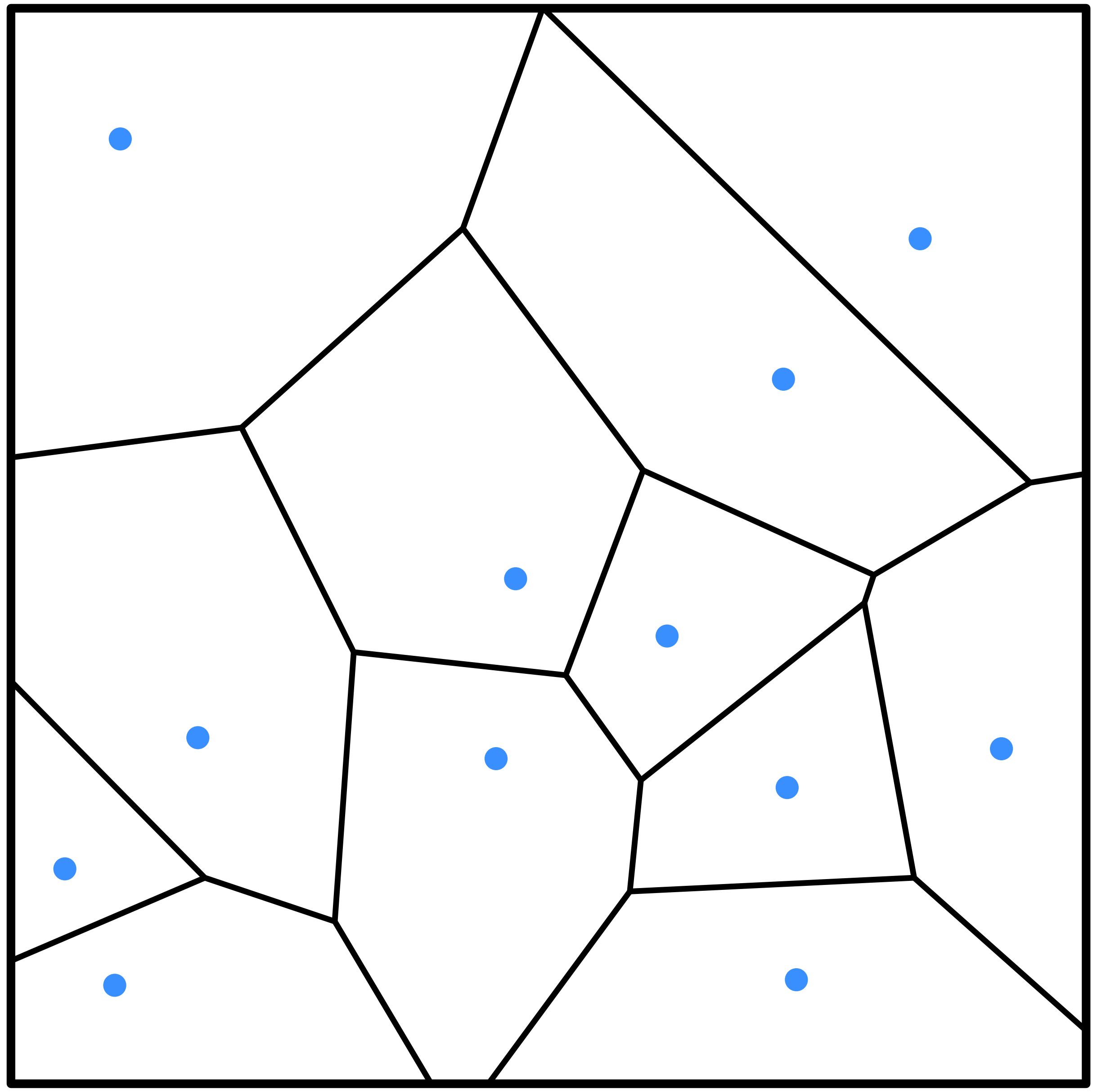
$d(q, P) = \|q - nn(q)\|$

Exact nearest neighbor



Exact nearest neighbor

Voronoi diagram



Exact nearest neighbor

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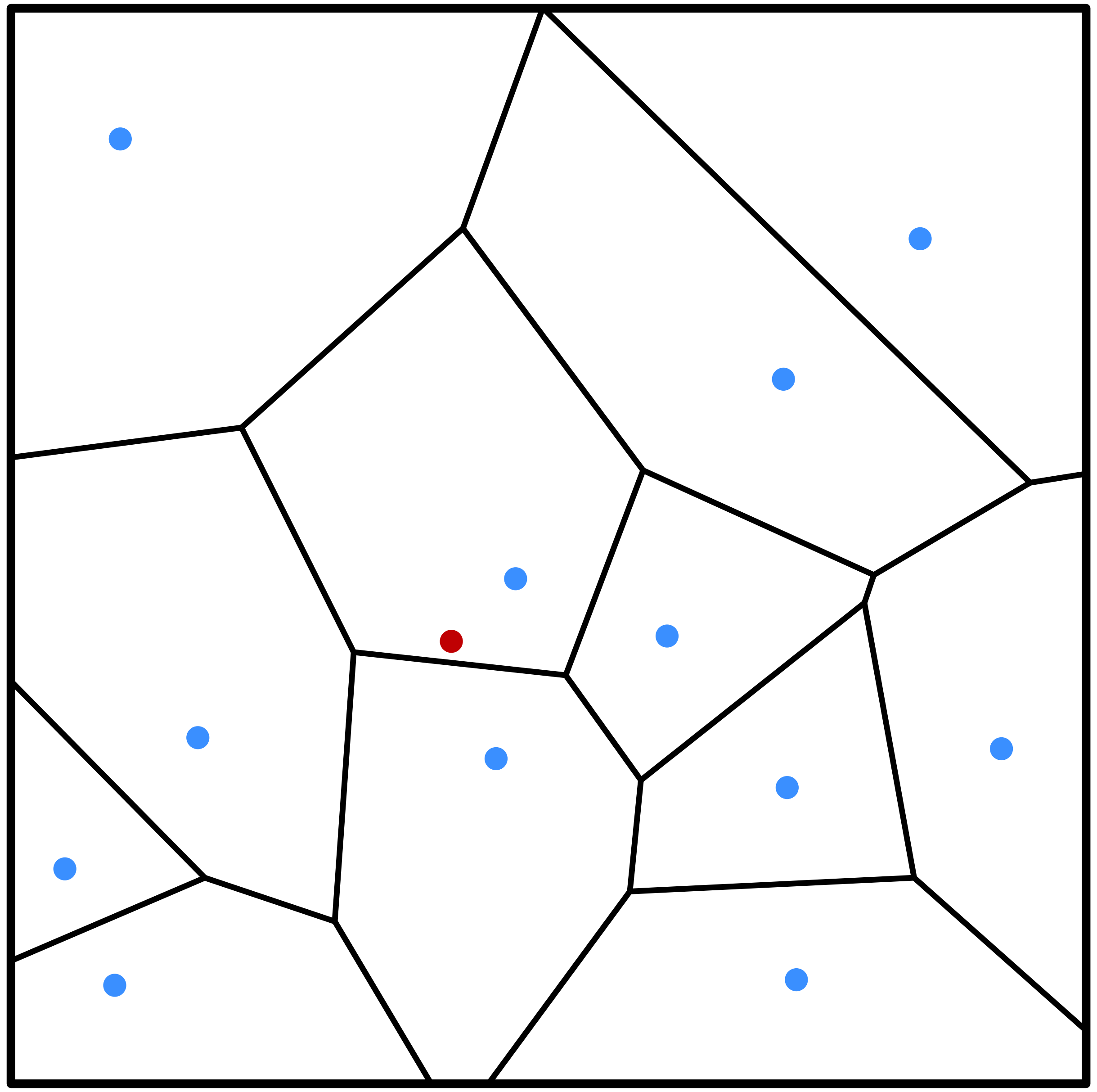

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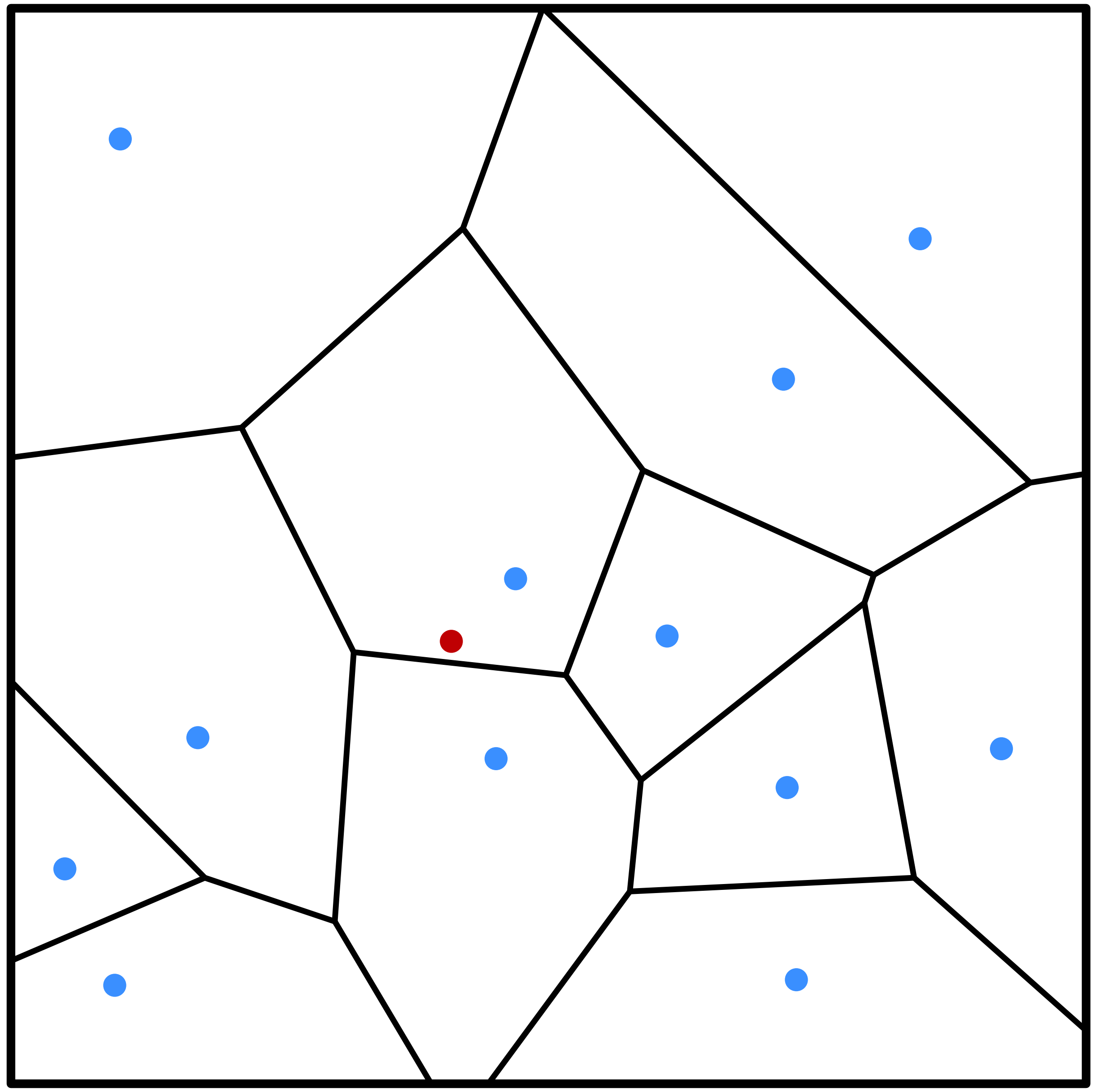
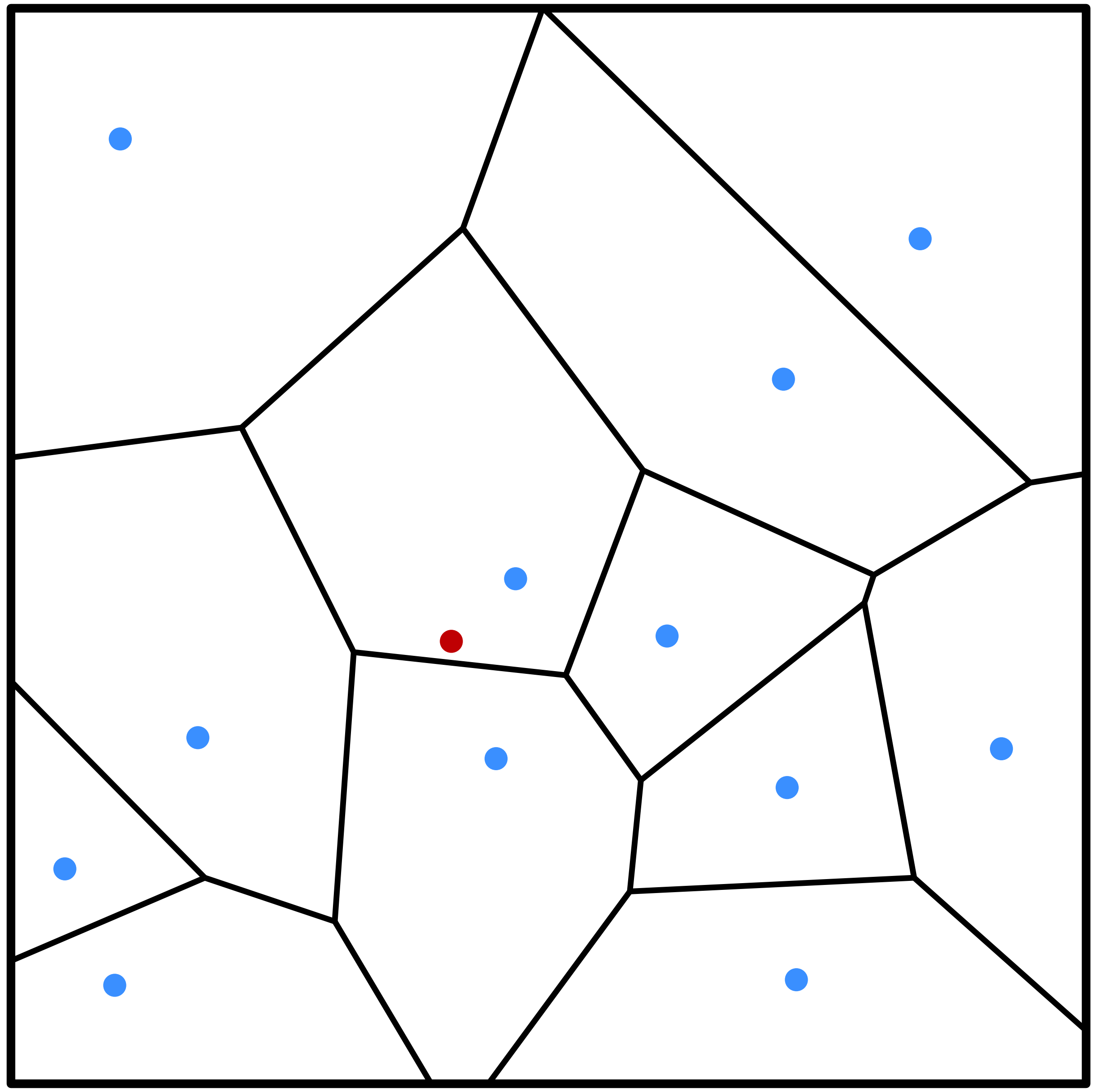


Computing the Voronoi diagram of P and preprocessing it for point-location queries requires roughly $O(n^{\lceil d/2 \rceil} + n \log n)$ time.

Exact nearest neighbor

Voronoi diagram



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Faster approximation?

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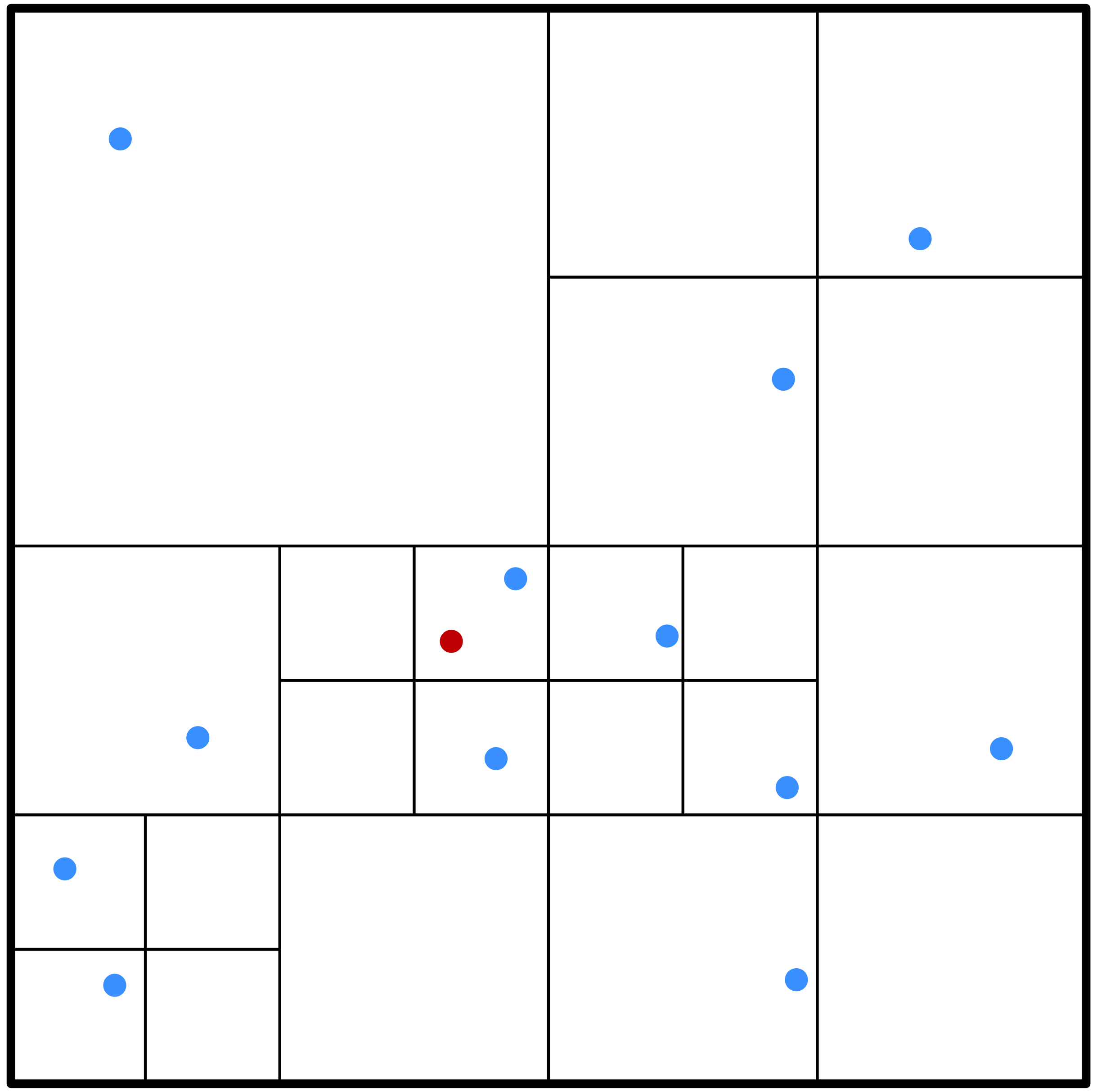

Computing the Voronoi diagram of P and preprocessing it for point-location queries requires roughly $O(n^{\lceil d/2 \rceil} + n \log n)$ time.

Faster approximation?

How about quadtrees?

