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Computational Intelligence

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Plan for Today

- Approximate Reasoning
- Fuzzy Control

So far:

- p: IF *X* is A THEN *Y* is B

  → R(x, y) = Imp(A(x), B(y))

rule as relation; fuzzy implication

- rule: IF *X* is A THEN *Y* is B

  fact: *X* is A'

  conclusion: *Y* is B'

  → $B'(y) = \sup_{x \in X} t(\, A'(x), R(x, y) \,)$

composition rule of inference

Thus:

- $B'(y) = \sup_{x \in X} t(\, A'(x), \mathrm{Imp}(\, A(x), B(y) \,) \,)$

given : fuzzy rule

input : fuzzy set A'

output : fuzzy set B'

special case:

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

$$B'(y) = \sup_{x \in X} t(\, A'(x), \mathrm{Imp}(\, A(x), B(y)\,)\,)$$

$$= \begin{cases} \sup\limits_{x \neq x_0} t(\, 0, \mathrm{Imp}(\, A(x), B(y)\,)\,) & \text{for } x \neq x_0 \\ t(\, 1, \mathrm{Imp}(\, A(x_0), B(y)\,)\,) & \text{for } x = x_0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } x \neq x_0 & \text{since } t(0, a) = 0 \\ \mathrm{Imp}(\, A(x_0), B(y)\,) & \text{for } x = x_0 & \text{since } t(a, 1) = a \end{cases}$$ [A1]

Lemma:

a) t(a, 1) = a

b) t(a, b) ≤ min { a, b }

c) t(0, a) = 0

Proof:

by a)

ad a) Identical to axiom 1 of t-norms.

ad b) From monotonicity (axiom 2) follows for b ≤ 1, that t(a, b) ≤ t(a, 1) = a.
Commutativity (axiom 3) and monotonicity lead in case of a ≤ 1 to
t(a, b) = t(b, a) ≤ t(b, 1) = b. Thus, t(a, b) is less than or
equal to a as well as b, which in turn implies t(a, b) ≤ min{ a, b }.

ad c) From b) follows 0 ≤ t(0, a) ≤ min { 0, a } = 0 and therefore t(0, a) = 0. ■

Multiple rules:

IF X is A_1, THEN Y is B_1 → $R_1(x, y) = \text{Imp}_1(\, A_1(x), B_1(y) \,)$
IF X is A_2, THEN Y is B_2 → $R_2(x, y) = \text{Imp}_2(\, A_2(x), B_2(y) \,)$
IF X is A_3, THEN Y is B_3 → $R_3(x, y) = \text{Imp}_3(\, A_3(x), B_3(y) \,)$
… …
IF X is A_n, THEN Y is B_n → $R_n(x, y) = \text{Imp}_n(\, A_n(x), B_n(y) \,)$
X is A'

Y is B'

Multiple rules for <u>fuzzy input</u>: A'(x) is given

$$\left.\begin{array}{l} B_1'(y) = \sup_{x \in X} t(\, A'(x), R_1(x, y) \,) \\ \ldots \\ B_n'(y) = \sup_{x \in X} t(\, A'(x), R_n(x, y) \,) \end{array}\right\}$$

aggregation of rules or local inferences necessary!

aggregate! ⇒ $B'(y) = \mathbf{aggr}\{\, B_1'(y), \ldots, B_n'(y) \,\}$, where $\mathbf{aggr} = \begin{cases} \min \\ \max \end{cases}$

FITA: "**F**irst **i**nference, **t**hen **a**ggregate!"

1. Each rule of the form IF *X* is A_k THEN *Y* is B_k must be transformed by an appropriate fuzzy implication $Imp_k(\cdot,\cdot)$ to a relation R_k :
$R_k(x, y) = Imp_k(A_k(x), B_k(y))$.
2. Determine $B_k'(y) = R_k(x, y) \circ A'(x)$ for all k = 1, …, n (local inference).
3. Aggregate to $B'(y) = \beta(B_1'(y), \ldots, B_n'(y))$.

FATI: "**F**irst **a**ggregate, **t**hen **i**nference!"

1. Each rule of the form IF *X* ist A_k THEN *Y* ist B_k must be transformed by an appropriate fuzzy implication $Imp_k(\cdot, \cdot)$ to a relation R_k :
$R_k(x, y) = Imp_k(A_k(x), B_k(y))$.
2. Aggregate $R_1, \ldots, R_n$ to a **superrelation** with aggregating function $\alpha(\cdot)$**:**
$R(x, y) = \alpha(R_1(x, y), \ldots, R_n(x, y))$.
3. Determine $B'(y) = R(x, y) \circ A'(x)$ w.r.t. superrelation (inference).

