

Computational Intelligence

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Computational Intelligence

Fakultät für Informatik

TU Dortmund

- Introduction to Artificial Neural Networks
 - McCulloch Pitts Neuron (MCP)
 - Minsky / Papert Perceptron (MPP)
 - Single Perceptron Learning

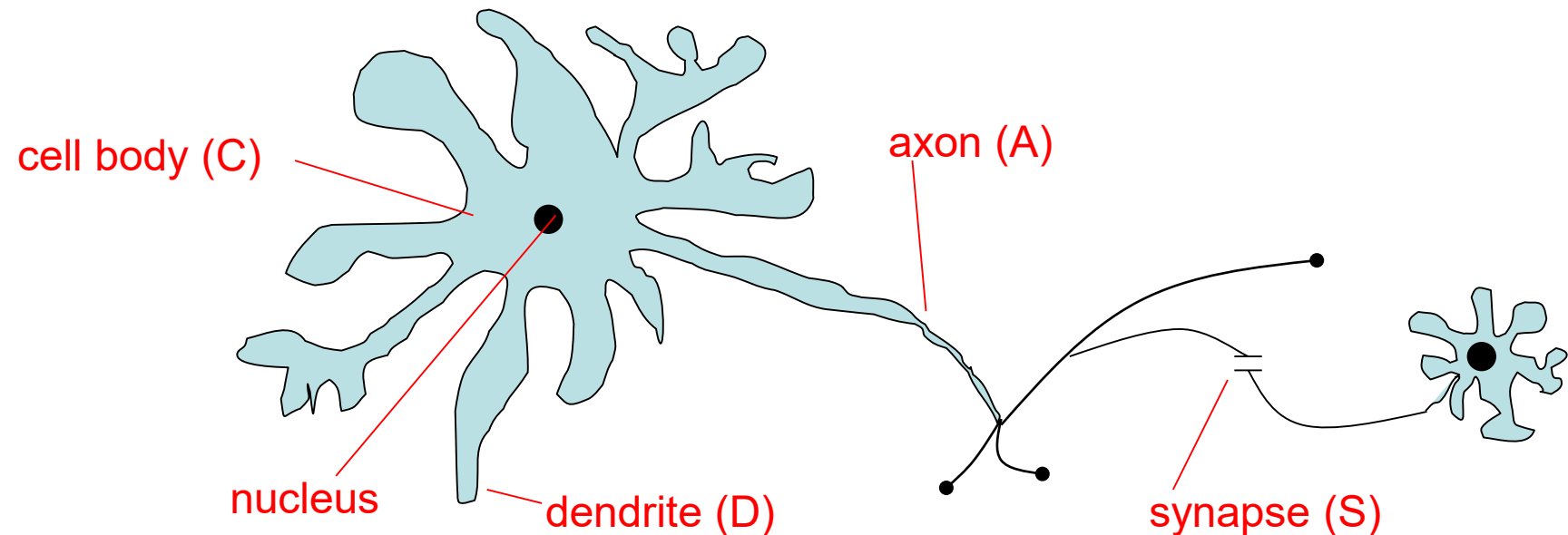
Biological Prototype

- Neuron
 - Information gathering (D)
 - Information processing (C)
 - Information propagation (A / S)