Approximate nearest neighbor (Bounded spread)

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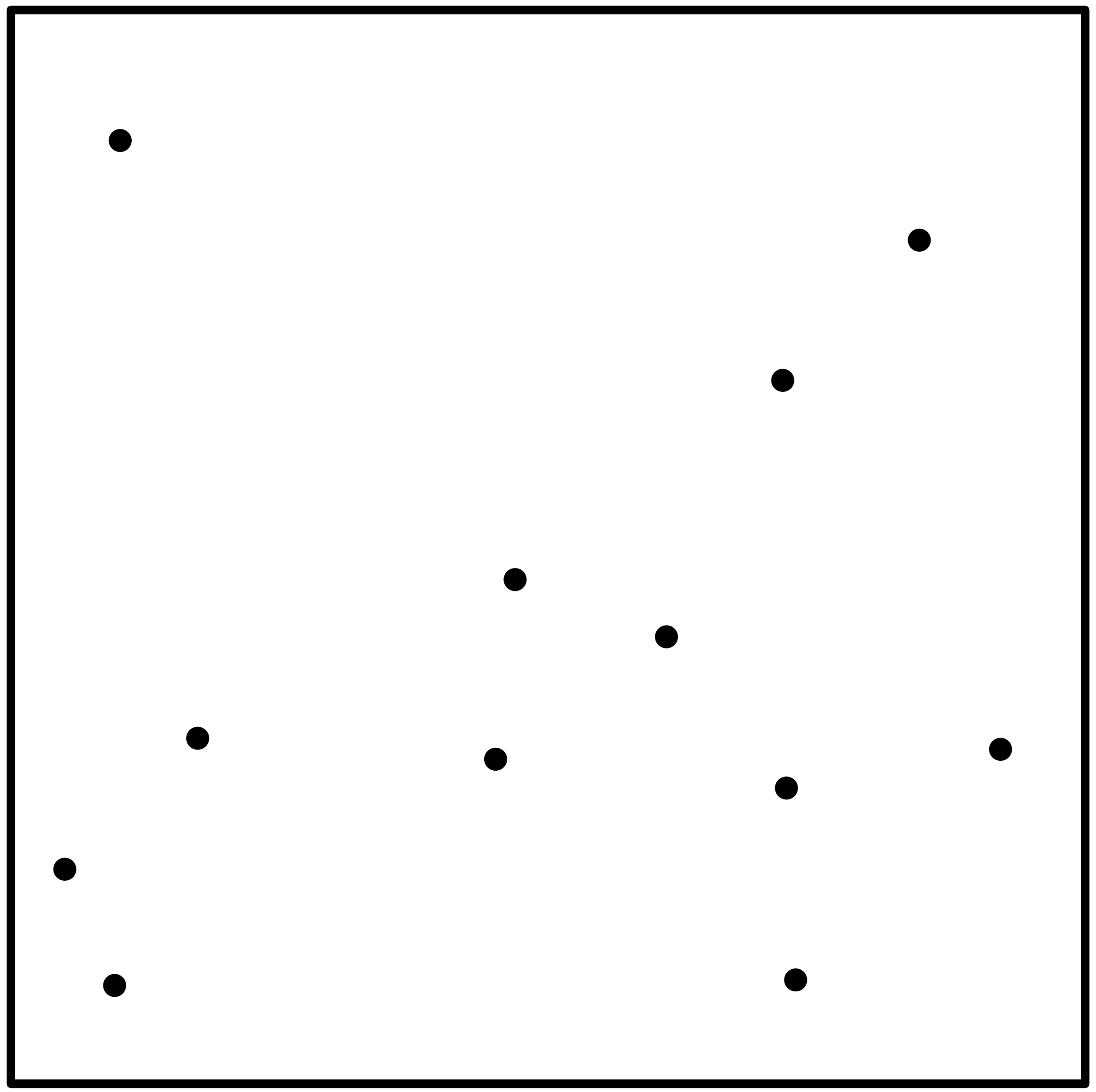
Setting:

- $P \subset [0,1]^d$, diameter$(P) = \Omega(1)$, $\Phi(P) = O(n^c)$, for constant c.
- $\mathcal{T}$: quadtree of P
- $\text{rep}_u \in P$: representative of node $u \in \mathcal{T}$
- $\varepsilon > 0$
- query point q

Approximate nearest neighbor (Bounded spread)



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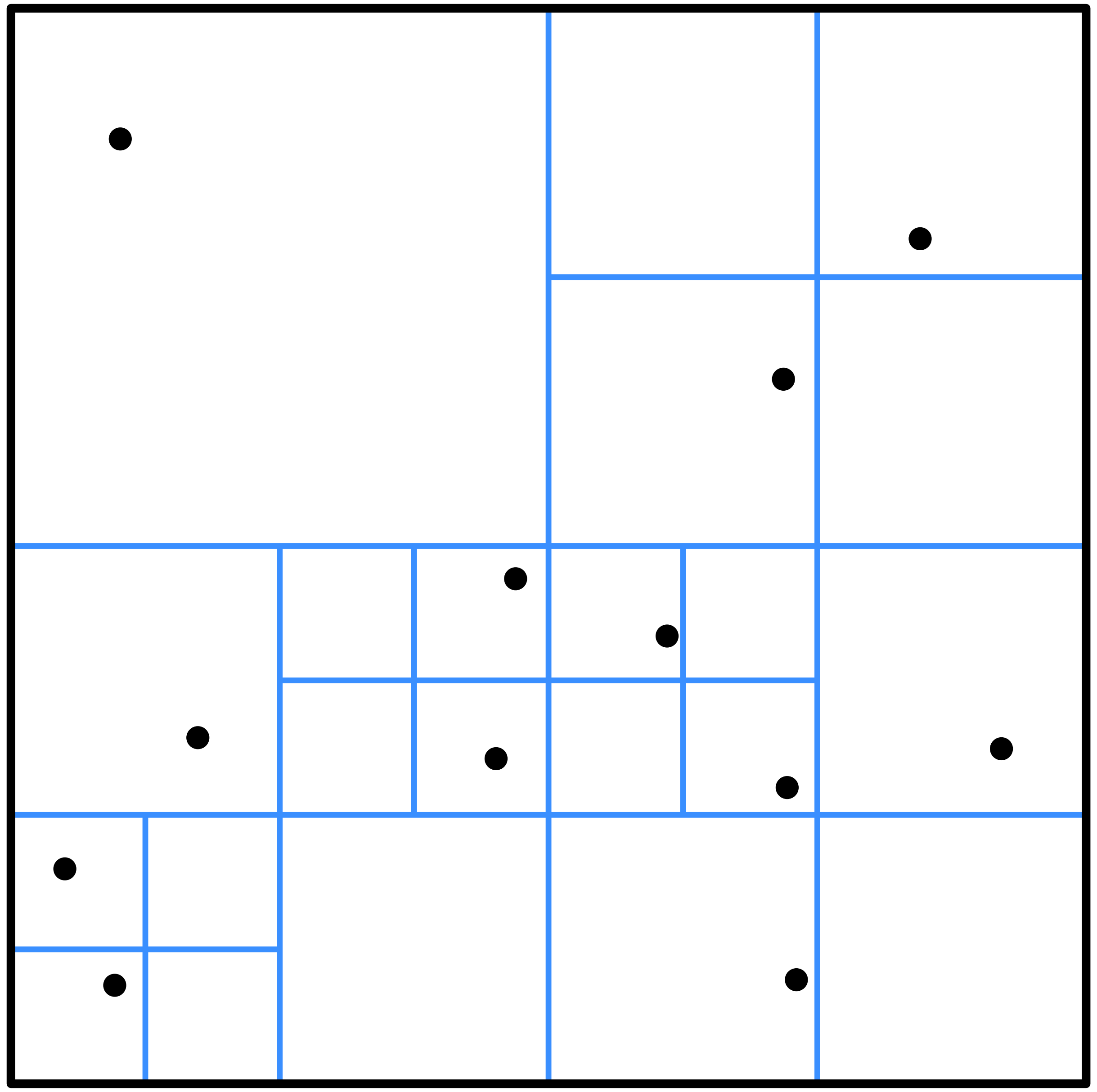

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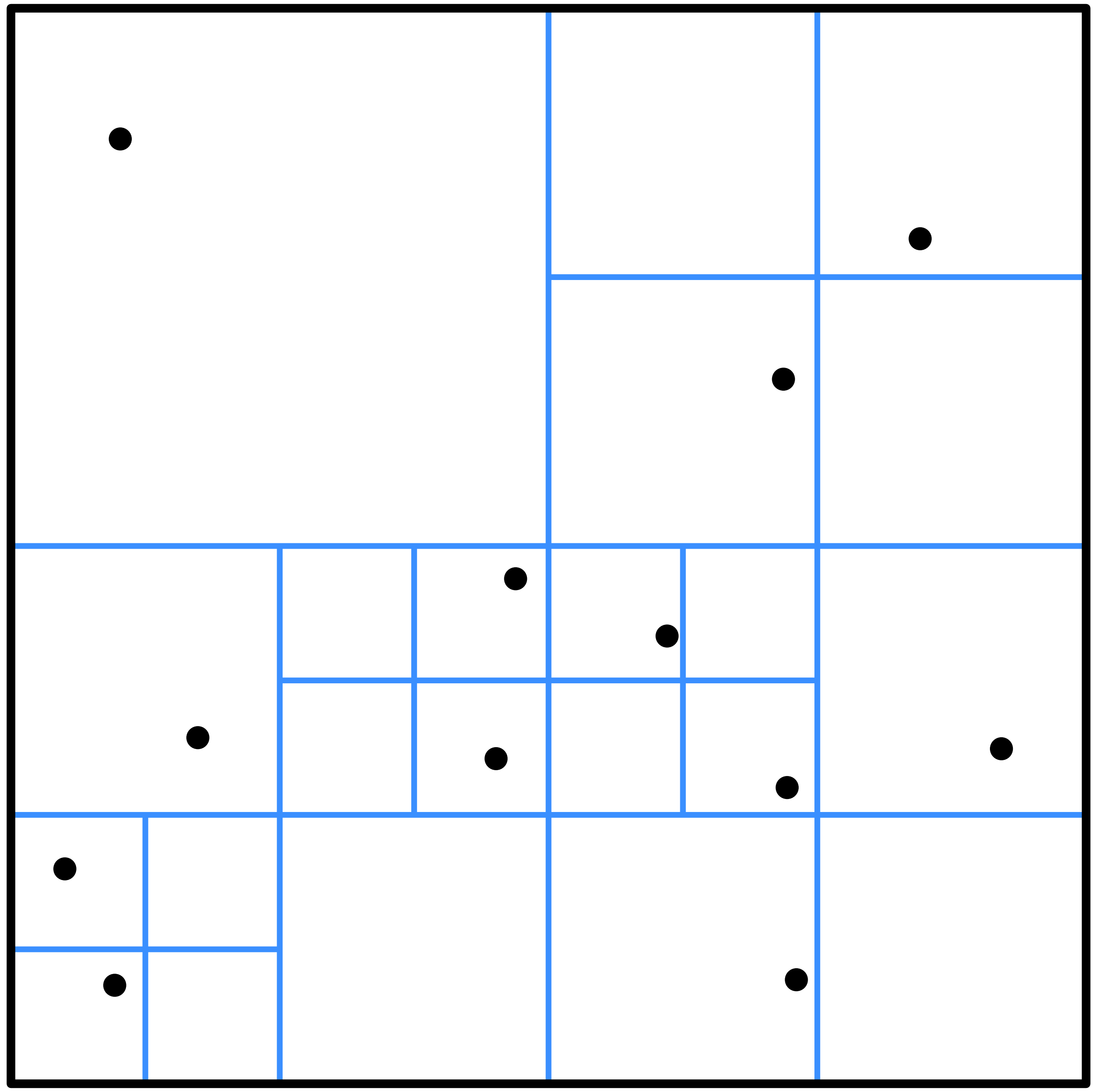

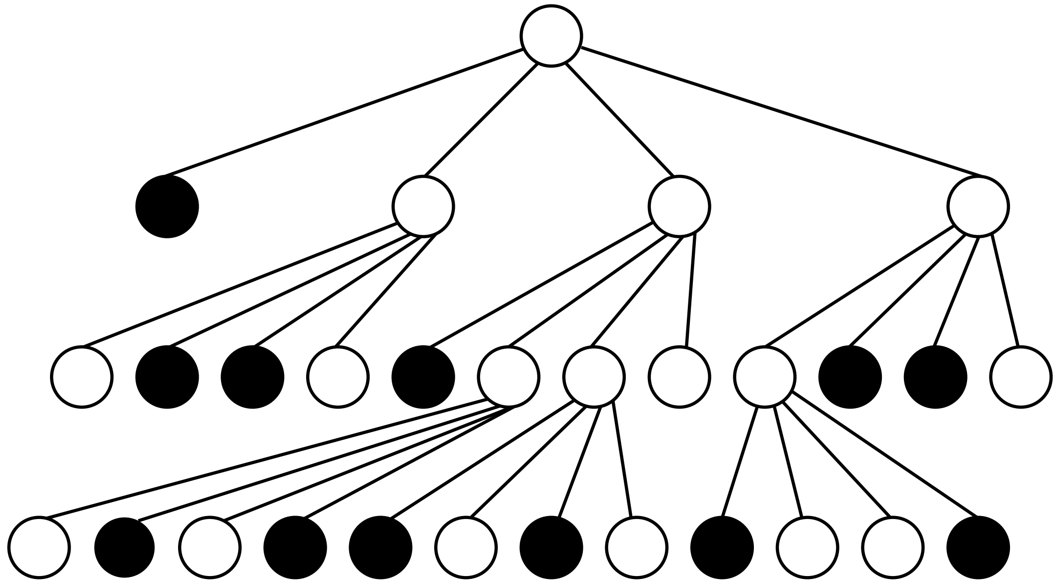

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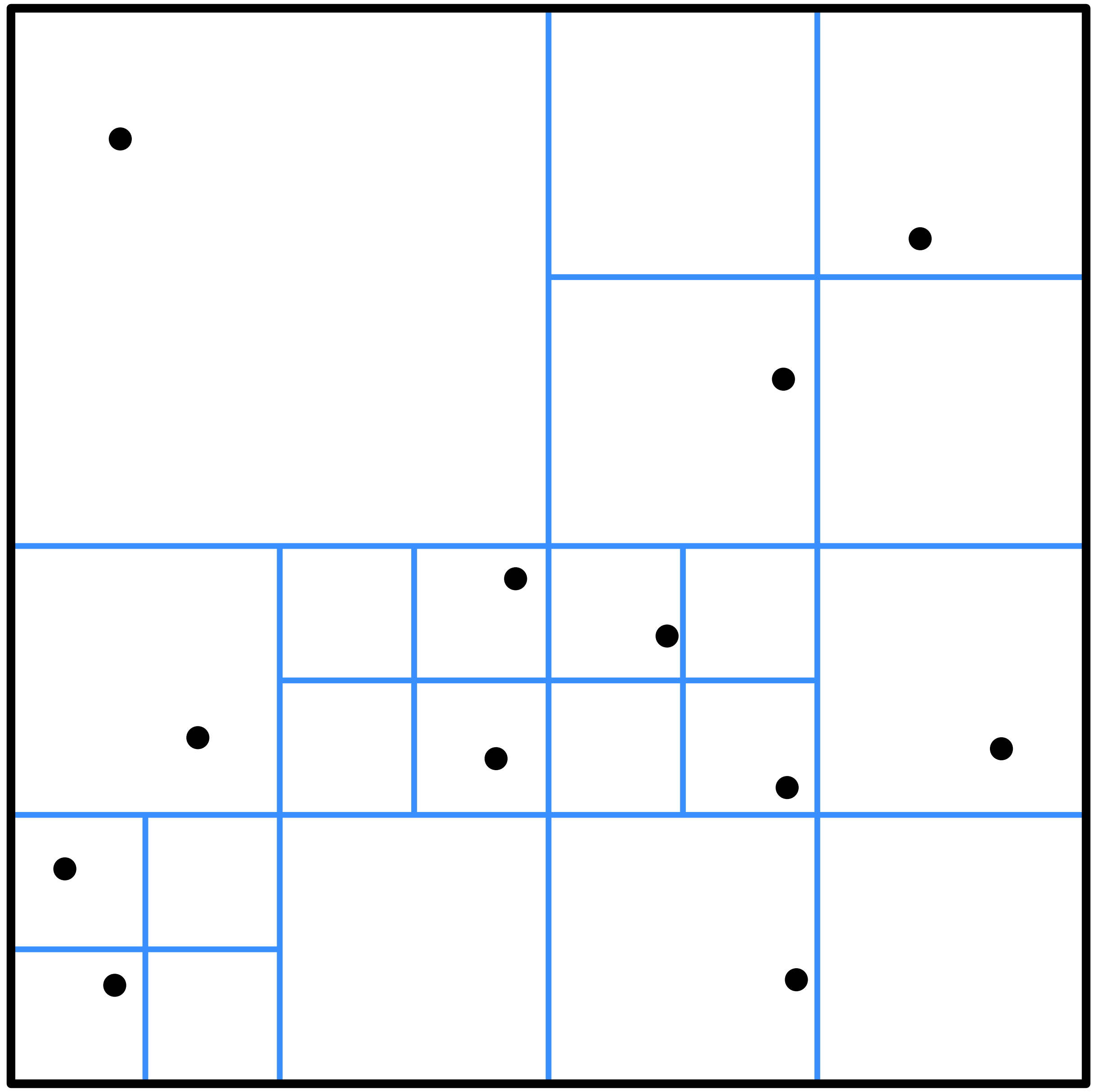
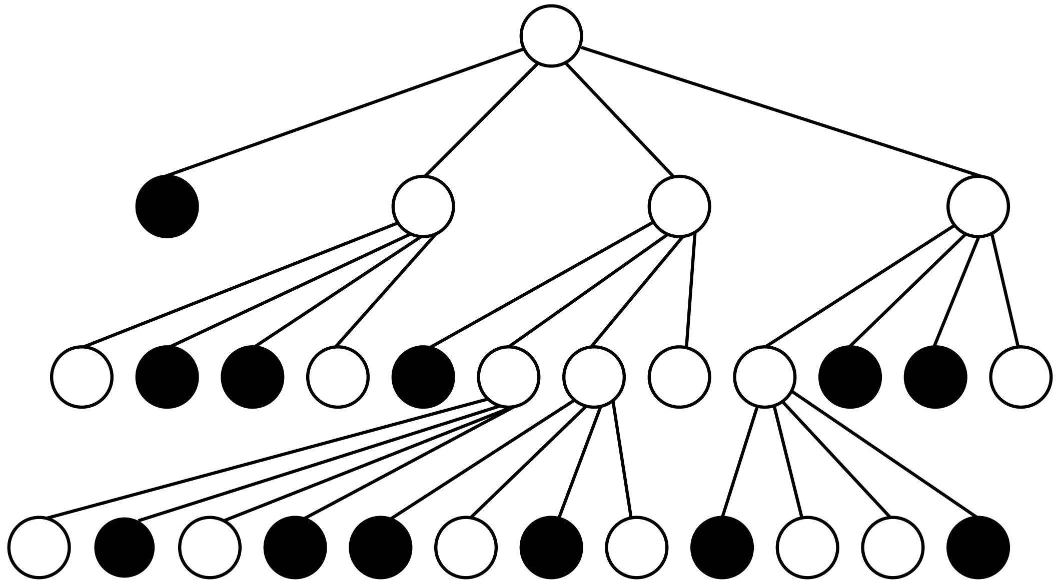
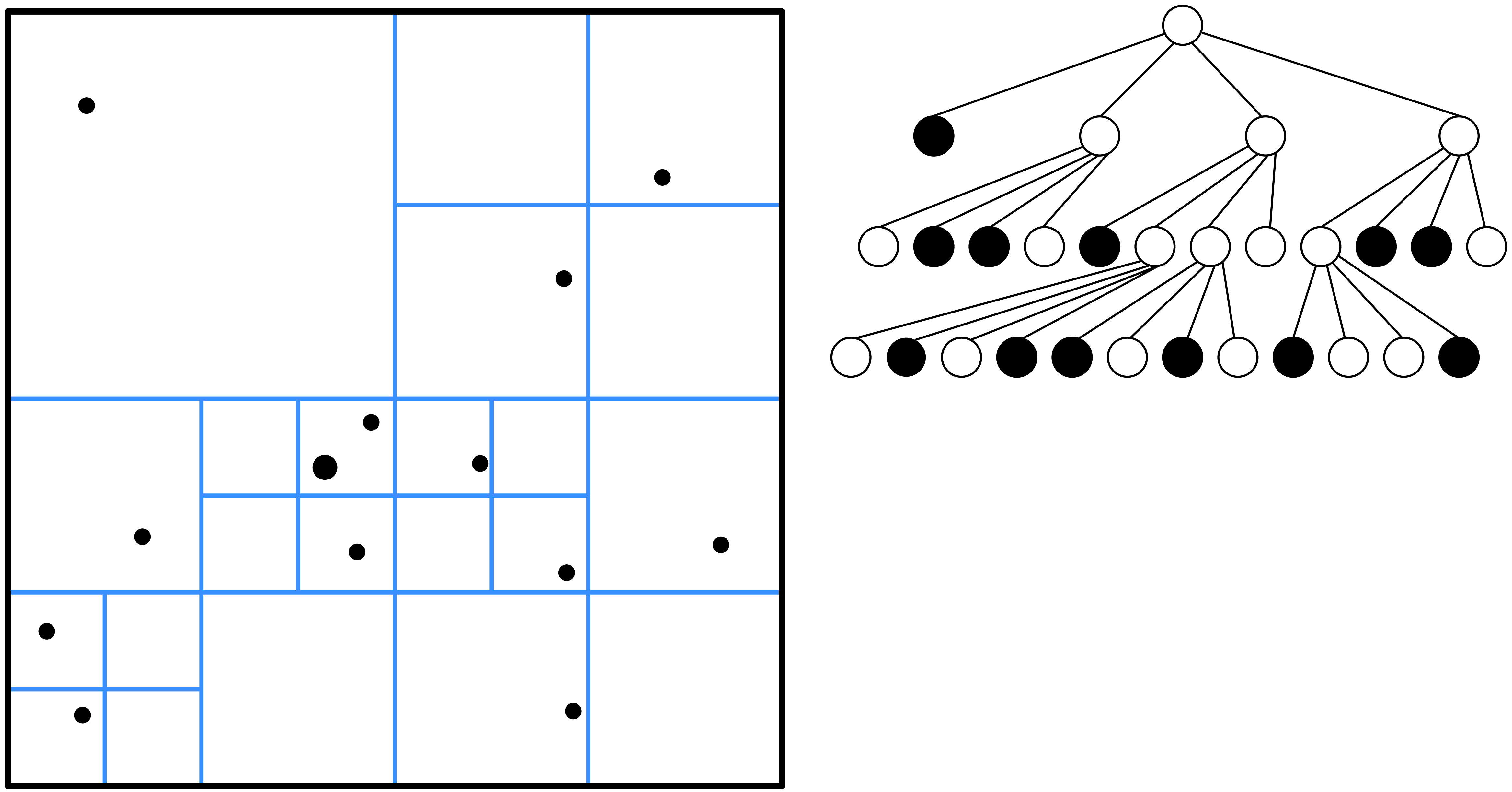