1. Which principle is better? FITA or FATI?

2. Equivalence of FITA and FATI ?

FITA:

$$
\begin{aligned}
B'(y) &= \beta(\, B_1'(y), \ldots, B_n'(y) \,) \\
&= \beta(\, R_1(x, y) \circ A'(x), \ldots, R_n(x, y) \circ A'(x) \,)
\end{aligned}
$$

FATI:

$$
\begin{aligned}
B'(y) &= R(x, y) \circ A'(x) \\
&= \alpha(\, R_1(x, y), \ldots, R_n(x, y) \,) \circ A'(x)
\end{aligned}
$$

→ general case: no further analysis without simplifying assumptions …

special case:

$$A'(x) = \begin{cases} 1 & \text{for } x = x_0 \\ 0 & \text{otherwise} \end{cases}$$

crisp input!

On the equivalence of FITA and FATI:

FITA: $B'(y) = \beta(\, B_1'(y), \ldots, B_n'(y)\,)$

$= \beta(\, Imp_1(A_1(x_0), B_1(y)\,), \ldots, Imp_n(A_n(x_0), B_n(y)\,)\,)$

FATI: $B'(y) = R(x, y) \circ A'(x)$

$= \sup_{x \in X} t(\, A'(x), R(x, y)\,)$ (from now: special case)

$= R(x_0, y)$

$= \alpha(\, Imp_1(\, A_1(x_0), B_1(y)\,), \ldots, Imp_n(\, A_n(x_0), B_n(y)\,)\,)$

FATI = FITA if sup-t-composition with same t-norm, **$\alpha(\cdot) = \beta(\cdot)$,** same $Imp_i()$, and ...

- **AND-connected premises**

IF $X_1 = A_{11}$ AND $X_2 = A_{12}$ AND … AND $X_m = A_{1m}$ THEN $Y = B_1$
…
IF $X_n = A_{n1}$ AND $X_2 = A_{n2}$ AND … AND $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$A_k(x_1,\ldots, x_m) = \min \{ A_{k1}(x_1), A_{k2}(x_2), \ldots, A_{km}(x_m) \}$ or in general: t-norm

- **OR-connected premises**

IF $X_1 = A_{11}$ OR $X_2 = A_{12}$ OR … OR $X_m = A_{1m}$ THEN $Y = B_1$
…
IF $X_n = A_{n1}$ OR $X_2 = A_{n2}$ OR … OR $X_m = A_{nm}$ THEN $Y = B_n$

reduce to single premise for each rule k:

$A_k(x_1,\ldots, x_m) = \max \{ A_{k1}(x_1), A_{k2}(x_2), \ldots, A_{km}(x_m) \}$ or in general: s-norm

important:

- if rules of the form **IF *X* is A THEN *Y* is B** interpreted as <u>logical</u> implication

  $\Rightarrow$ R(x, y) = Imp(A(x), B(y)) makes sense

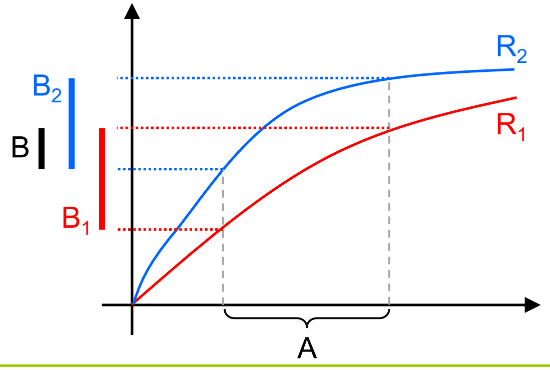
- we obtain: $B'(y) = \sup_{x \in X} t(\, A'(x), R(x, y) \,)$

interpretation of output set B'(y):

- B'(y) is the set of values that are possible under the particular rule
- each rule leads to a different restriction of the values that are possible
- must determine set of values that are possible for **all** rules

$\Rightarrow$ resulting fuzzy sets $B'_k(y)$ obtained from single rules must be mutually <u>intersected</u>!

$\Rightarrow$ aggregation via $B'(y) = \textbf{min} \{ B_1'(y), \ldots, B_n'(y) \}$



important:

- if rules of the form **IF *X* is A THEN *Y* is B** are <u>not</u> interpreted as <u>logical</u> implications, then the function Fct(•) in

$$R(x, y) = \text{Fct}(A(x), B(y))$$

can be chosen as required for desired interpretation.

- frequent choice (especially in fuzzy control):

  - $R(x, y) = \min \{ A(x), B(y) \}$ Mamdani – "implication"
  - $R(x, y) = A(x) \cdot B(y)$ Larsen – "implication"

⇒ of course, they are no implications but specific t-norms!

⇒ thus, if <u>relation R(x, y) is given</u>, then the *composition rule of inference*

$$\boxed{B'(y) = A'(x) \circ R(x, y) = \sup_{x \in X} \min \{ A'(x), R(x, y) \}}$$

still can lead to a conclusion via fuzzy logic.

example: [JM96, S. 244ff.]