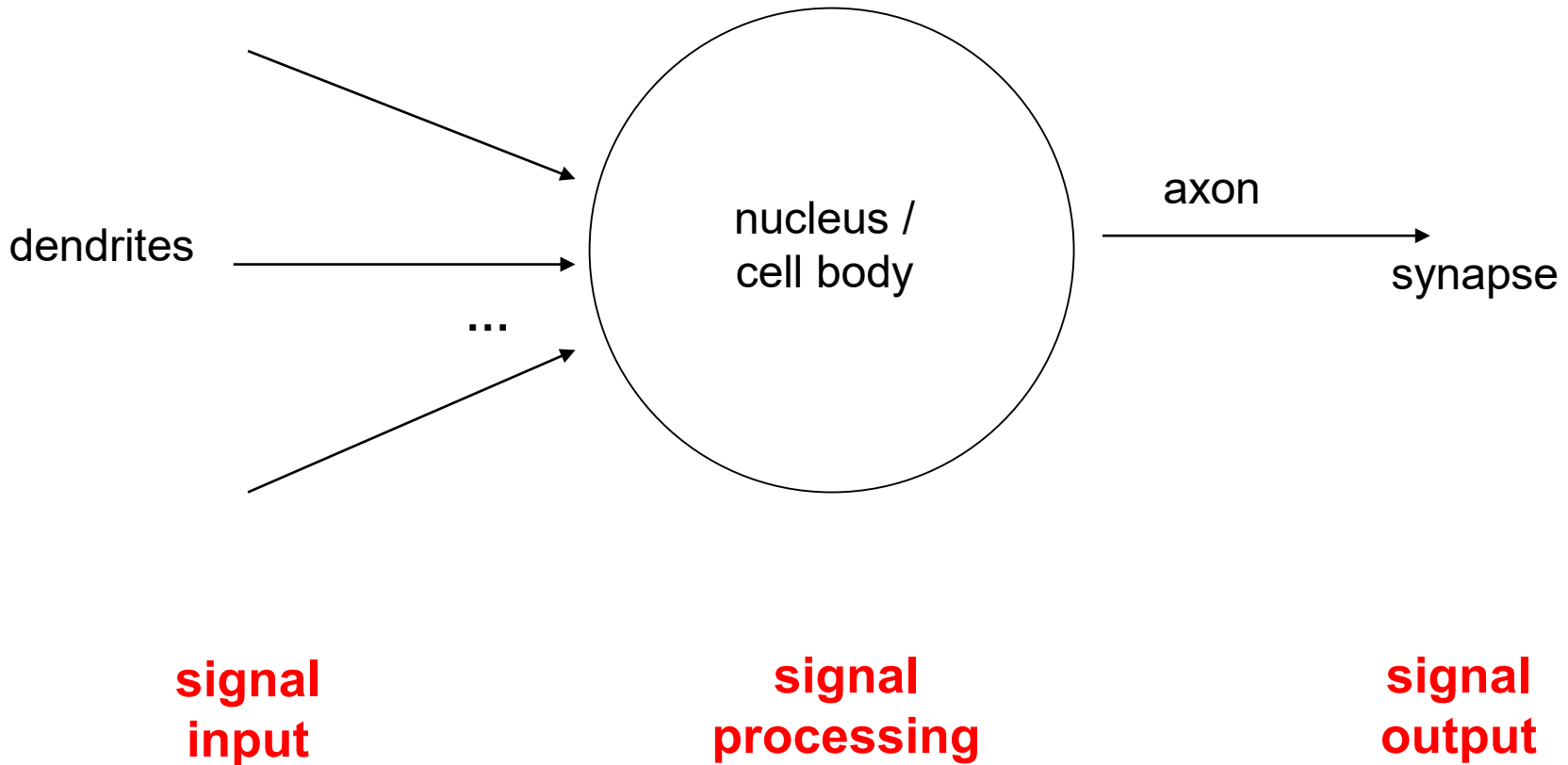
human being: 10^{12} neurons

electricity in mV range

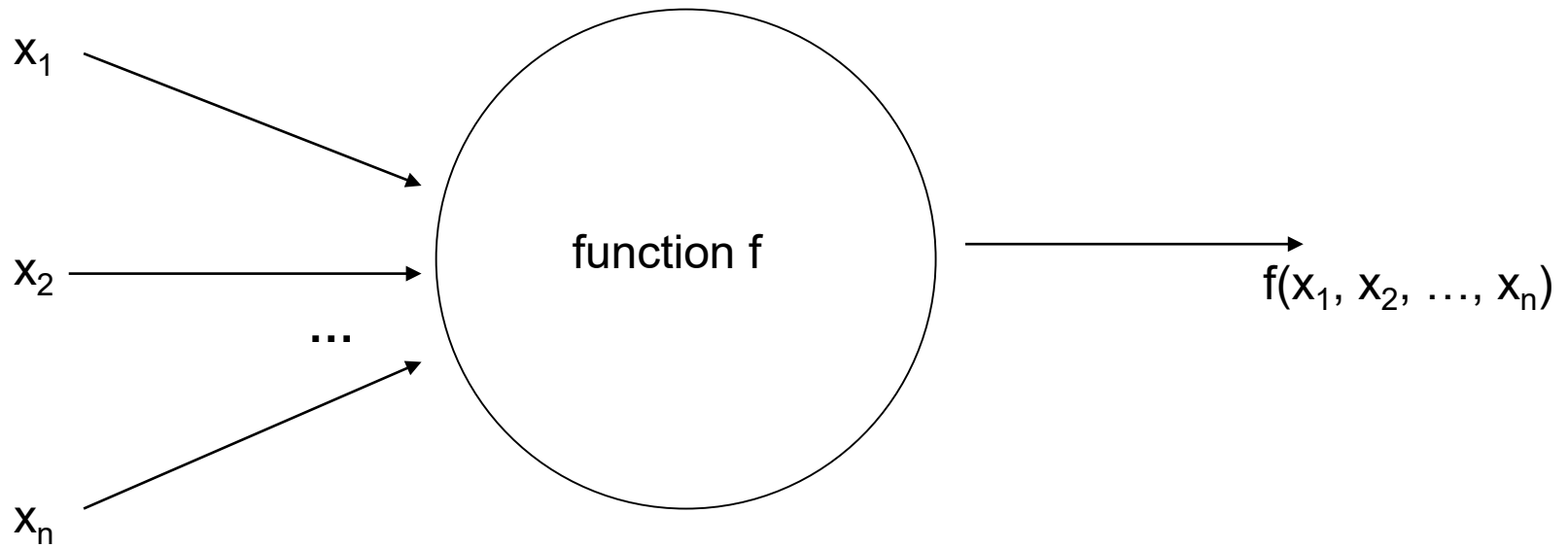
speed: 120 m / s



Abstraction



Model



McCulloch-Pitts-Neuron 1943:

$$x_i \in \{0, 1\} =: B$$

$$f: B^n \rightarrow B$$

1943: Warren McCulloch / Walter Pitts

- description of neurological networks
 - modell: McCulloch-Pitts-Neuron (MCP)
- basic idea:
 - neuron is either active or inactive
 - skills result from **connecting** neurons
- considered static networks
 - (i.e. connections had been constructed and not learnt)

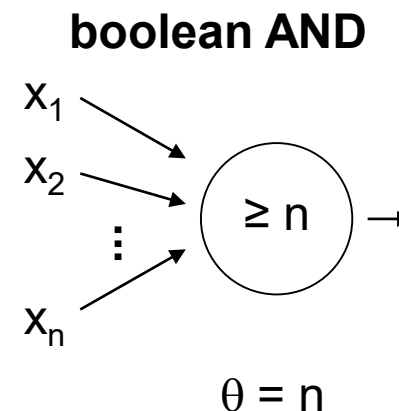
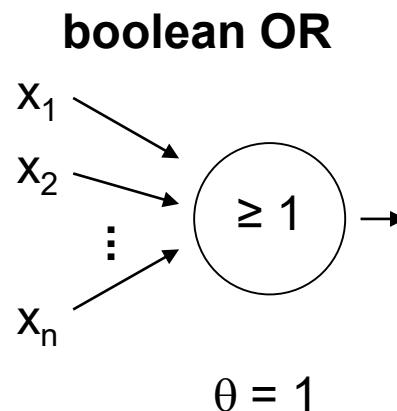
McCulloch-Pitts-Neuron

n binary input signals x_1, \dots, x_n