Questions:

How long does point location take in a quadtree? How long in a compressed quadtree?

Approximate nearest neighbor (Bounded spread)



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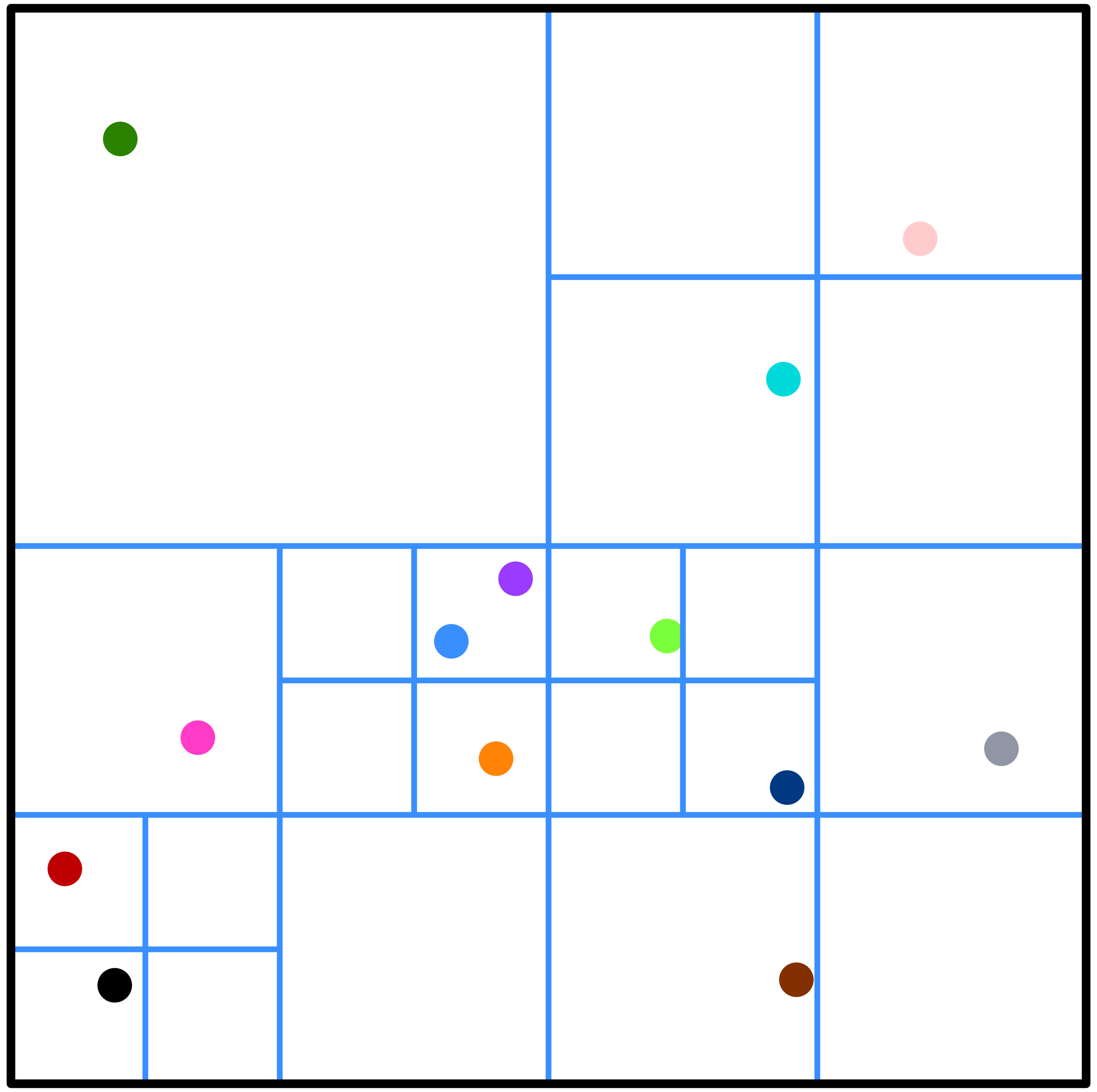

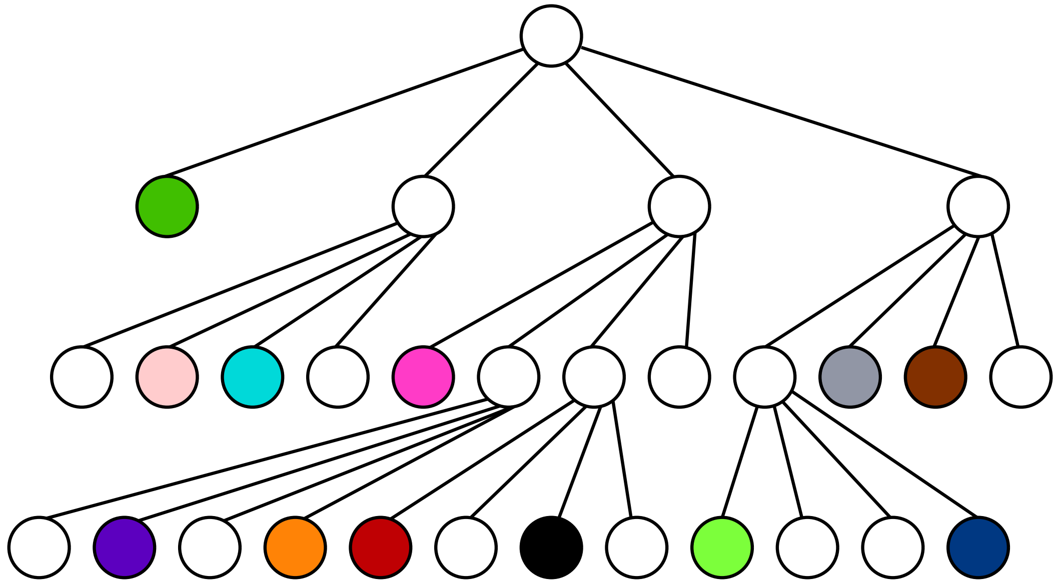

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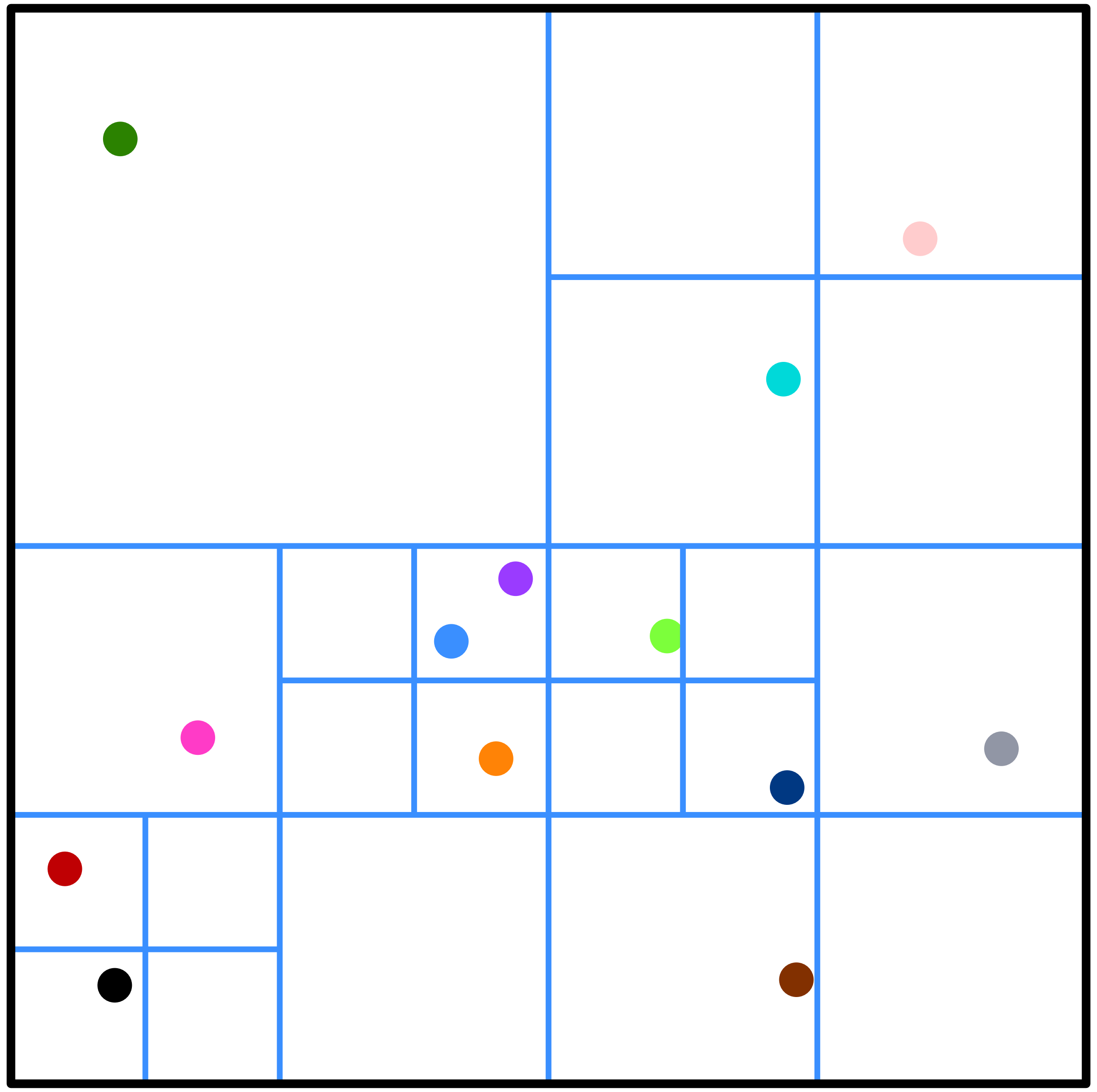

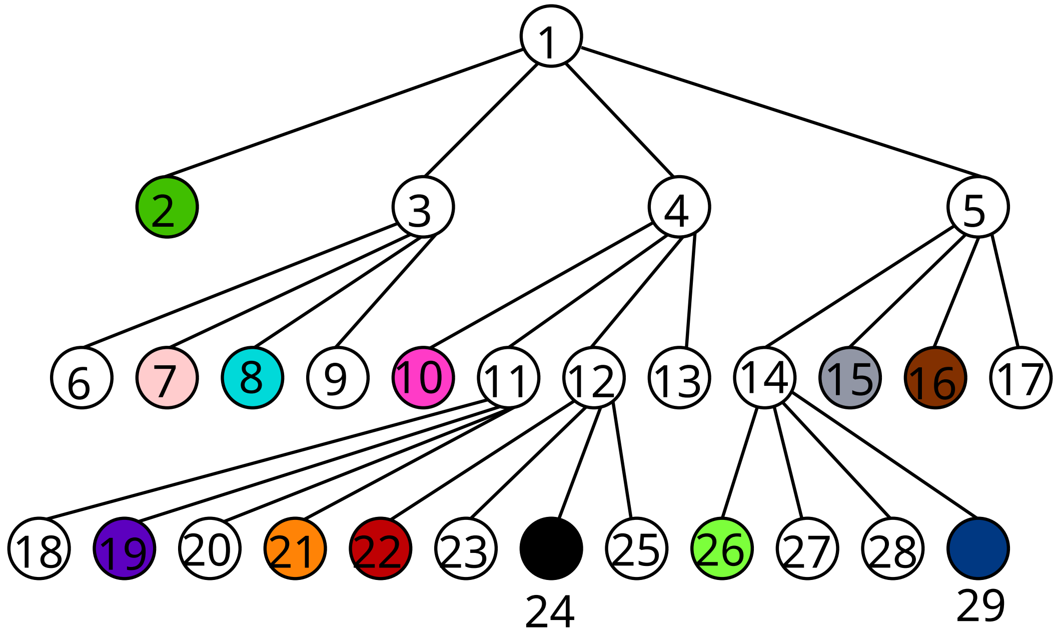

Ideas for ANN?

Approximate nearest neighbor (Bounded spread)

Algorithm ideas

recursive: start at root (like point location).

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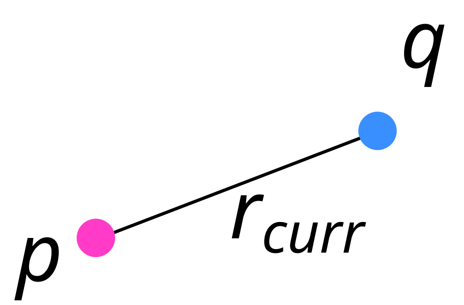
maintain best candidate p for nearest neighbor and distance $r_{curr} = d(q, p)$



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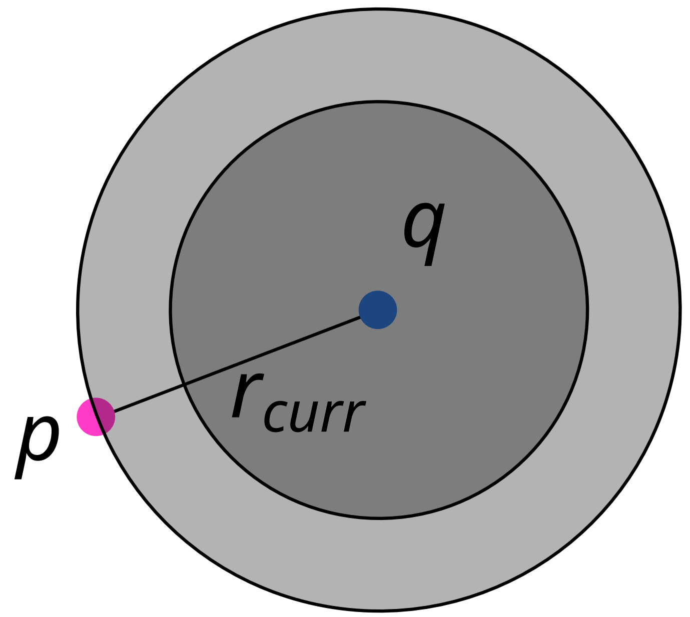
only recurse on nodes/squares that could decrease distance significantly



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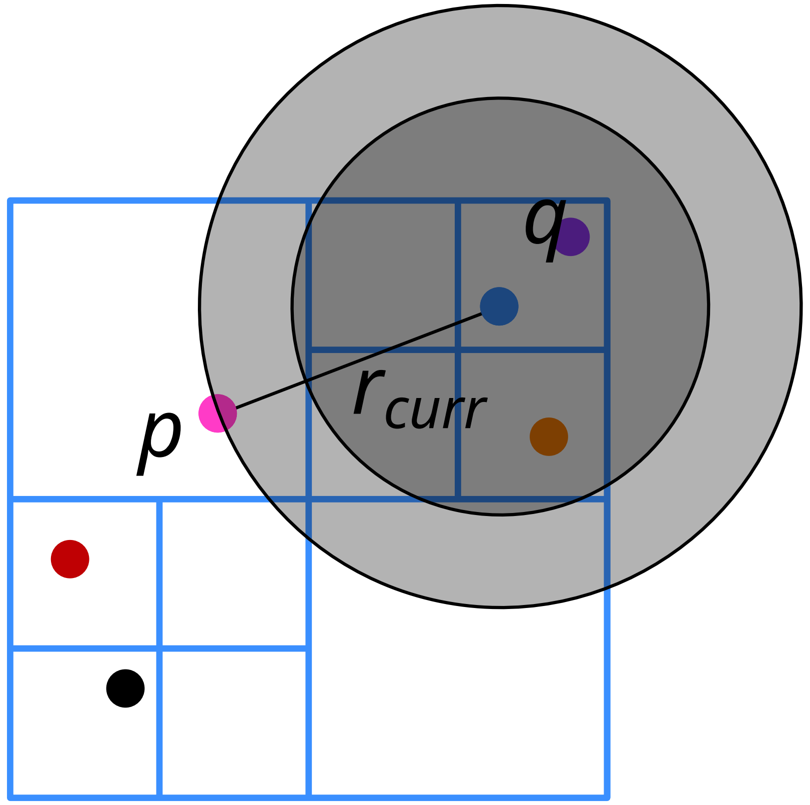

could contain $s \in P$ with $||q - s|| < (1 - \varepsilon/2)r_{curr}$

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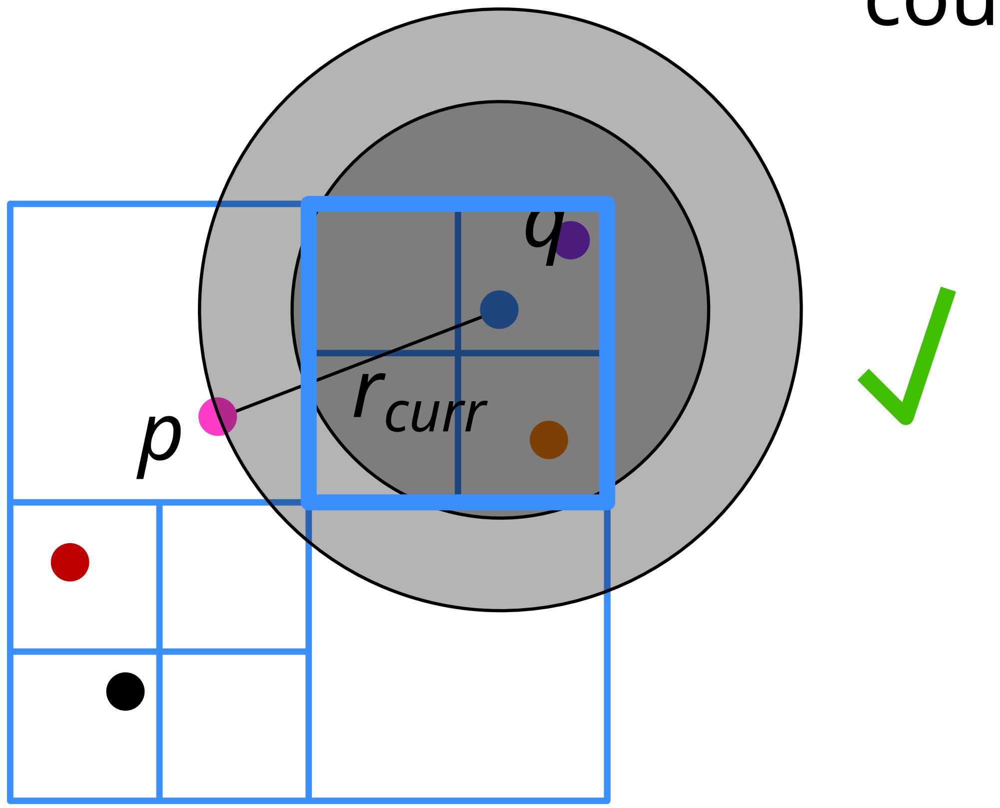