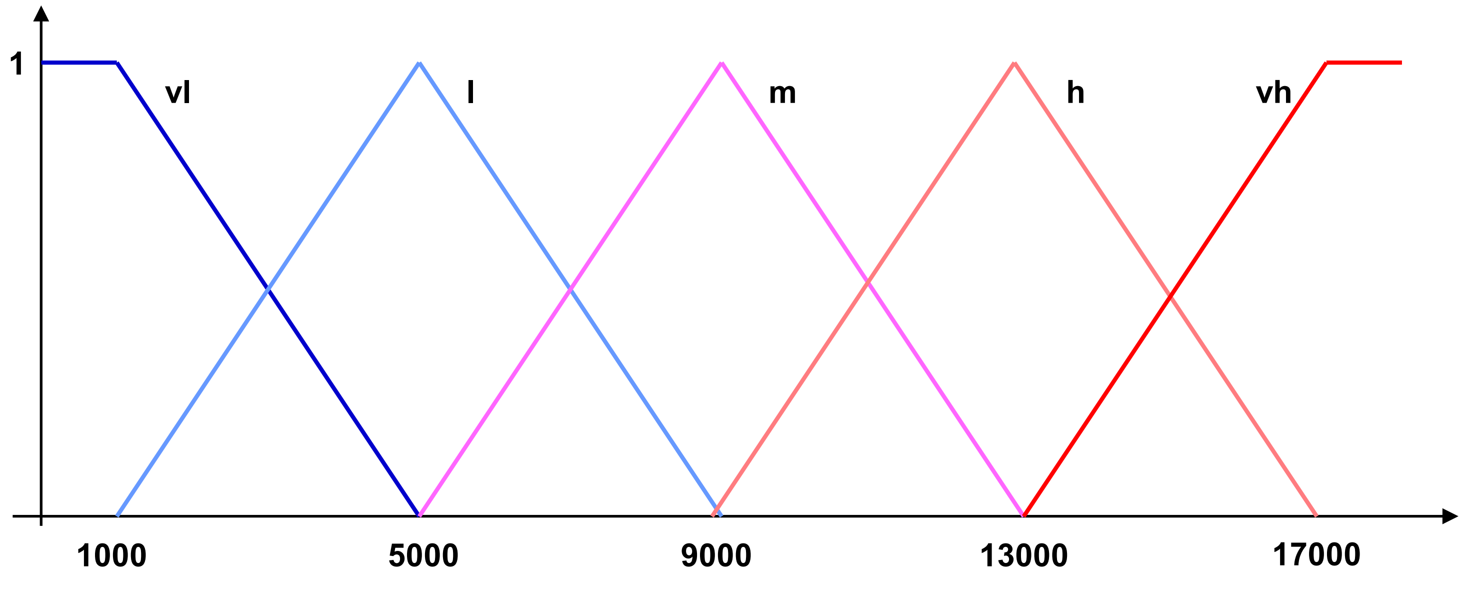
industrial drill machine → control of cooling supply

modelling

linguistic variable : **rotation speed**

linguistic terms : ***very low***, ***low***, ***medium***, ***high***, ***very high***

ground set : X with $0 \le x \le 18000$ [rpm]



example: (continued)