threshold $\theta > 0$

$$f(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i \geq \theta \\ 0 & \text{else} \end{cases}$$

⇒ can be realized:



McCulloch-Pitts-Neuron

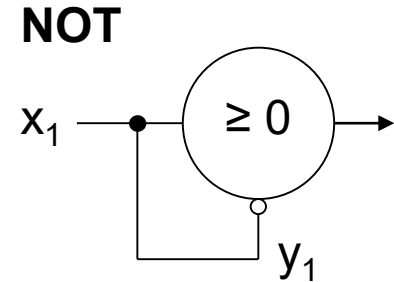
n binary input signals x_1, \dots, x_n

threshold $\theta > 0$

in addition: m binary inhibitory signals y_1, \dots, y_m

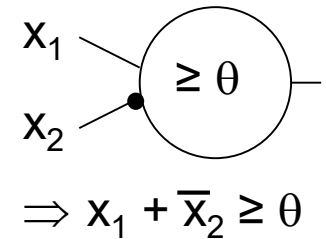
$$\tilde{f}(x_1, \dots, x_n; y_1, \dots, y_m) = f(x_1, \dots, x_n) \cdot \prod_{j=1}^m (1 - y_j)$$

- if at least one $y_j = 1$, then output = 0
- otherwise:
 - sum of inputs \geq threshold, then output = 1
 - else output = 0



Assumption:

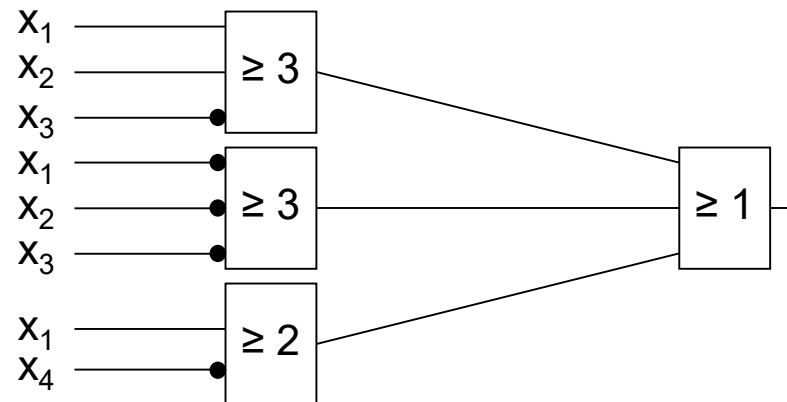
inputs also available in inverted form, i.e. \exists inverted inputs.