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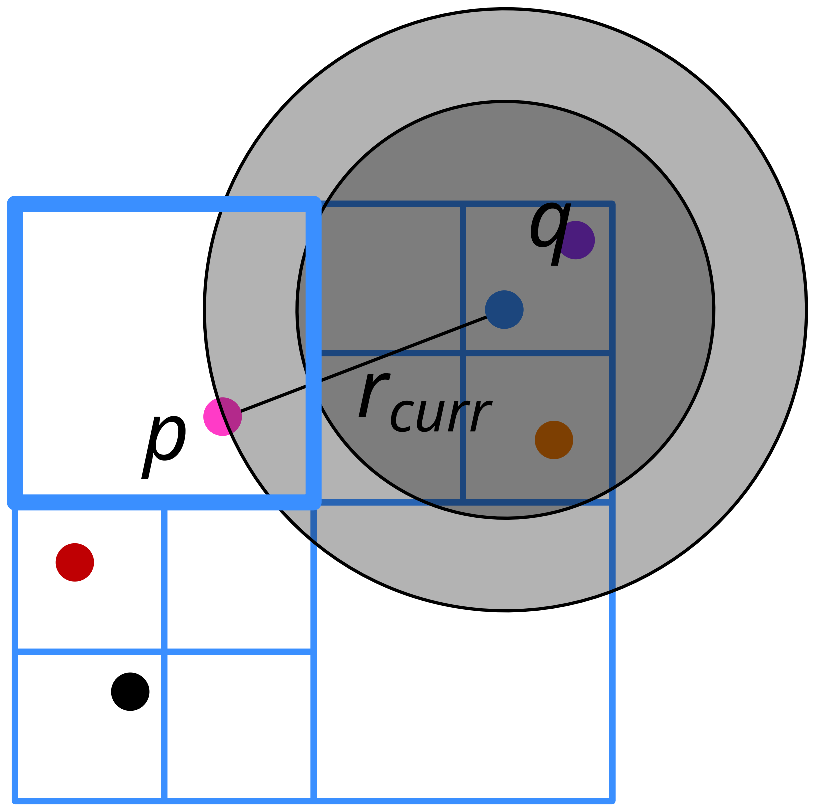

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✓

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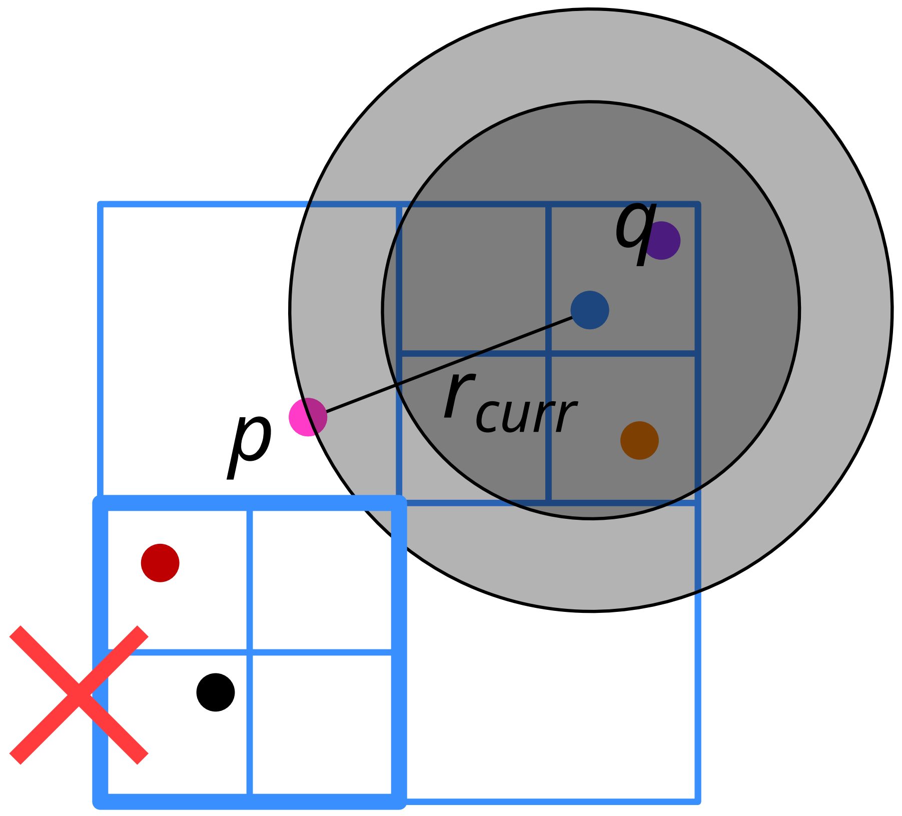

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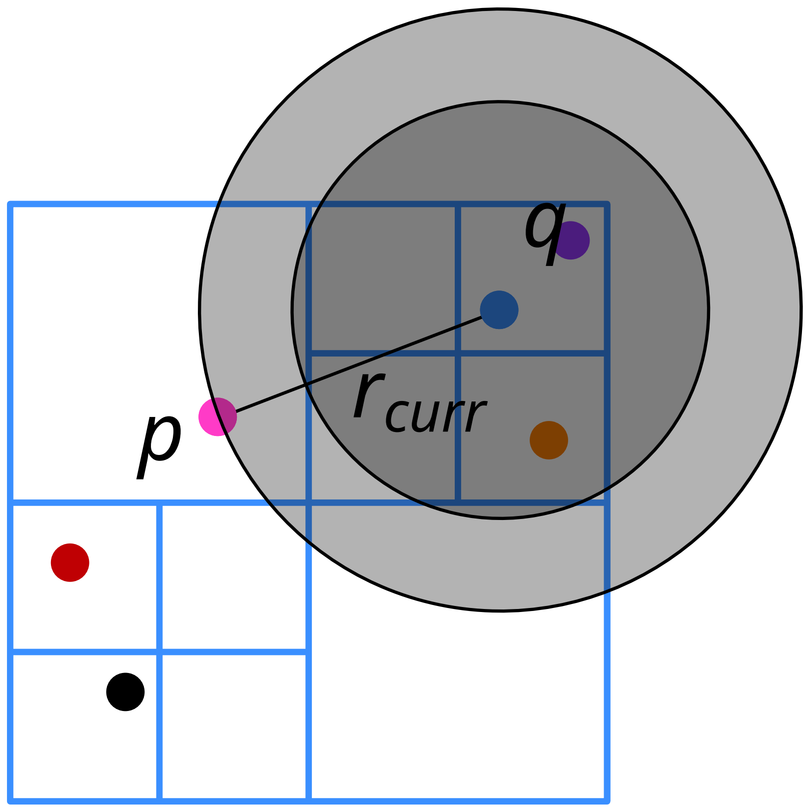

could contain $s \in P$ with $||q - s|| < (1 - \varepsilon/2)r_{curr}$

ignore cell w if $||q - \text{rep}_w|| - diam(\square_w) > (1 - \varepsilon/2)r_{curr}$

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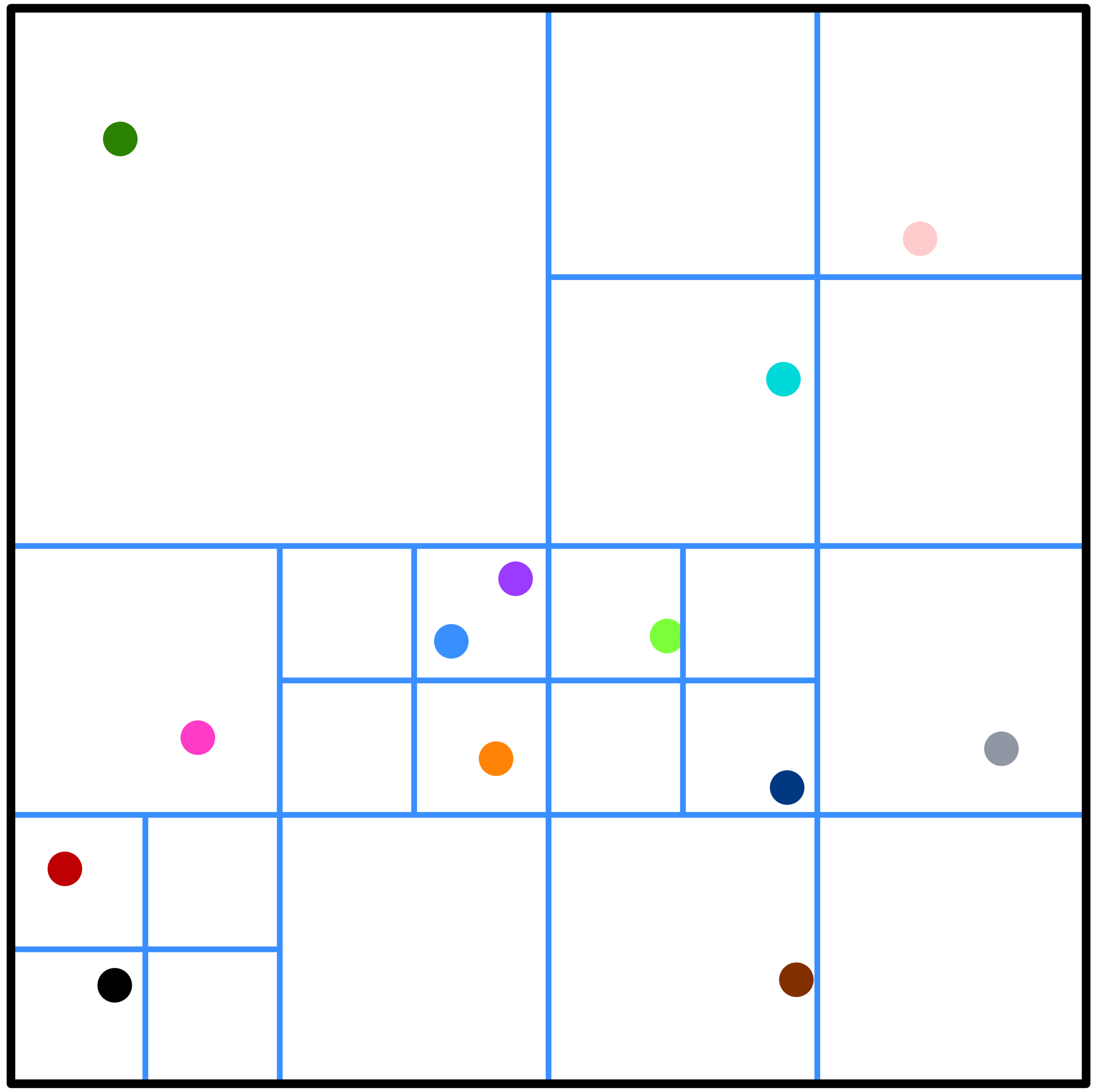

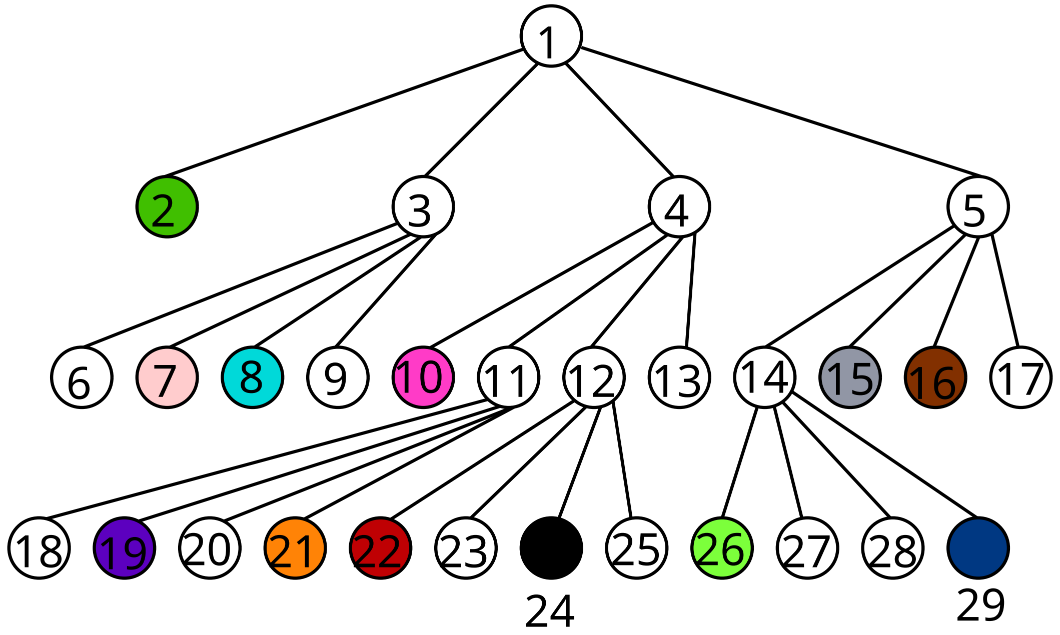

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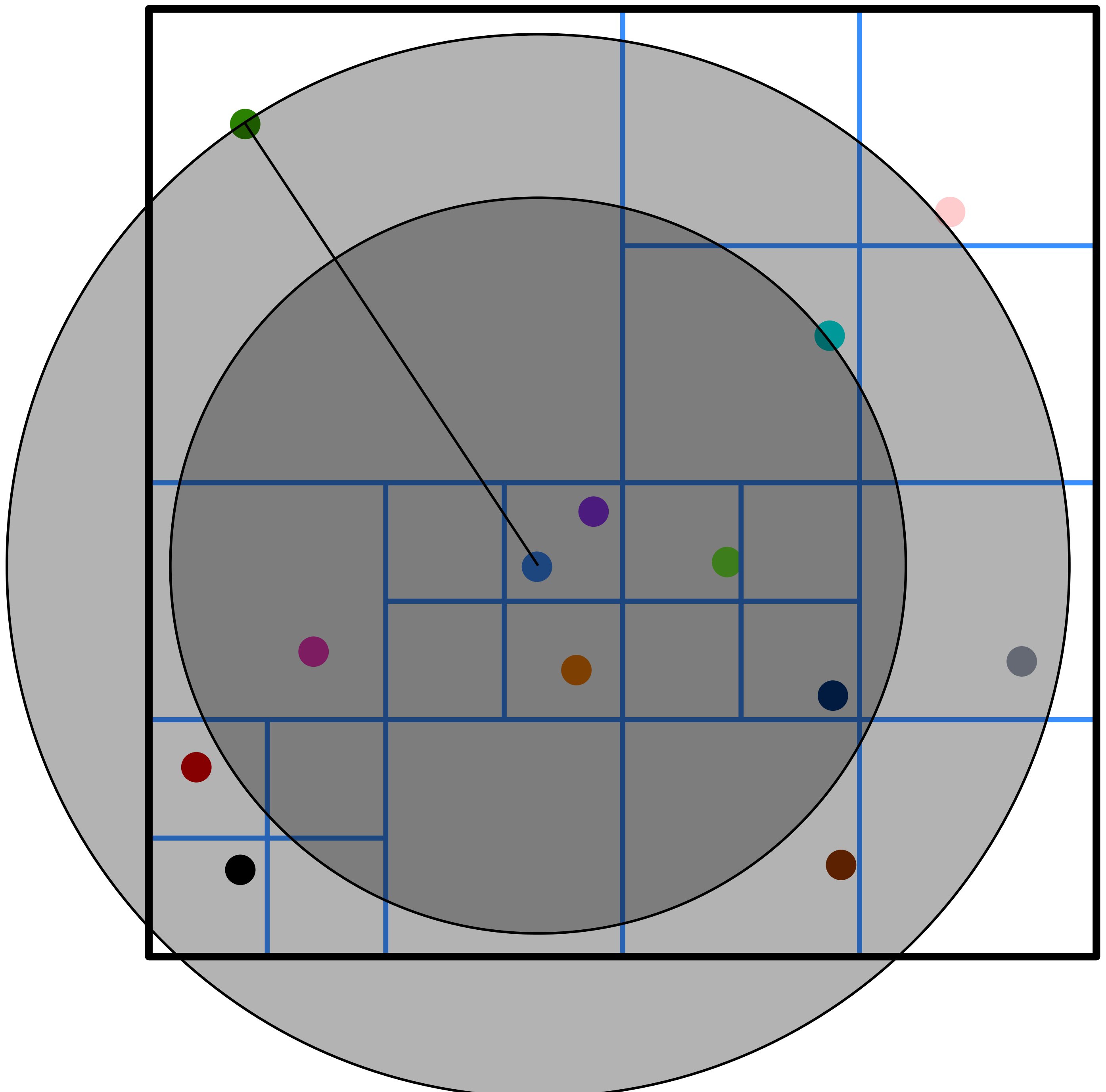
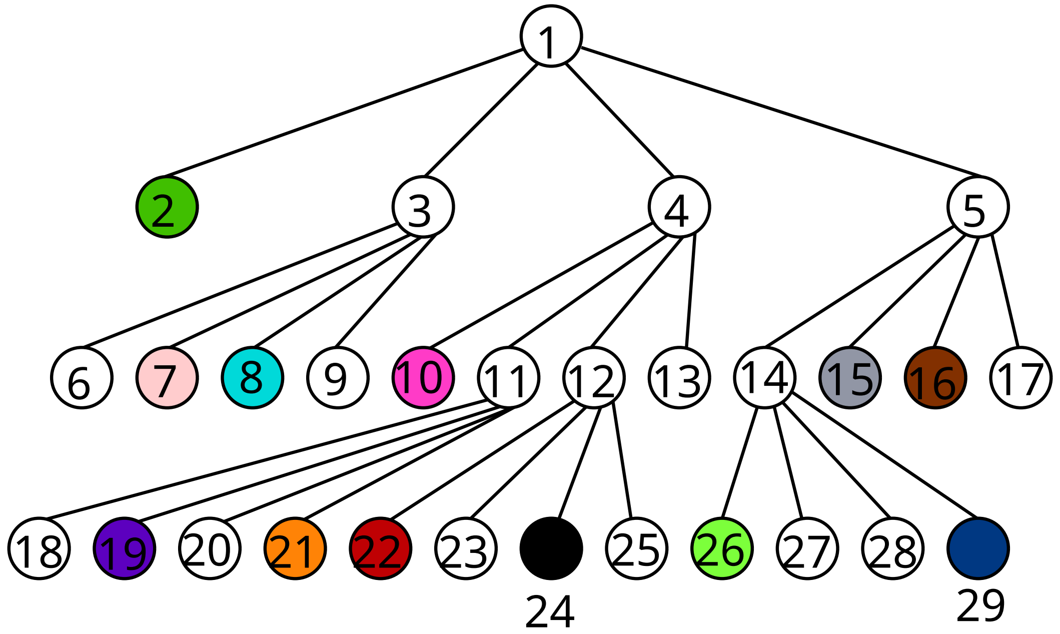
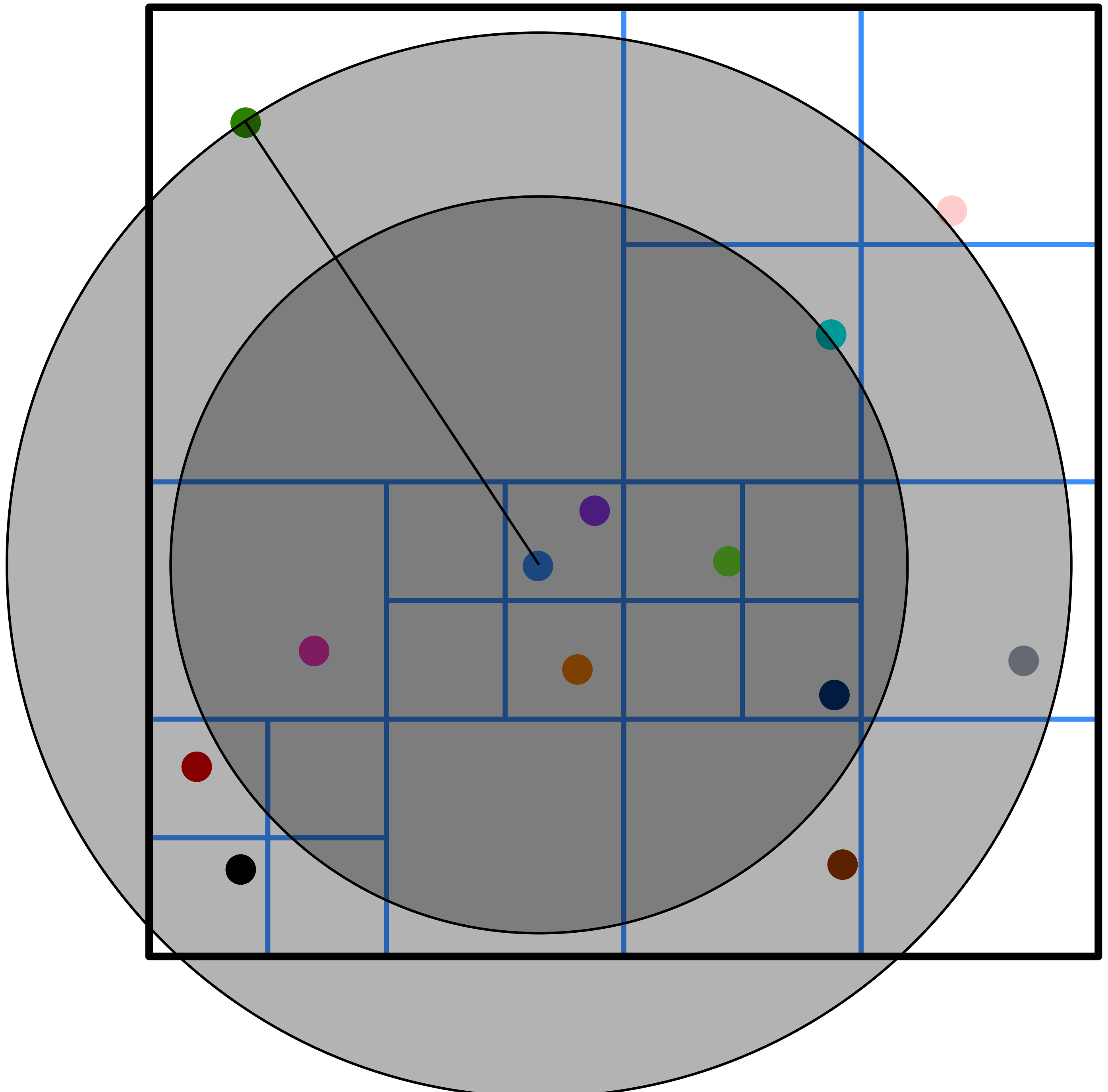
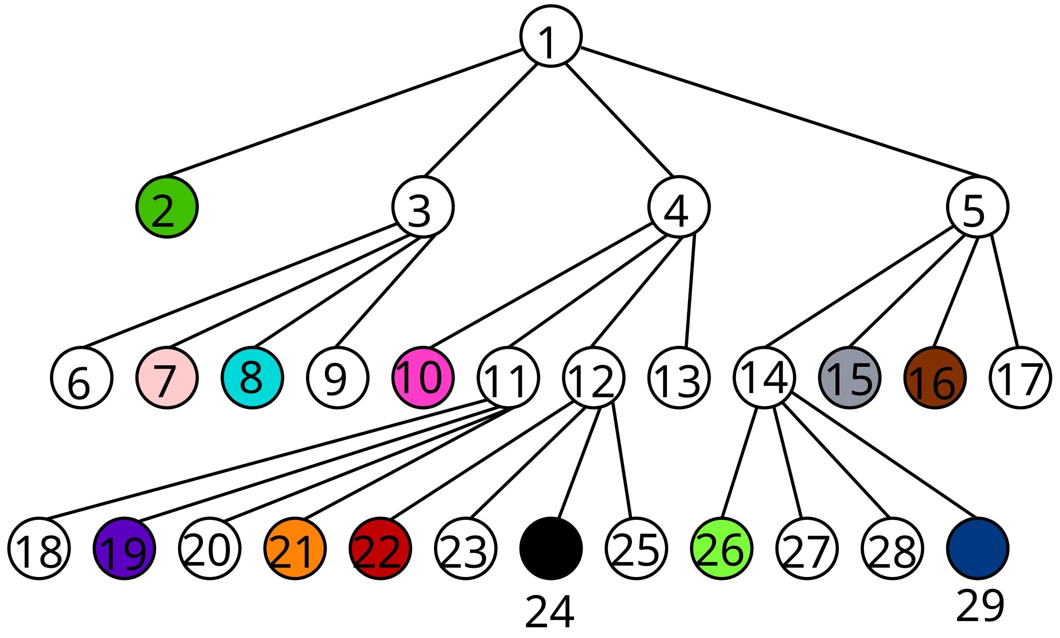




$A_0 = \{1\}$
$\text{rep}_1 = 2$
$p = 2$

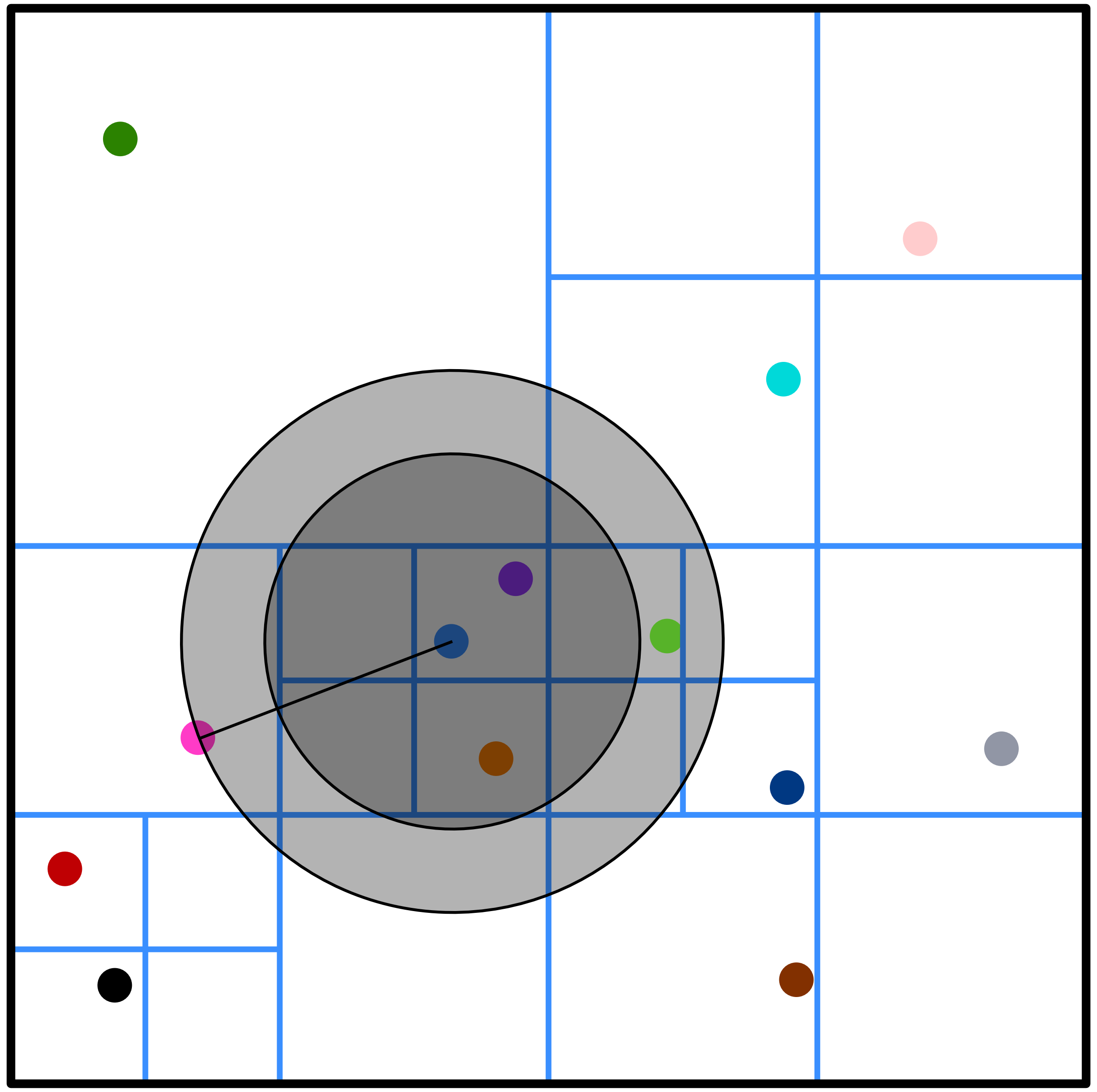
Approximate nearest neighbor (Bounded spread)

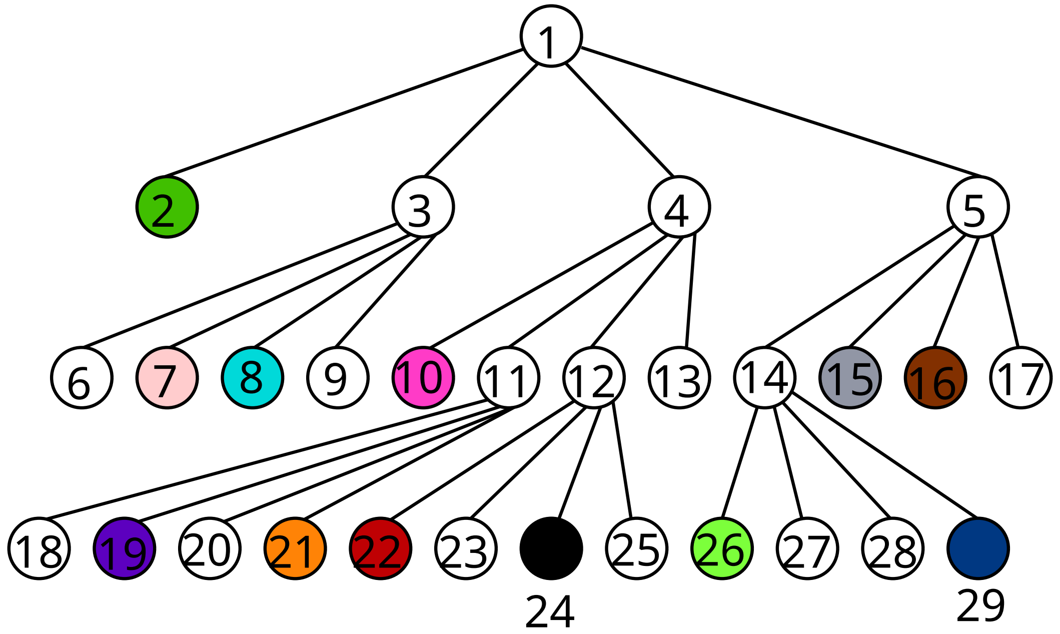




$A_1 = \{2, 3, 4, 5\}$
$\text{rep}_3 = 7$, $\text{rep}_4 = 10$, $\text{rep}_5 = 15$
$p = 2$

Approximate nearest neighbor (Bounded spread)





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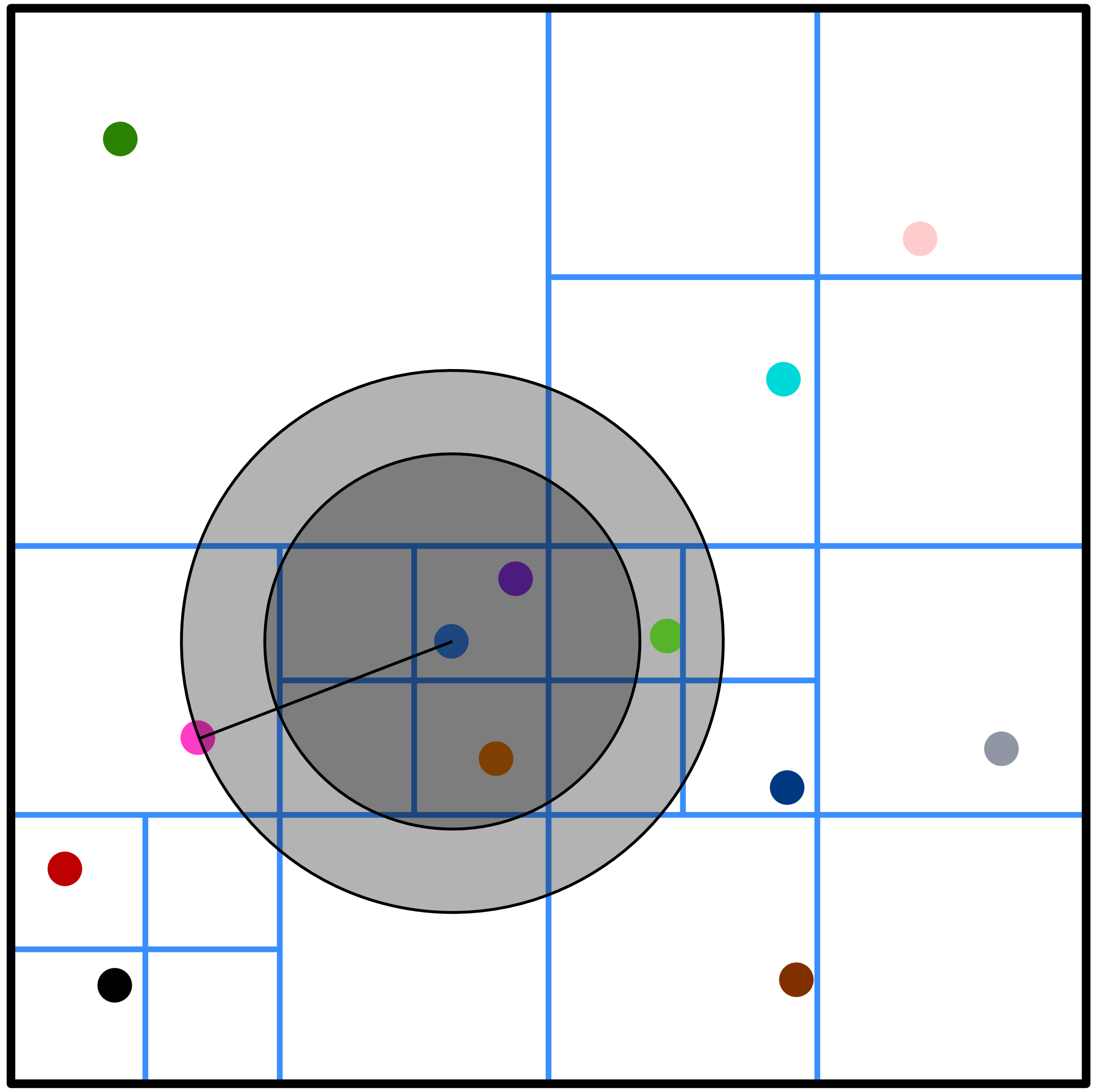
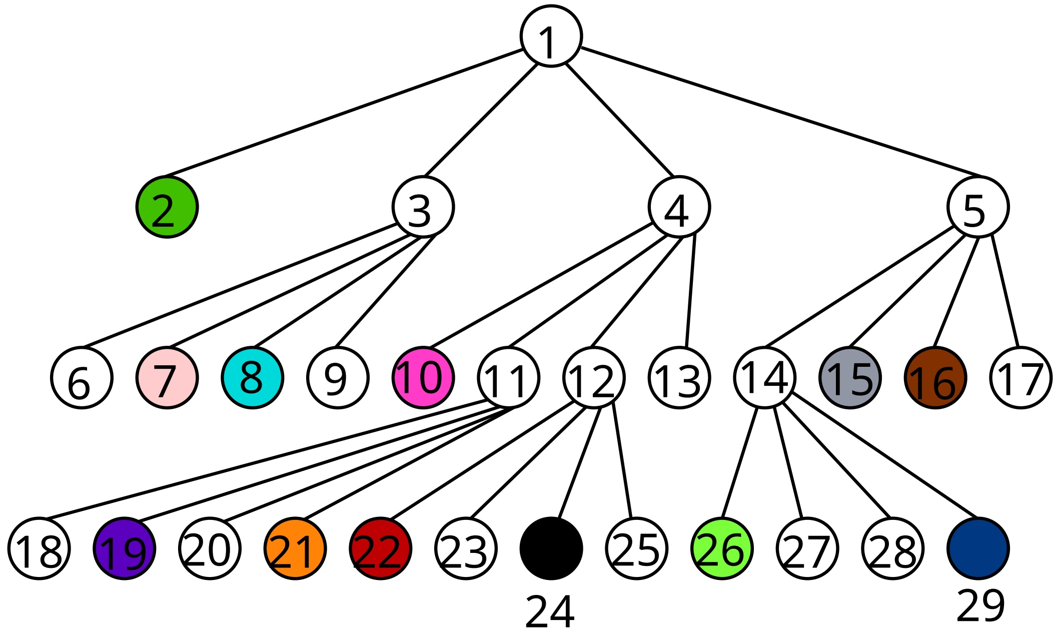
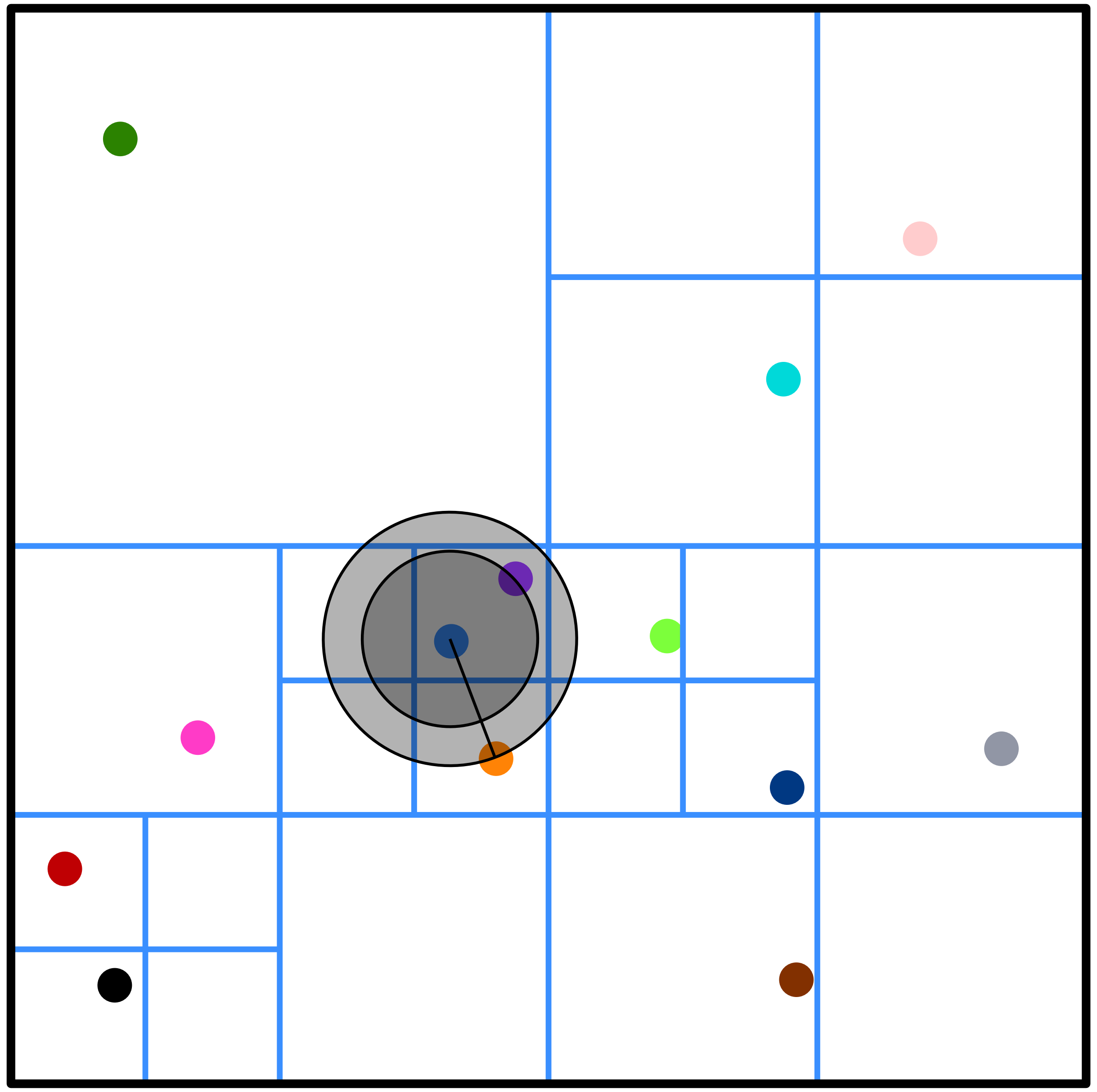
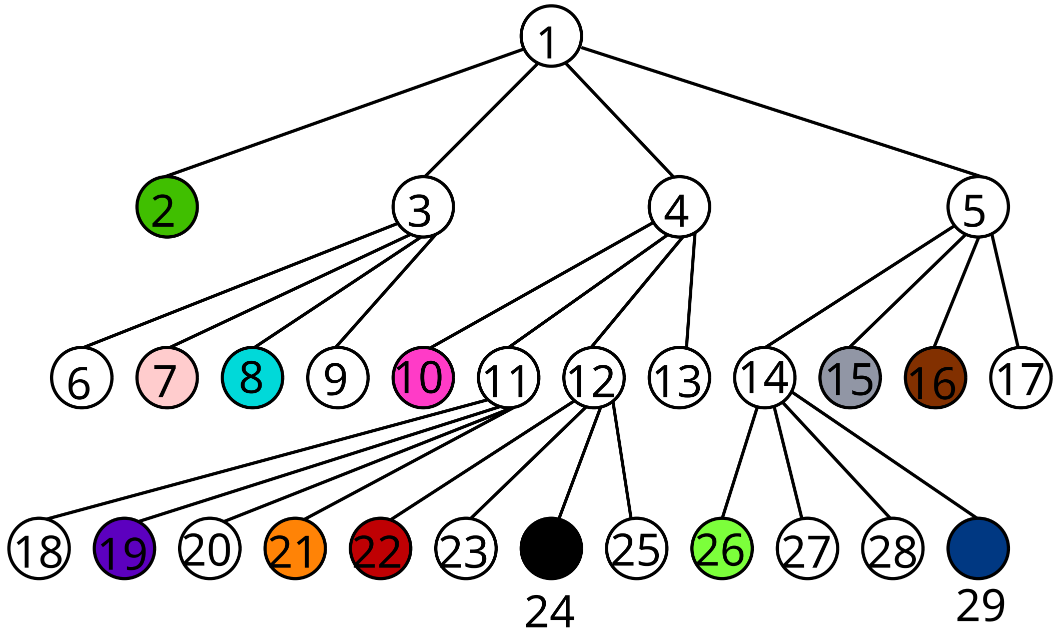