Theorem:

Every logical function $F: B^n \rightarrow B$ can be simulated with a two-layered McCulloch/Pitts net.

Example: $F(x) = x_1x_2\bar{x}_3 \vee \bar{x}_1\bar{x}_2\bar{x}_3 \vee x_1\bar{x}_4$



Proof: (by construction)

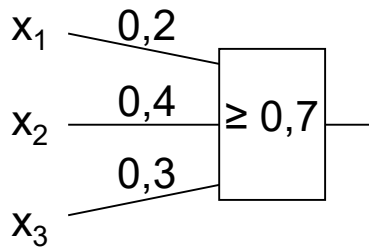
Every boolean function F can be transformed in disjunctive normal form

\Rightarrow 2 layers (AND - OR)

1. Every clause gets a decoding neuron with $\theta = n$
 \Rightarrow output = 1 only if clause satisfied (AND gate)
2. All outputs of decoding neurons
are inputs of a neuron with $\theta = 1$ (OR gate)

q.e.d.

Generalization: inputs with weights



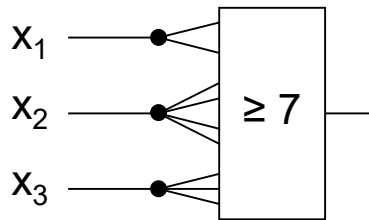
fires 1 if

$$0,2 x_1 + 0,4 x_2 + 0,3 x_3 \geq 0,7 \quad | \cdot 10$$

$$2 x_1 + 4 x_2 + 3 x_3 \geq 7$$



duplicate inputs!



⇒ equivalent!

Theorem:

Weighted and unweighted MCP-nets are equivalent for weights $\in \mathbb{Q}^+$.

Proof:

„ \Rightarrow “ Let $\sum_{i=1}^n \frac{a_i}{b_i} x_i \geq \frac{a_0}{b_0}$ with $a_i, b_i \in \mathbb{N}$

Multiplication with $\prod_{i=1}^n b_i$ yields inequality with coefficients in \mathbb{N}

Duplicate input x_i , such that we get $a_i b_1 b_2 \cdots b_{i-1} b_{i+1} \cdots b_n$ inputs.

Threshold $\theta = a_0 b_1 \cdots b_n$

„ \Leftarrow “

Set all weights to 1.

q.e.d.

Conclusion for MCP nets:

- + feed-forward: able to compute any Boolean function
- + recursive: able to simulate DFA (**d**eterministic **f**inite **a**utomaton)
- very similar to conventional logical circuits
- difficult to construct
- no good learning algorithm available

Perceptron (Rosenblatt 1958)

→ complex model → reduced by Minsky & Papert to what is “necessary“

→ Minsky-Papert perceptron (MPP), 1969 → essential difference: $x \in [0, 1] \subset \mathbb{R}$

What can a single MPP do?

$$w_1 x_1 + w_2 x_2 \geq \theta \begin{cases} \text{Y} & \rightarrow 1 \\ \text{N} & \rightarrow 0 \end{cases}$$

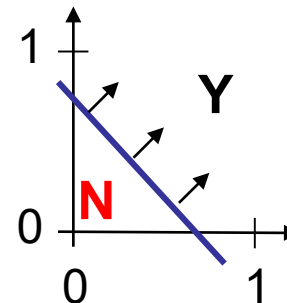
isolation of x_2 yields:

$$x_2 \geq \frac{\theta}{w_2} - \frac{w_1}{w_2} x_1 \begin{cases} \text{Y} & \rightarrow 1 \\ \text{N} & \rightarrow 0 \end{cases}$$

Example:

$$0,9 x_1 + 0,8 x_2 \geq 0,6$$

$$\Leftrightarrow x_2 \geq \frac{3}{4} - \frac{9}{8} x_1$$

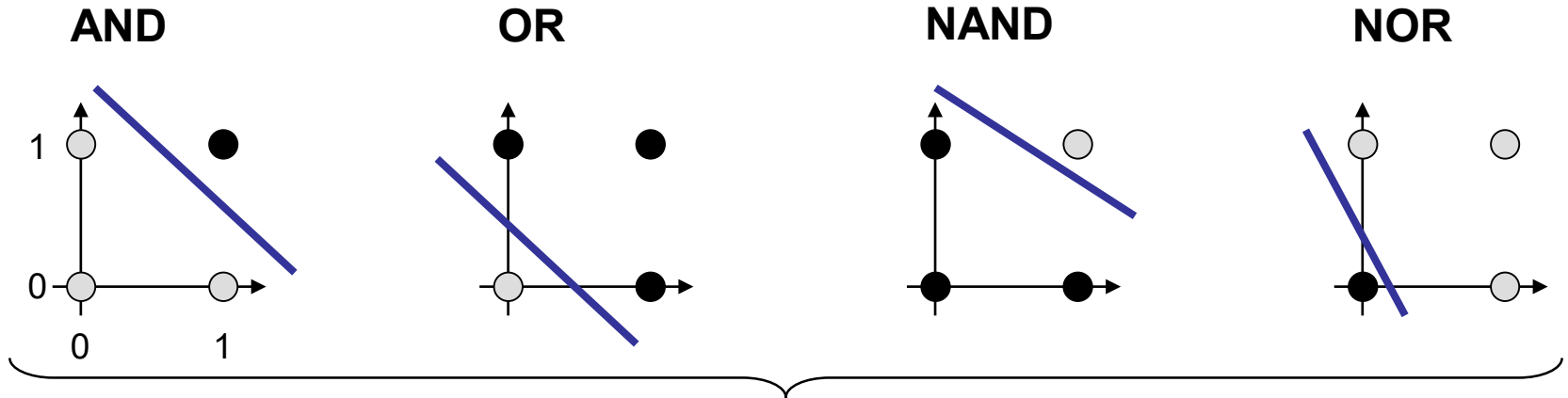


separating line

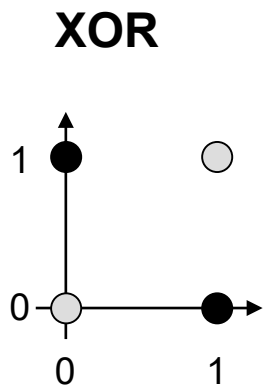
separates \mathbb{R}^2

in 2 classes

○ = 0 ● = 1



→ MPP at least as powerful as MCP neuron!



?

x_1	x_2	xor
0	0	0
0	1	1
1	0	1
1	1	0

$\Rightarrow 0 < \theta$

$\Rightarrow w_2 \geq \theta$

$\Rightarrow w_1 \geq \theta$

$\Rightarrow w_1 + w_2 < \theta$

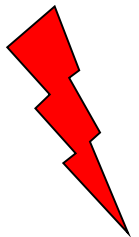
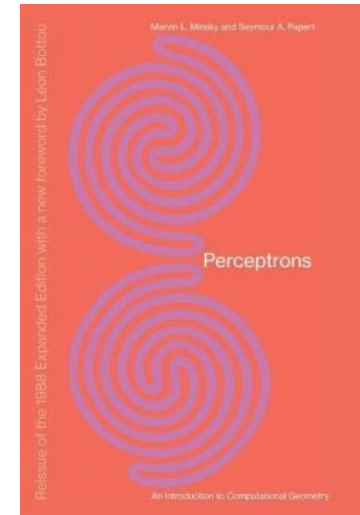
$w_1, w_2 \geq \theta > 0$
 $\Rightarrow w_1 + w_2 \geq 2\theta$

contradiction!

$w_1 x_1 + w_2 x_2 \geq \theta$

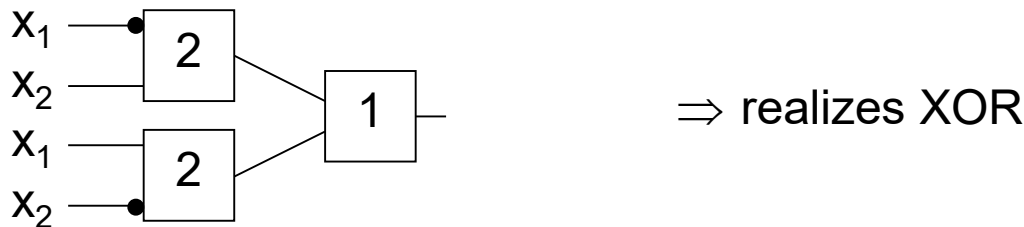
1969: Marvin Minsky / Seymour Papert

- book *Perceptrons* → analysis math. properties of perceptrons
- disillusioning result:
perceptions fail to solve a number of trivial problems!
 - XOR Problem
 - Parity Problem
 - Connectivity Problem
- “conclusion“: all artificial neurons have this kind of weakness!
⇒ research in this field is a scientific dead end!
- consequence: research funding for ANN cut down extremely (~ 15 years)



how to leave the „dead end“:

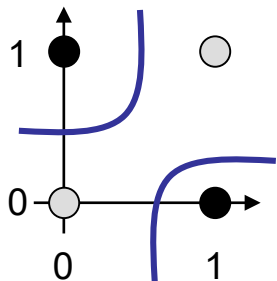
1. Multilayer Perceptrons:



2. Nonlinear separating functions:

XOR

$$g(x_1, x_2) = 2x_1 + 2x_2 - 4x_1x_2 - 1 \quad \text{with} \quad \theta = 0$$



$$g(0,0) = -1$$

$$g(0,1) = +1$$

$$g(1,0) = +1$$

$$g(1,1) = -1$$

How to obtain weights w_i and threshold θ ?

as yet: by construction

example: NAND-gate

x_1	x_2	NAND
0	0	1
0	1	1
1	0	1
1	1	0

$$\Rightarrow 0 \geq \theta$$

$$\Rightarrow w_2 \geq \theta$$

$$\Rightarrow w_1 \geq \theta$$

$$\Rightarrow w_1 + w_2 < \theta$$

requires solution of a system of linear inequalities ($\in P$)

(e.g.: $w_1 = w_2 = -2, \theta = -3$)

now: by „learning“ / training

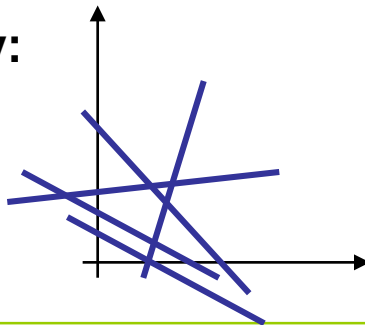
Perceptron Learning

Assumption: test examples with correct I/O behavior available

Principle:

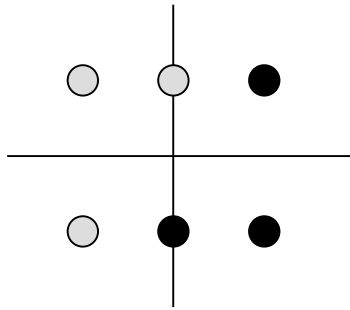
- (1) choose initial weights in arbitrary manner
- (2) feed in test pattern
- (3) if output of perceptron wrong, then change weights
- (4) goto (2) until correct output for all test patterns

graphically:



→ translation and rotation of separating lines

Example



$$P = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \quad \bullet$$

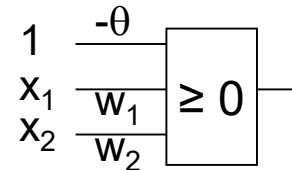
$$N = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \circ$$

threshold as a weight: $w = (\theta, w_1, w_2)'$



$$P = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$N = \left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$



$$w_1x_1 + w_2x_2 \geq \theta \Leftrightarrow w_1x_1 + w_2x_2 - \underbrace{\theta}_{w_0} \cdot \underbrace{1}_{x_0} \geq 0$$

⇒ separating hyperplane:

$$H(w) = \{ x : h(x;w) = 0 \}$$

where

$$h(x;w) = w'x = w_0x_0 + w_1x_1 + \dots + w_nx_n$$

⇒ origin $0 \in H(w)$ since $h(0;w) = 0$

Perceptron Learning

P: set of positive examples

→ output 1

N: set of negative examples

→ output 0

threshold θ integrated in weights

1. choose w_0 at random, $t = 0$
2. choose arbitrary $x \in P \cup N$
3. if $x \in P$ and $w_t'x > 0$ then **goto 2**
if $x \in N$ and $w_t'x \leq 0$ then **goto 2**
4. if $x \in P$ and $w_t'x \leq 0$ then
 $w_{t+1} = w_t + x$; $t++$; **goto 2**
5. if $x \in N$ and $w_t'x > 0$ then
 $w_{t+1} = w_t - x$; $t++$; **goto 2**
6. stop? If I/O correct for all examples!

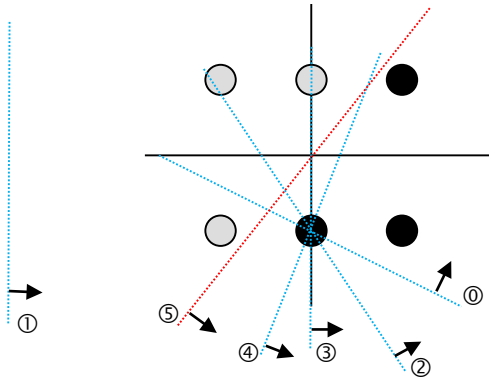
} I/O correct!