$A_2 = \{8, 10, 11, 14\}$
$\mathrm{rep}_{11} = 21, \mathrm{rep}_{14} = 26$
$p = 10$

Approximate nearest neighbor (Bounded spread)





$A_2 = \{8, 10, 11, 14\}$
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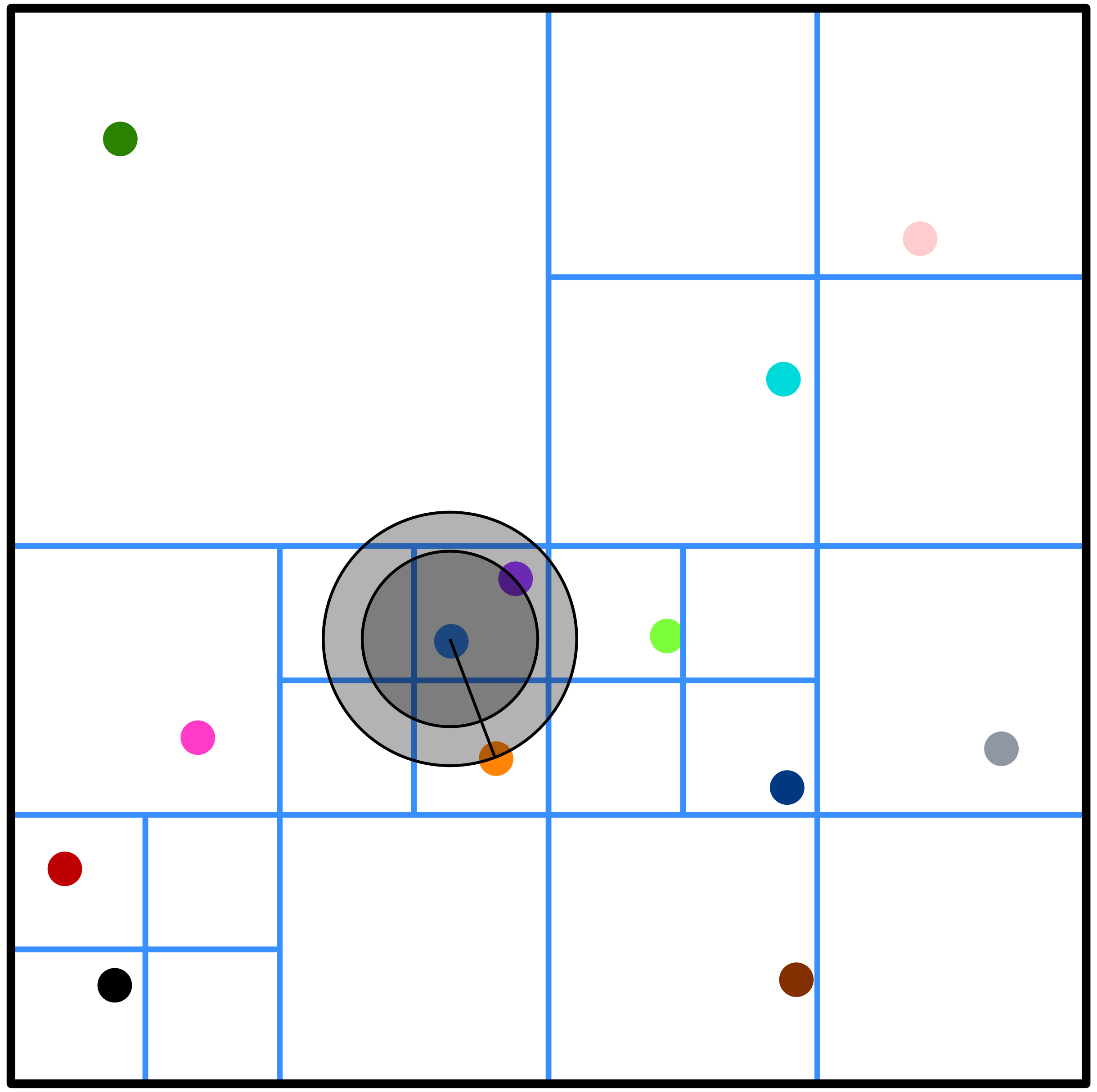
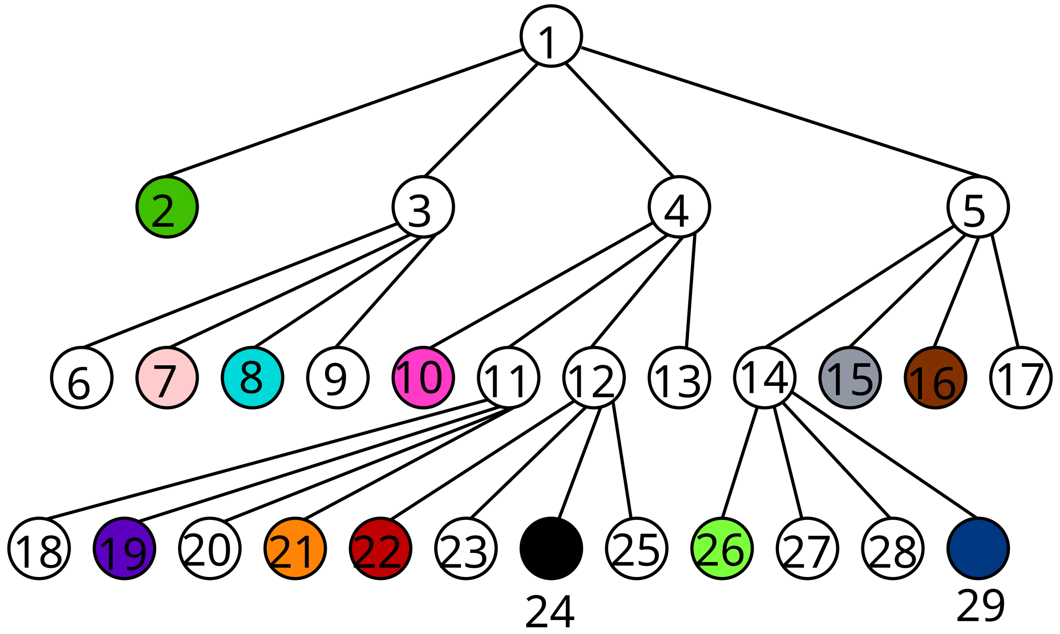
$p = 21$

Approximate nearest neighbor (Bounded spread)

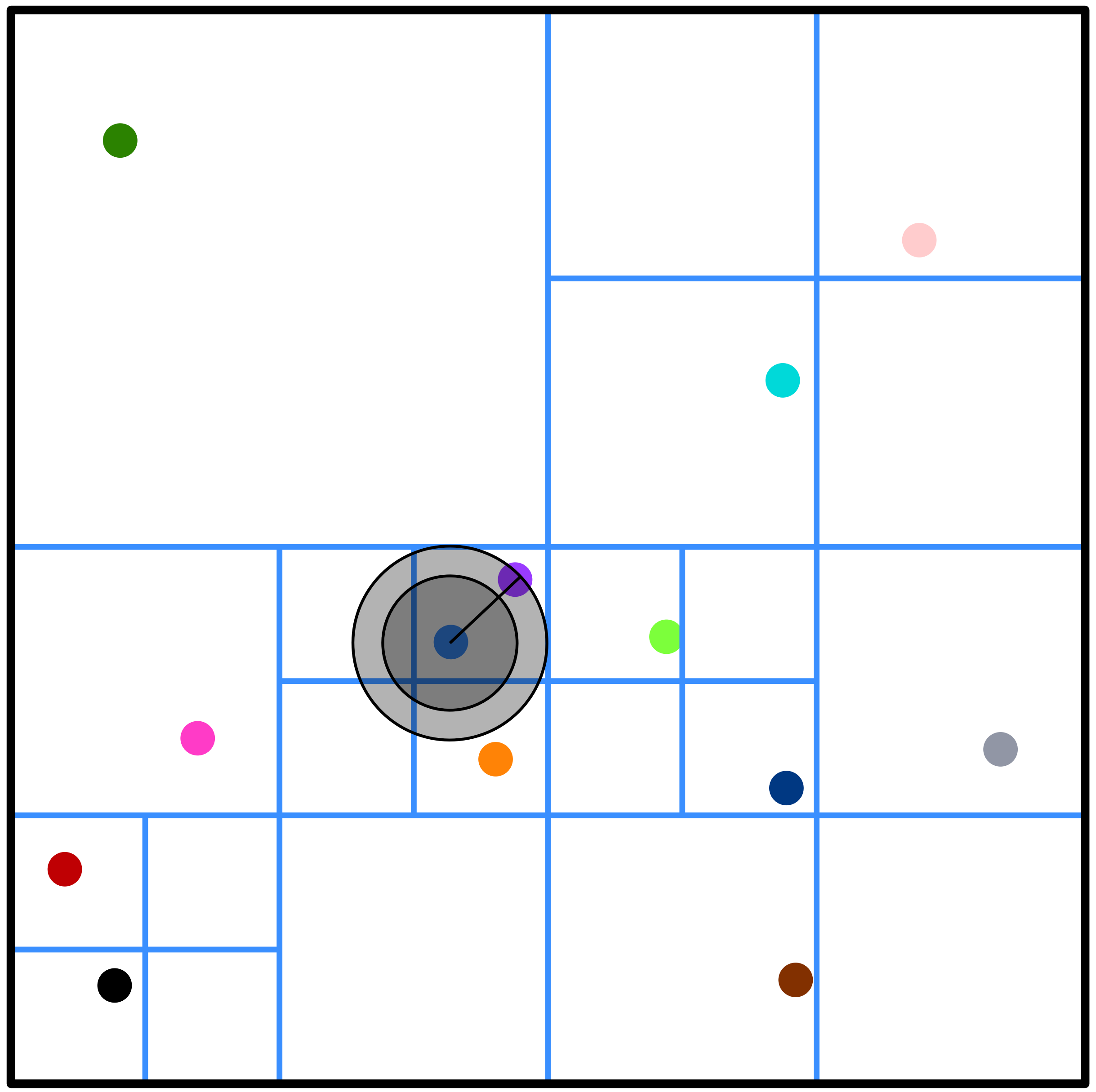


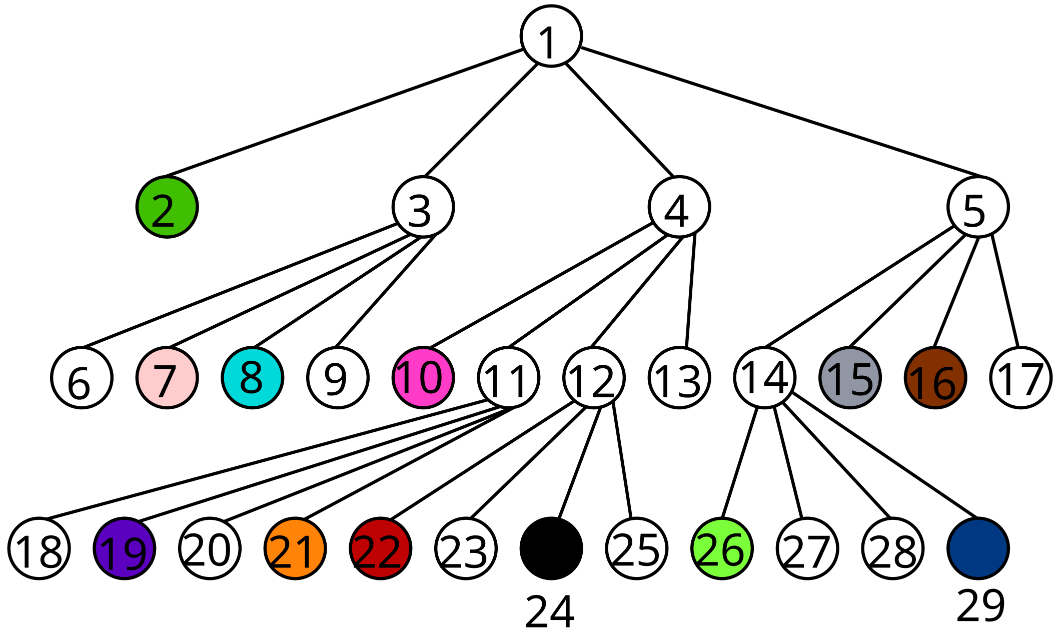


$$A_3 = \{21, 19, 26\}$$

$$p = 21$$

Approximate nearest neighbor (Bounded spread)





$A_3 = \{21, 19, 26\}$

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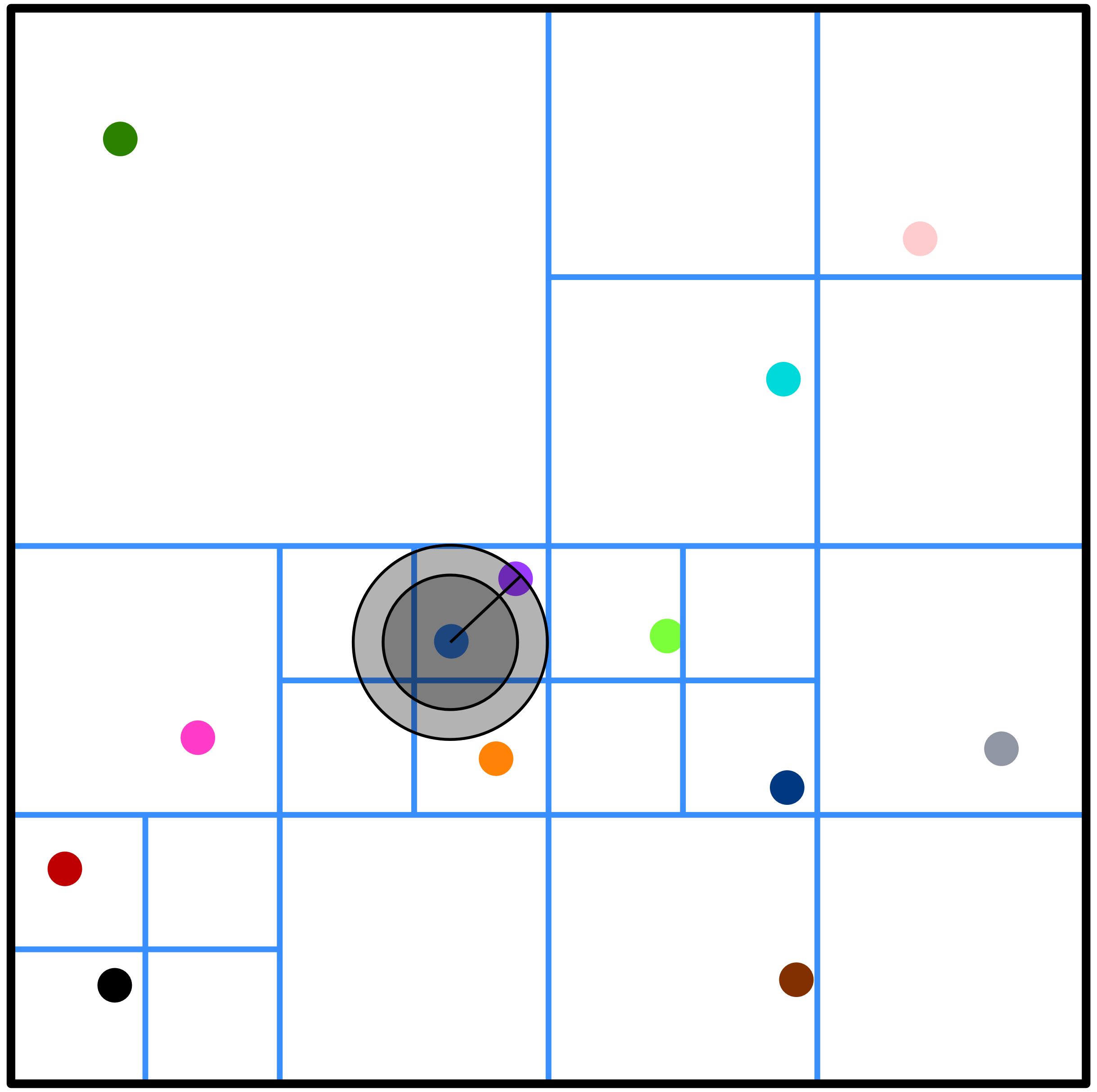
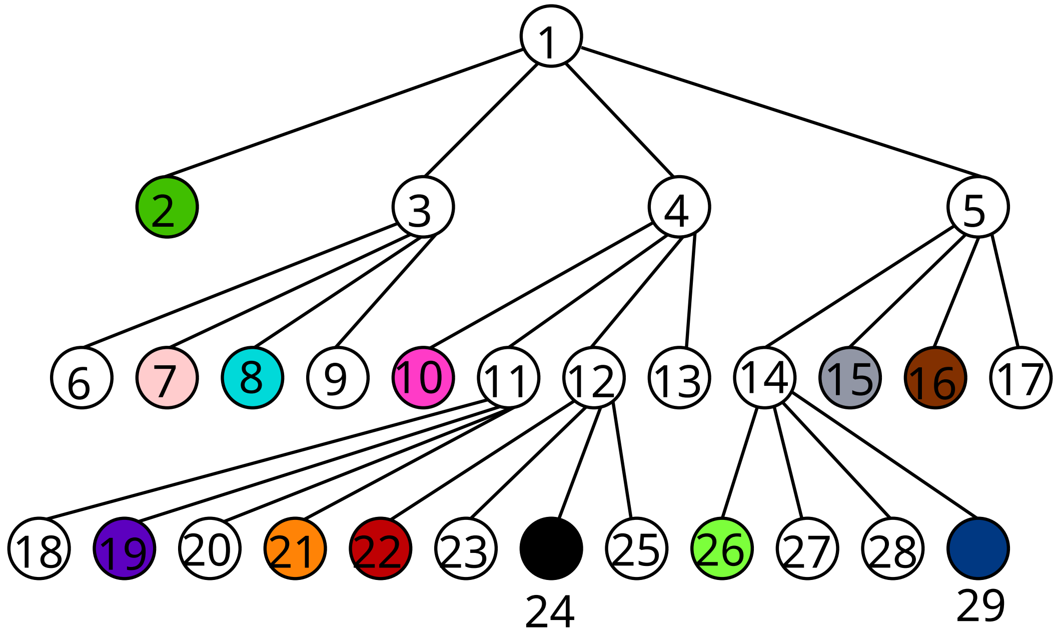
Approximate nearest neighbor (Bounded spread)





Question: How do we analyze the running time?

Approximate nearest neighbor (Bounded spread)

Running time main ideas

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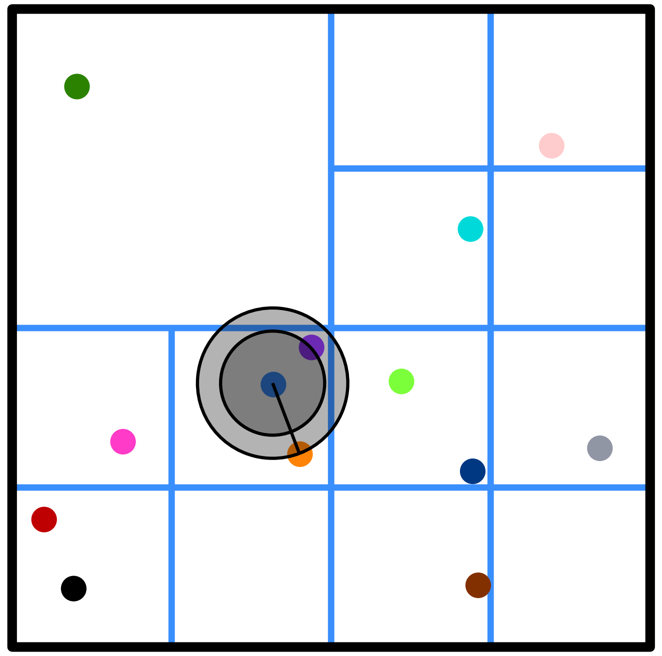
Running time main ideas

1. as long as cells > $d(q, P)$ only $O(1)$ cells per level



Approximate nearest neighbor (Bounded spread)

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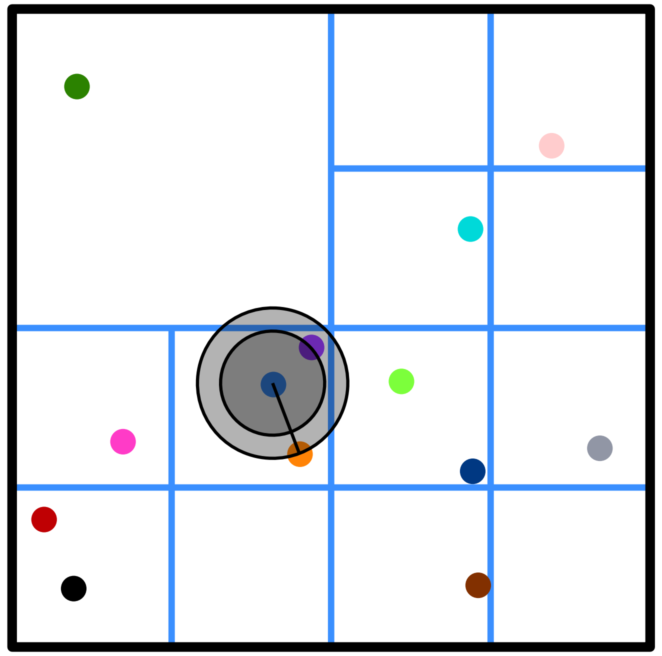
$$\# \text{ such cells} = O(\text{height}) = O(\log \Phi(P))$$



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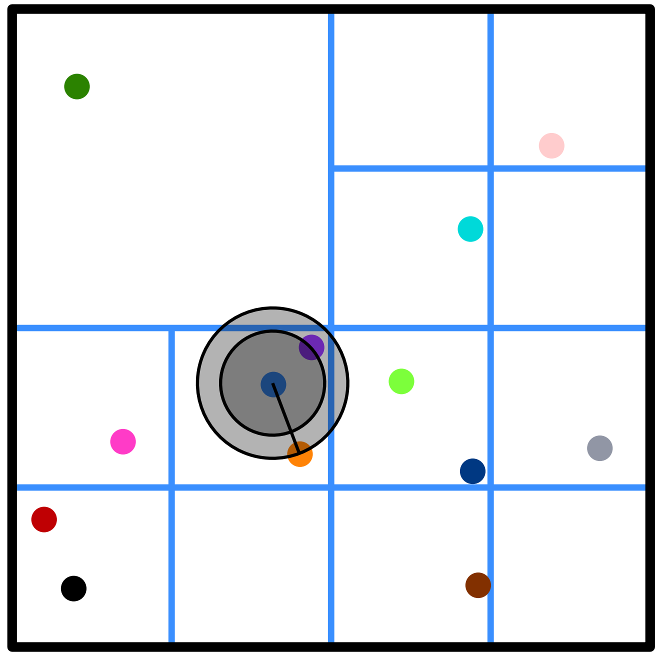
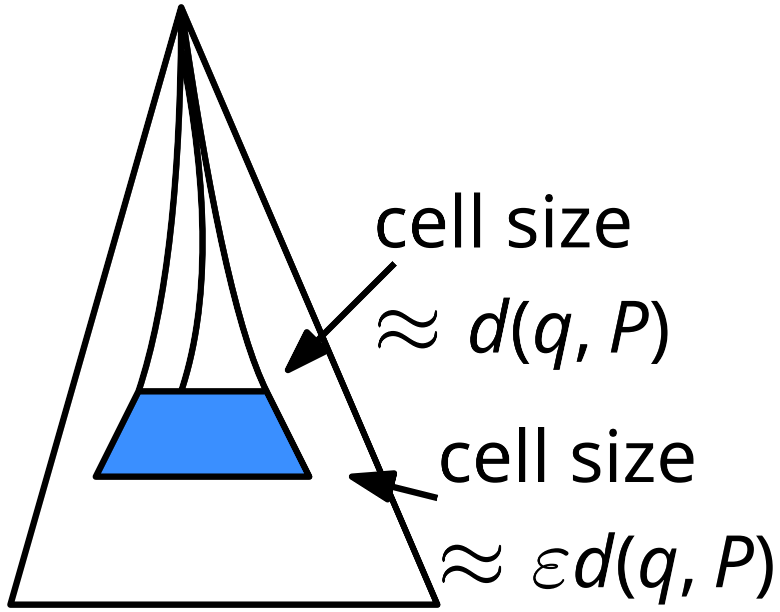


2. ends when cells have size $\varepsilon d(q, P)$



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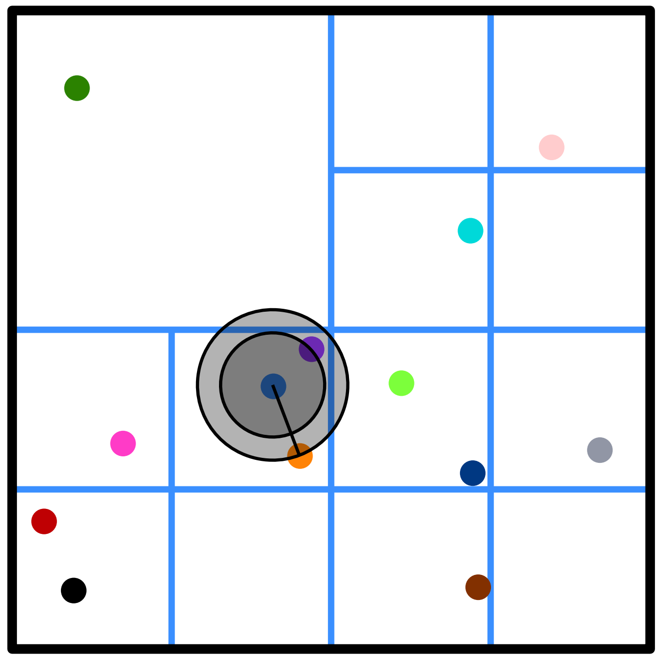

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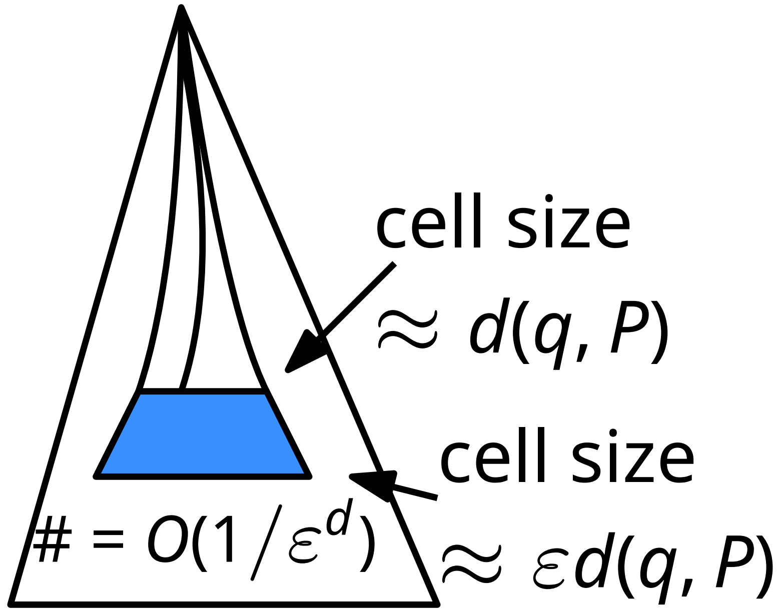
   # cells in last levels = $O(1/\varepsilon^d)$



Approximate nearest neighbor (Bounded spread)

Running time

$r := d(q, P)$

Claim: node w with square σ with $diam(\sigma) < (\varepsilon/4)r$ is not further considered

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$r := d(q, P)$

Claim: node w with square σ with $diam(\sigma) < (\varepsilon/4)r$ is not further considered

$\|q - rep_w\| - diam(\sigma_w) \geq \|q - rep_w\| - (\varepsilon/4)r$

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Running time

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Claim: node w with square σ with $diam(\sigma) < (\varepsilon/4)r$ is not further considered

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Approximate nearest neighbor (Bounded spread)

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Summary:

A $(1 + \varepsilon)$-ANN query on a quadtree takes $O(1/\varepsilon^d + \log \Phi(P))$ time.

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How about unbounded spread?

Overview

1. Introduction
2. ANN with quadtree (bounded spread)
3. Why low-quality approximation helps for unbounded spread
4. Low-quality approximation

low-quality approximation $\rightarrow$ unbounded spread

Assume we can compute p that is $4n$-ANN of q.

low-quality approximation $\to$ unbounded spread

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$R := \|p - q\|$, $L := \lfloor \log R \rfloor$

Algorithm

1. Compute $4n$-approximation

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in short:
running time (without step 1) = $O(1/\varepsilon^2 + \log(R/r)) = O(1/\varepsilon^2 + \log n)$

Overview

1. Introduction
2. ANN with quadtree (bounded spread)
3. Why low-quality approximation helps for unbounded spread
4. Low-quality approximation:

    ring separator tree

    later lecture: shifting

Low-quality approximate nearest neighbour search

The Ring Separator Tree

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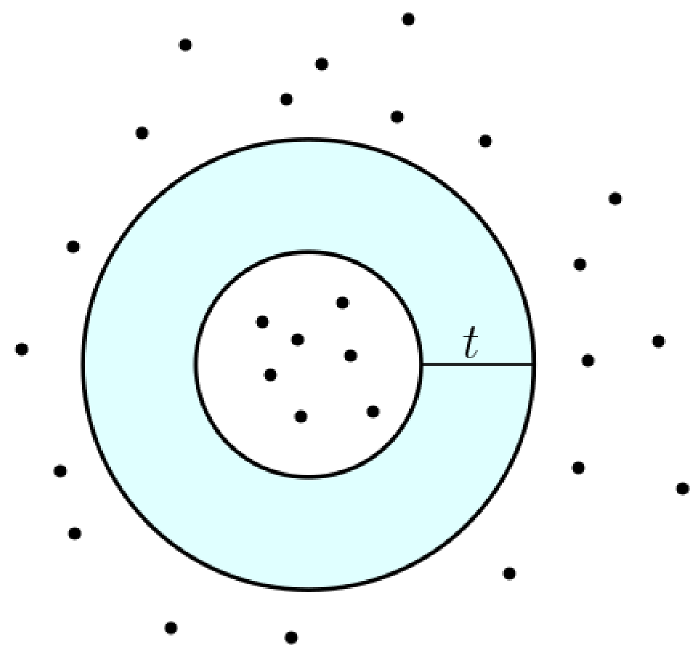

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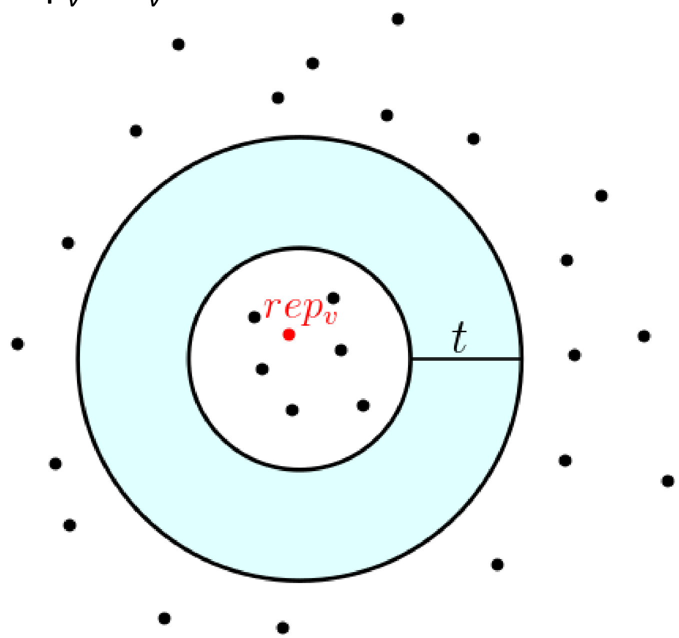
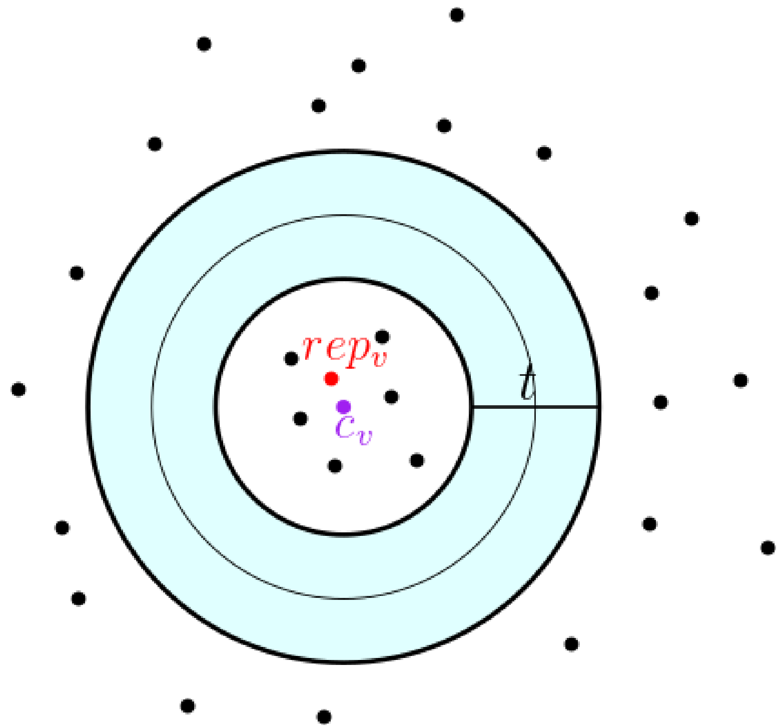


Search procedure

Given query point q:



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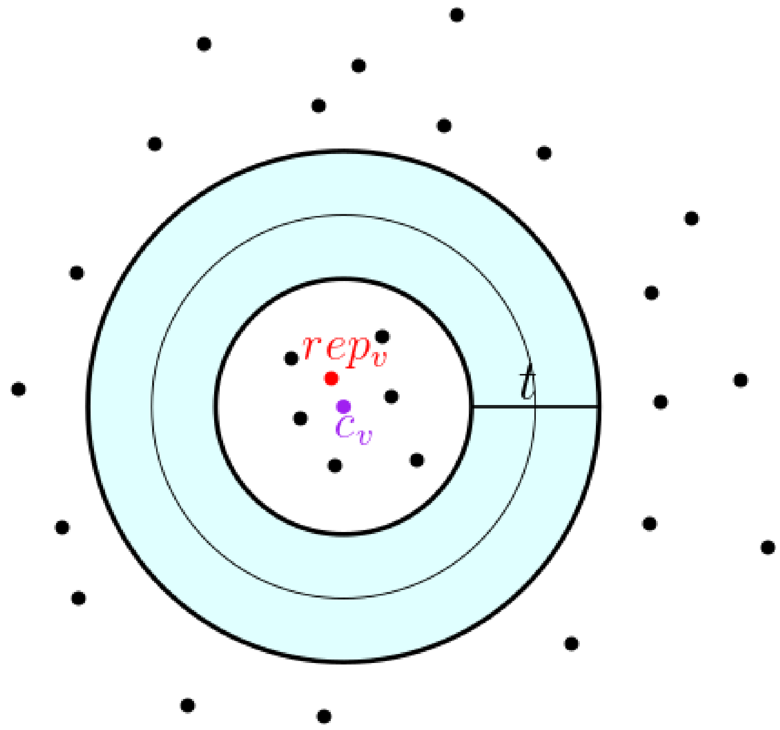

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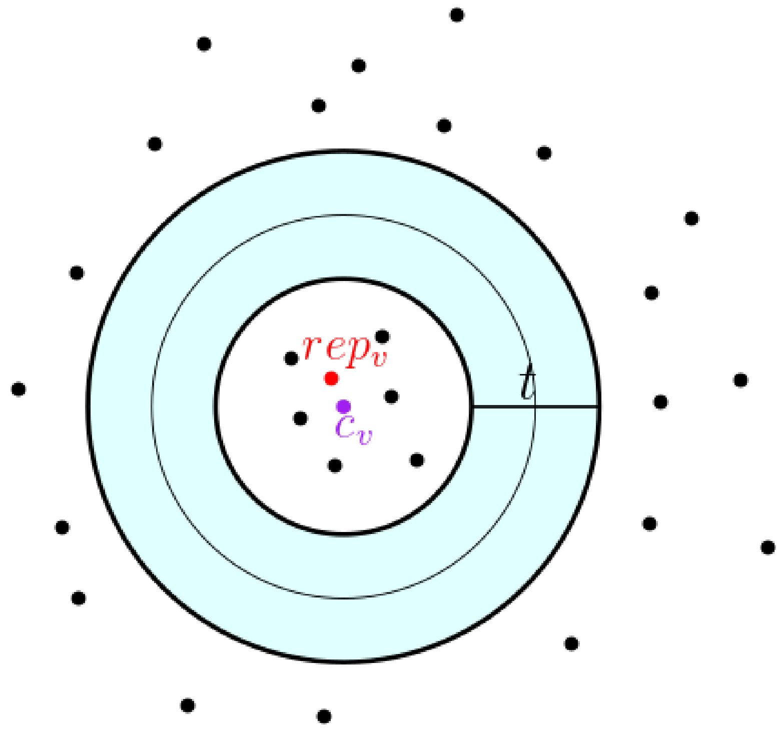

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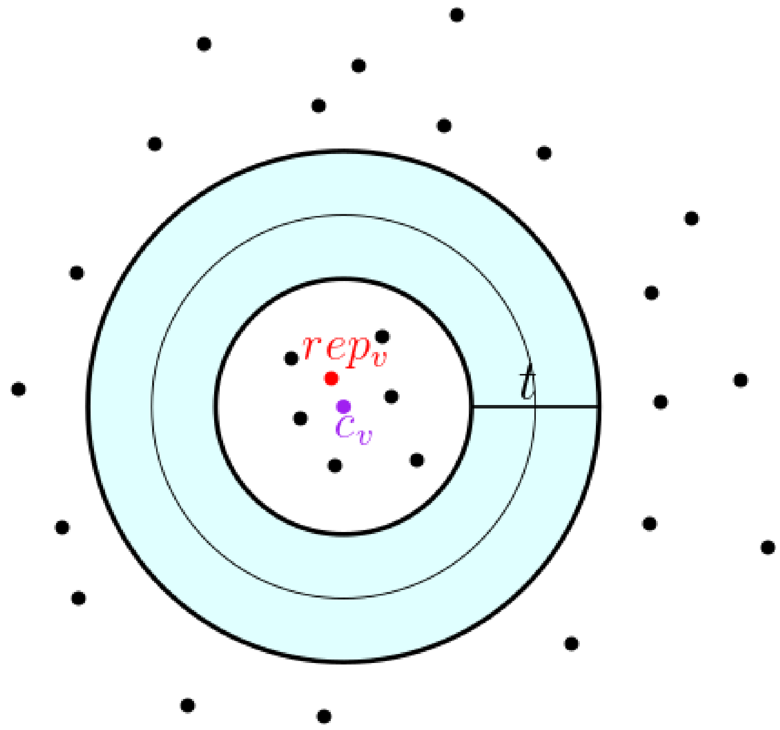

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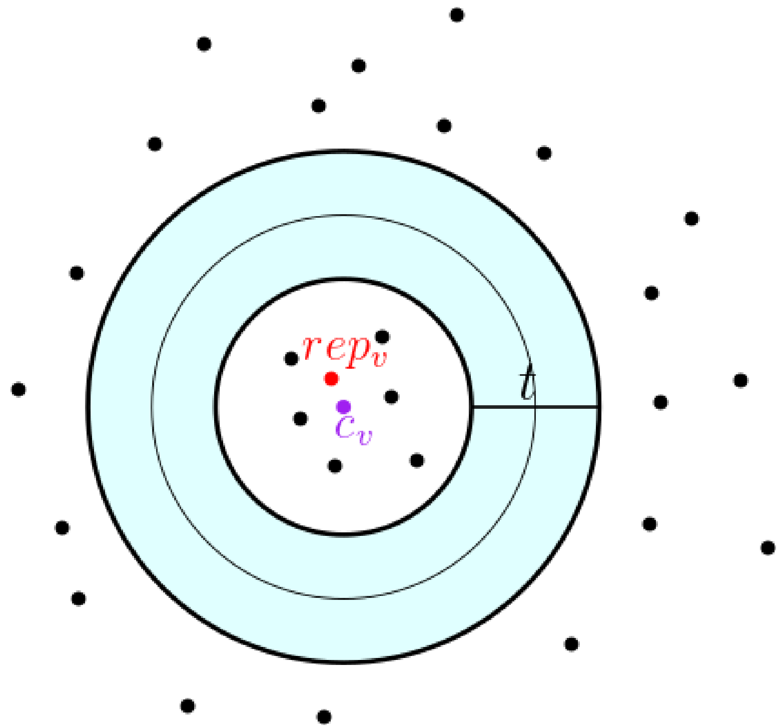

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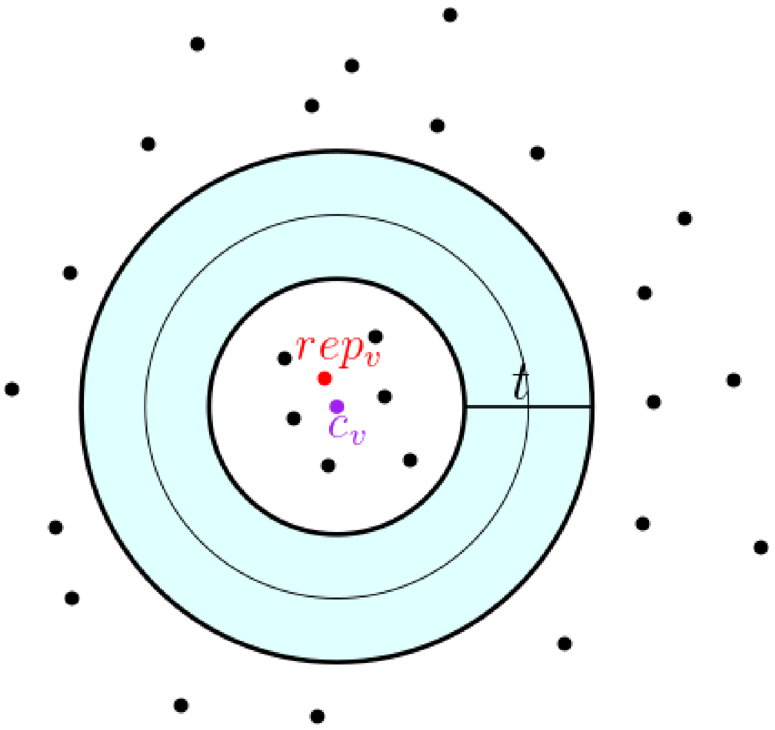

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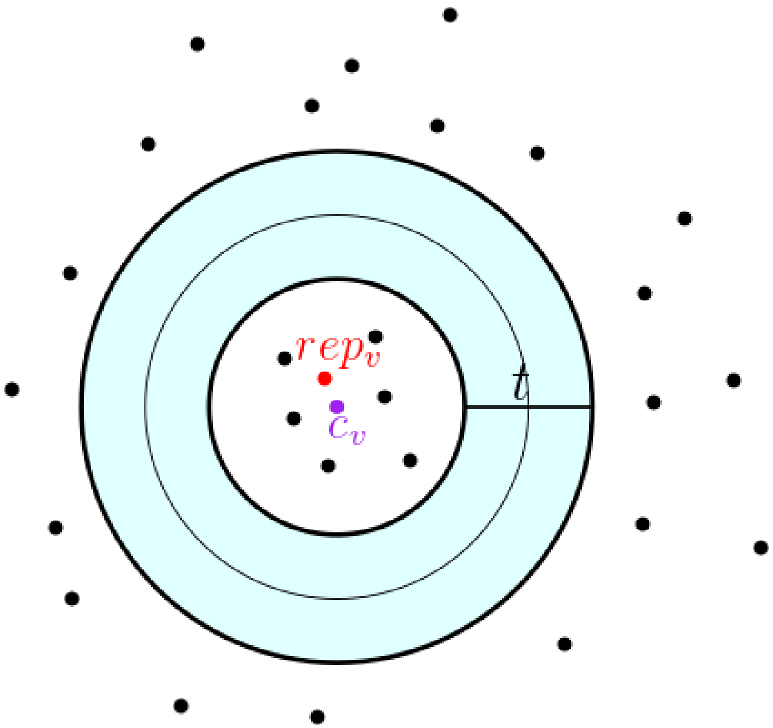

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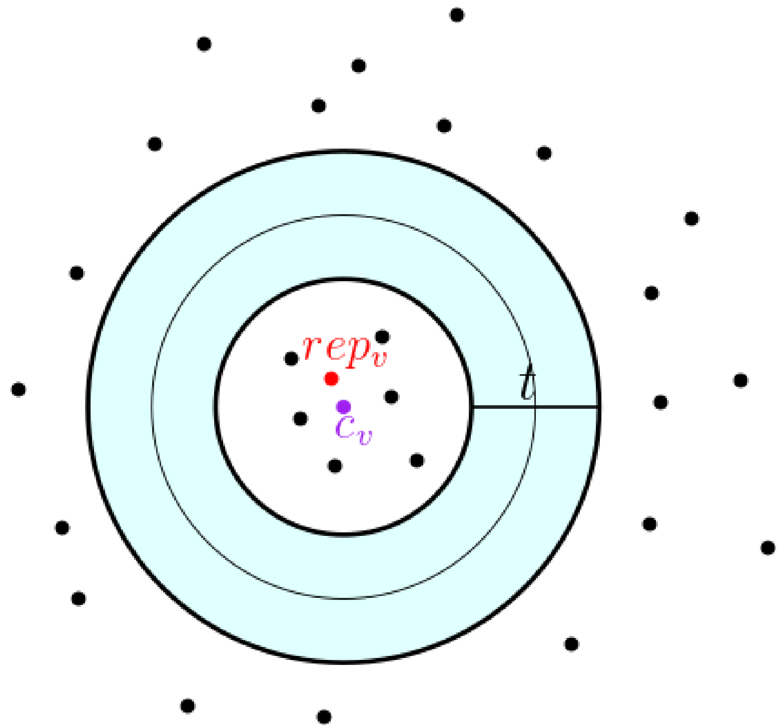

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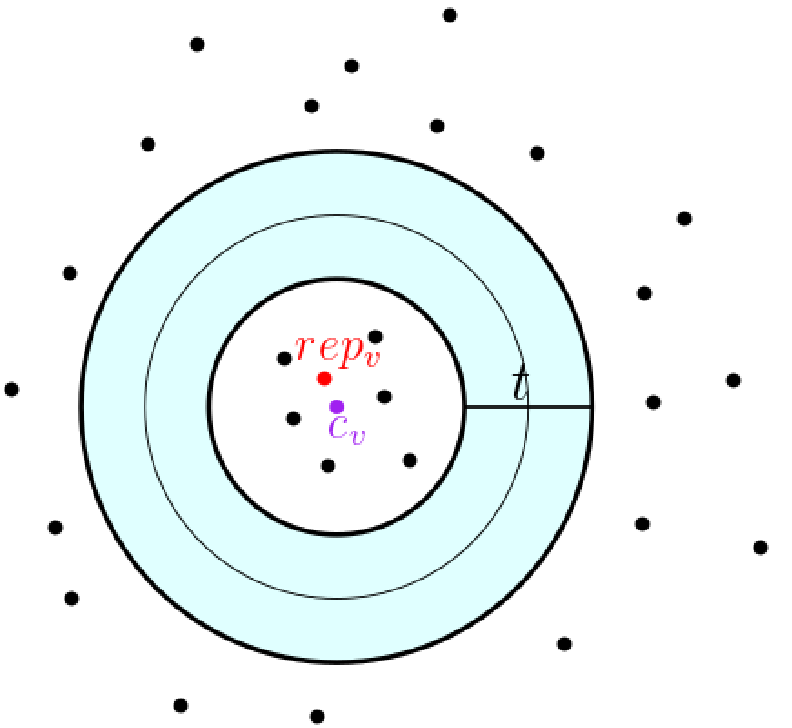
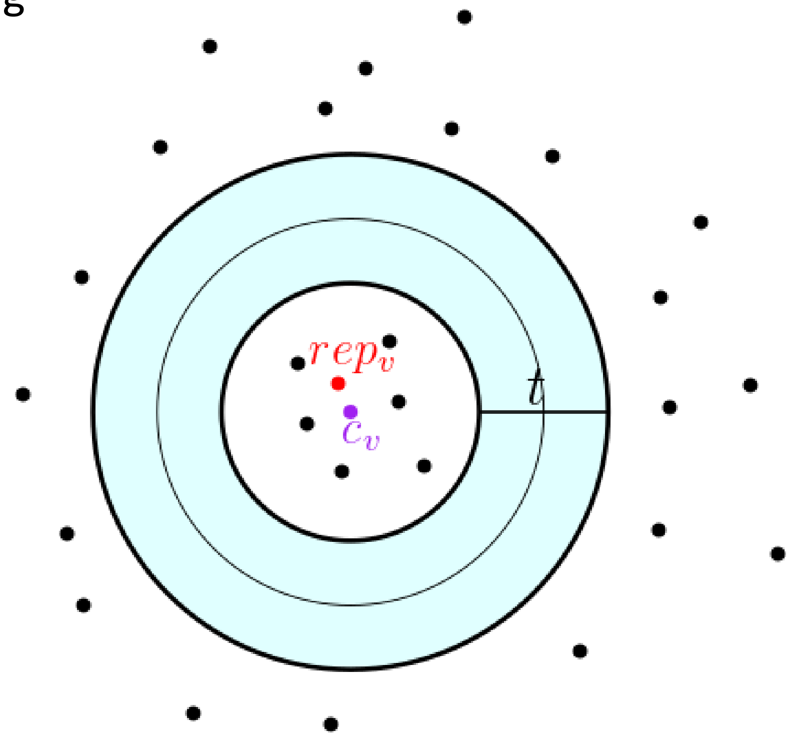
return r_{curr}



Intuition

Case distinction:

- query point is inside the inner ball
- query point is outside the enlarged ball
- query point is in the ring



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There must be a nearby point, ignore the outside

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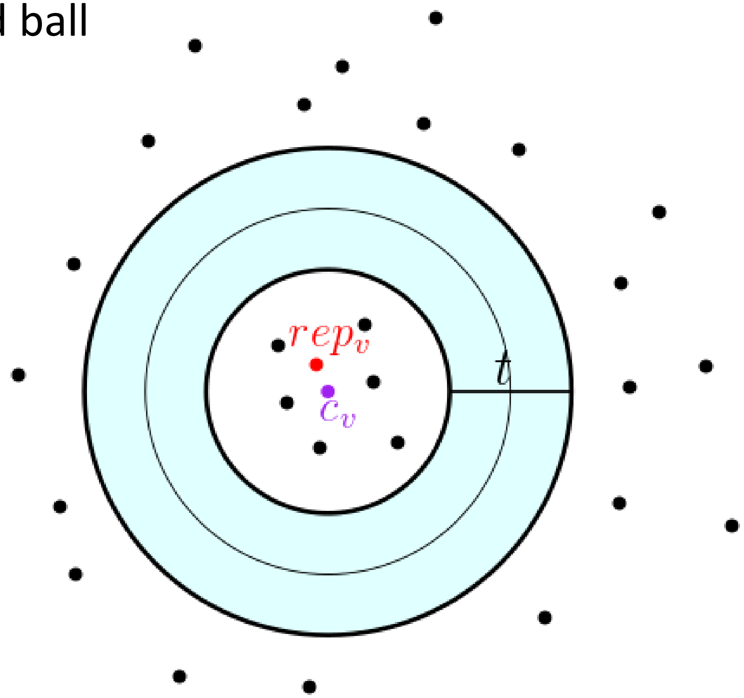

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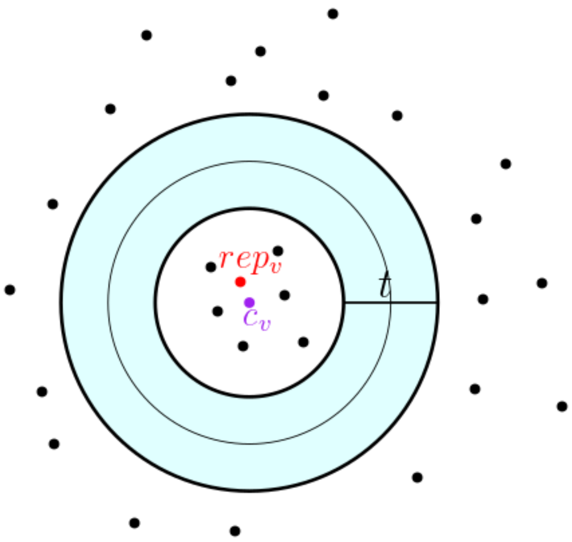

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- query point is inside the inner ball
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  - The inner points are far away, ignore the inside
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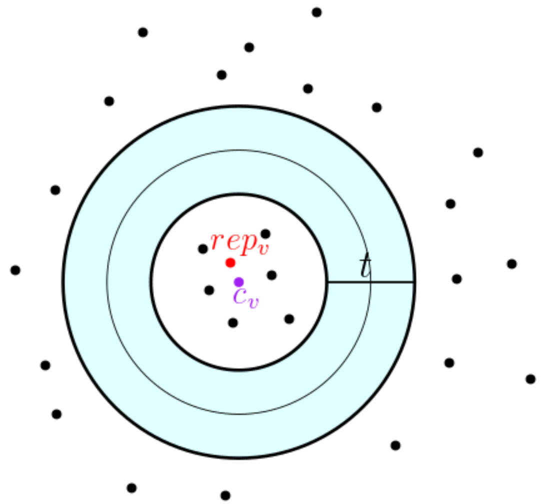
  - Split the ring halfway, group with inner or outer set
  - Slightly worse constants



Proof outline

Let P be a set of n points in R^d. One can preprocess it in O(n log n) time, such that given a query point $q \in R^d$, one can return a (1 + 4n)-ANN of q in P in O(log n) time.

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Step 3: combine previous steps to reach above goal