} let $w'x \leq 0$, should be > 0 !
 $(w+x)'x = w'x + x'x > w'x$

} let $w'x > 0$, should be ≤ 0 !
 $(w-x)'x = w'x - x'x < w'x$

remark: if separating $H(w^*)$ exists, then algorithm converges, is finite (but in worst case: exponential runtime)

Example



suppose initial vector of weights is

$$w^{(0)} = (1, \frac{1}{2}, 1)'$$

```
> w = SPL(m, c(1, 0.5, 1))
[1] 1.0 0.5 1.0
[1] 2.0 0.5 0.0
[1] 1.0 1.5 1.0
[1] 0.0 2.5 0.0
[1] -1.0 2.5 -1.0
[1] 0.0 2.5 -2.0
```

```
SPL <- function(m,w) {
  print(w)
  repeat {
    OK <- TRUE
    for (i in 1:nrow(m)) {
      x <- m[i,]
      s <- x[1]*w[1]+x[2]*w[2]+x[3]*w[3]
      if (s <= 0) {
        OK <- FALSE
        w <- w + x
        print(w) # show every change
      }
    }
    if (OK) break;
  }
  return(w)
}
```

```
m <- matrix( # only positive examples
  c(c( 1,1,1),c( 1,1,-1),c( 1,0,-1),
    c(-1,1,1),c(-1,1,-1),c(-1,0,-1)),
  nrow=6,byrow=TRUE)
```

Acceleration of Perceptron Learning

Assumption: $x \in \{0, 1\}^n \Rightarrow \|x\| = \sum_{i=1}^n |x_i| \geq 1$ for all $x \neq (0, \dots, 0)$

Let $B = P \cup \{-x : x \in N\}$ (only positive examples)

If classification incorrect, then $w'x < 0$. 

Consequently, size of error is just $\delta = -w'x > 0$.

$\Rightarrow w_{t+1} = w_t + (\delta + \varepsilon) x$ for $\varepsilon > 0$ (small) corrects error in a single step, since

$$\begin{aligned}
 w'_{t+1}x &= (w_t + (\delta + \varepsilon) x)' x \\
 &= \underbrace{w'_t x}_{- \delta} + (\delta + \varepsilon) x'x \\
 &= -\delta + \delta \|x\|^2 + \varepsilon \|x\|^2 \\
 &= \underbrace{\delta (\|x\|^2 - 1)}_{\geq 0} + \underbrace{\varepsilon \|x\|^2}_{> 0} > 0 \quad \checkmark
 \end{aligned}$$

Generalization:

Assumption: $x \in \mathbb{R}^n \Rightarrow \|x\| > 0$ for all $x \neq (0, \dots, 0)$

as before: $w_{t+1} = w_t + (\delta + \varepsilon) x$ for $\varepsilon > 0$ (small) and $\delta = -w'_t x > 0$

$$\Rightarrow w'_{t+1} x = \underbrace{\delta (\|x\|^2 - 1)}_{< 0 \text{ possible!}} + \underbrace{\varepsilon \|x\|^2}_{> 0}$$

Claim: Scaling of data does not alter classification task (if threshold 0)!

Let $\ell = \min \{ \|x\| : x \in B \} > 0$

Set $\hat{x} = \frac{x}{\ell} \Rightarrow$ set of scaled examples \hat{B}

$$\Rightarrow \|\hat{x}\| \geq 1 \Rightarrow \|\hat{x}\|^2 - 1 \geq 0 \Rightarrow w'_{t+1} \hat{x} > 0 \quad \checkmark$$

Theorem:

Let $X = P \cup N$ with $P \cap N = \emptyset$ be training patterns (P: positive; N: negative examples). Suppose training patterns are embedded in \mathbb{R}^{n+1} with threshold 0 and origin $0 \notin X$.

If separating hyperplane $H(w)$ exists,
then scaling of data does not alter classification task!

Proof:

Suppose $\exists x \in P \cup N$ with $\|x\| < 1$ and let $\ell = \min\{\|x\| : x \in P \cup N\} > 0$.

Set $\hat{x} = \frac{1}{\ell} x$ so that $\hat{P} = \{\frac{x}{\ell} : x \in P\}$ and $\hat{N} = \{\frac{x}{\ell} : x \in N\}$.

Suppose $\exists w$ with $\forall \hat{x} \in \hat{P} : w' \hat{x} > 0$ and $\forall \hat{x} \in \hat{N} : w' \hat{x} \leq 0$.

Then holds:

$$w' \hat{x} > 0 \Leftrightarrow w' \frac{x}{\ell} > 0 \Leftrightarrow w' x > 0$$

$$w' \hat{x} \leq 0 \Leftrightarrow w' \frac{x}{\ell} \leq 0 \Leftrightarrow w' x \leq 0$$

q.e.d.

There exist numerous variants of Perceptron Learning Methods.

Theorem: (Duda & Hart 1973)

If rule for correcting weights is $w_{t+1} = w_t + \gamma_t x$ (i.e., if $w'_t x < 0$) and

1. $\forall t \geq 0 : \gamma_t \geq 0$

2. $\sum_{t=0}^{\infty} \gamma_t = \infty$

3. $\lim_{m \rightarrow \infty} \frac{\sum_{t=0}^m \gamma_t^2}{\left(\sum_{t=0}^m \gamma_t \right)^2} = 0$

then $w_t \rightarrow w^*$ for $t \rightarrow \infty$ with $\forall x: x'w^* > 0$. ■

e.g.: $\gamma_t = \gamma > 0$ or $\gamma_t = \gamma / (t+1)$ for $\gamma > 0$

as yet: *Online Learning*

→ Update of weights after each training pattern (if necessary)

now: *Batch Learning*

→ Update of weights only after test of all training patterns

→ Update rule:

$$w_{t+1} = w_t + \gamma \sum_{\substack{w_t^i x < 0 \\ x \in B}} x \quad (\gamma > 0)$$

vague assessment in literature:

- advantage : „usually faster“
- disadvantage : „needs more memory“ ← just a single vector!

find weights by means of optimization

Let $F(w) = \{ x \in B : w'x < 0 \}$ be the set of patterns incorrectly classified by weight w .

Objective function:
$$f(w) = - \sum_{x \in F(w)} w'x \rightarrow \min!$$

Optimum: $f(w) = 0$ iff $F(w)$ is empty

Possible approach: gradient method

$$w_{t+1} = w_t - \gamma \nabla f(w_t) \quad (\gamma > 0)$$

converges to a local minimum (dep. on w_0)

Gradient method

$$w_{t+1} = w_t - \gamma \nabla f(w_t)$$

negative gradient points in direction
of steepest descent of function $f(\cdot)$

Gradient $\nabla f(w) = \left(\frac{\partial f(w)}{\partial w_1}, \frac{\partial f(w)}{\partial w_2}, \dots, \frac{\partial f(w)}{\partial w_n} \right)$

Caution:

Indices i of w_i
here denote
components of
vector w ; they are
not the iteration
counters!

$$\begin{aligned} \frac{\partial f(w)}{\partial w_i} &= -\frac{\partial}{\partial w_i} \sum_{x \in F(w)} w'x = -\frac{\partial}{\partial w_i} \sum_{x \in F(w)} \sum_{j=1}^n w_j \cdot x_j \\ &= - \sum_{x \in F(w)} \underbrace{\frac{\partial}{\partial w_i} \left(\sum_{j=1}^n w_j \cdot x_j \right)}_{x_i} = - \sum_{x \in F(w)} x_i \end{aligned}$$

Gradient method

thus:

$$\begin{aligned}
 \text{gradient } \nabla f(w) &= \left(\frac{\partial f(w)}{\partial w_1}, \frac{\partial f(w)}{\partial w_2}, \dots, \frac{\partial f(w)}{\partial w_n} \right)' \\
 &= \left(-\sum_{x \in F(w)} x_1, -\sum_{x \in F(w)} x_2, \dots, -\sum_{x \in F(w)} x_n \right)' \\
 &= -\sum_{x \in F(w)} x
 \end{aligned}$$

$$\Rightarrow w_{t+1} = w_t + \gamma \sum_{x \in F(w_t)} x$$

gradient method \Leftrightarrow batch learning

How difficult is it

(a) to find a separating hyperplane, provided it exists?

(b) to decide, that there is no separating hyperplane?

Let $B = P \cup \{-x : x \in N\}$ (only positive examples), $w_i \in \mathbb{R}$, $\theta \in \mathbb{R}$, $|B| = m$

For every example $x_i \in B$ should hold:

$x_{i1} w_1 + x_{i2} w_2 + \dots + x_{in} w_n \geq \theta$ → trivial solution $w_i = \theta = 0$ to be excluded!

Therefore additionally: $\eta \in \mathbb{R}$

$x_{i1} w_1 + x_{i2} w_2 + \dots + x_{in} w_n - \theta - \eta \geq 0$

Idea: maximize η s.t. constraints → if $\eta^* > 0$, then solution found

Matrix notation:

$$A = \begin{pmatrix} x'_1 & -1 & -1 \\ x'_2 & -1 & -1 \\ \vdots & \vdots & \vdots \\ x'_m & -1 & -1 \end{pmatrix} \quad z = \begin{pmatrix} w \\ \theta \\ \eta \end{pmatrix}$$

Linear Programming Problem:

$$f(z_1, z_2, \dots, z_n, z_{n+1}, z_{n+2}) = z_{n+2} \rightarrow \max!$$

$$\text{s.t. } Az \geq 0$$

solved by e.g. Kamarkar
algorithm in **polynomial time**

If $z_{n+2} = \eta > 0$, then weights and threshold are given by z .

Otherwise separating hyperplane does not exist!