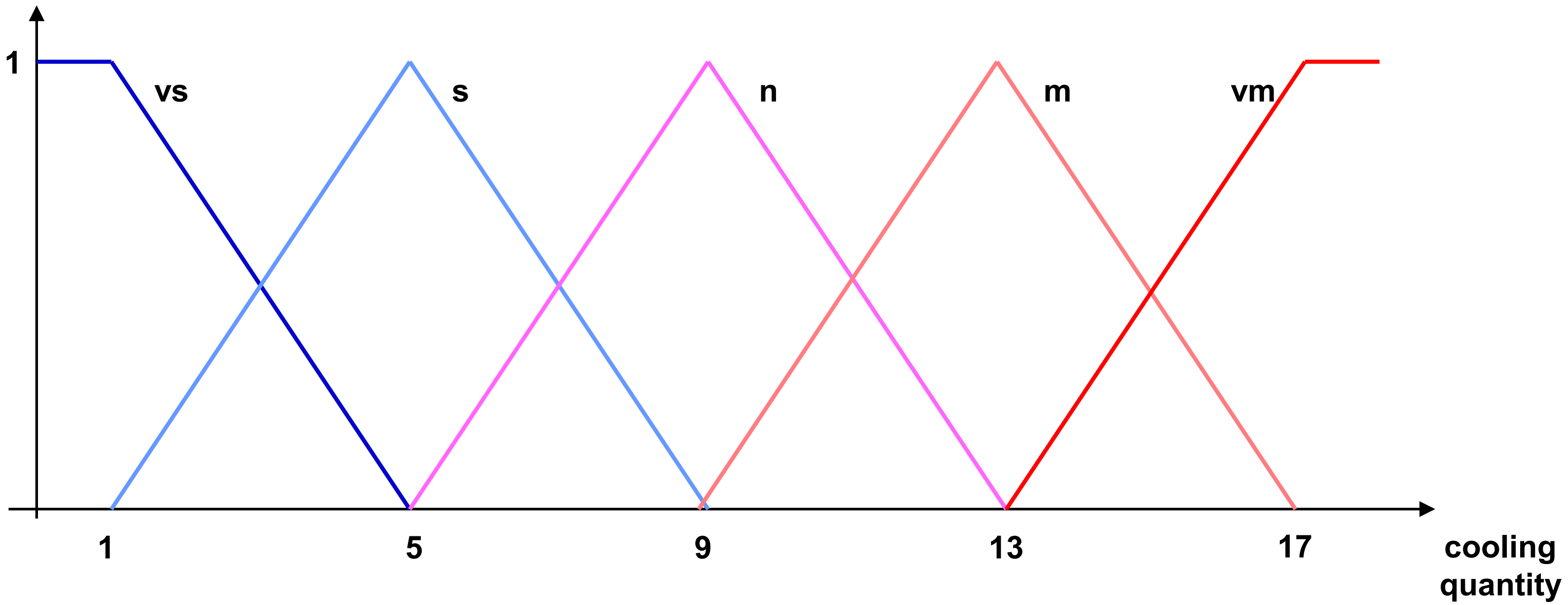
industrial drill machine → control of cooling supply

modelling

linguistic variable : **cooling quantity**

linguistic terms : ***very small***, ***small***, ***normal***, ***much***, ***very much***

ground set : Y with $0 \le y \le 18$ [liter / time unit]



<u>example:</u> (continued)

industrial drill machine → control of cooling supply

<u>rule base</u>

`IF` **rotation speed** `IS`	***very low***	`THEN` **cooling quantity** `IS`	***very small***
	low		***small***
	medium		***normal***
	high		***much***
	very high		***very much***

↑ sets S_{vl}, S_l, S_m, S_h, S_{vh}

"rotation <u>speed</u>"

↑ sets C_{vs}, C_s, C_n, C_m, C_{vm}

"<u>cooling</u> quantity"

<u>example:</u> (continued)

industrial drill machine → control of cooling supply

1. **input**: crisp value $x_0 = 10\,000 \text{ min}^{-1}$ (<u>not</u> a fuzzy set!)

   → **fuzzyfication** = determine membership for each fuzzy set over X

   → yields S' = (0, 0, ¾, ¼, 0) via $x \mapsto (\, S_{vl}(x_0), S_l(x_0), S_m(x_0), S_h(x_0), S_{vh}(x_0) \,)$

2. FITA: local **inference** ⇒ note: Imp(0,a) = 1 (axiom 3)

   S_{vl}: $C'_{vs}(y) = \text{Imp}(\, 0 \,,\, C_{vs}(y) \,)$

   S_l: $C'_s(y) = \text{Imp}(0, C_s(y) \,)$

   S_m: $C'_n(y) = \text{Imp}(¾, C_n(y) \,)$

   S_h: $C'_m(y) = \text{Imp}(¼ \,,\, C_m(y) \,)$

   S_{vh}: $C'_{vm}(y) = \text{Imp}(\, 0 \,,\, C_{vm}(y) \,)$

Must we replace logical Imp() by technical relation?

example: (continued)

industrial drill machine → control of cooling supply

in case of <u>control task</u> typically **no logic-based interpretation**:

→ max-aggregation and

→ relation R(x,y) not interpreted as implication.

often: R(x,y) = min(A(x), B(y))   „Mamdani controller“

2. FITA: local **inference**

$$S_{vl}: \quad C'_{vs}(y) = \min(0, C_{vs}(y)) = 0$$
$$S_{l}: \quad C'_{s}(y) = \min(0, C_{s}(y)) = 0$$
$$S_{m}: \quad C'_{n}(y) = \min(\tfrac{3}{4}, C_{n}(y)) \geq 0$$
$$S_{h}: \quad C'_{m}(y) = \min(\tfrac{1}{4}, C_{m}(y)) \geq 0$$
$$S_{vh}: \quad C'_{vm}(y) = \min(0, C_{vm}(y)) = 0$$

⇒ since min(0,a) = 0 and max-aggr. we only need to consider C_n and C_m

example: (continued)

industrial drill machine → control of cooling supply

3. aggregation:

$C'(y) = \texttt{aggr}\{ C'_n(y), C'_m(y) \} = \max\{ \min(\frac{3}{4}, C_n(y)), \min(\frac{1}{4}, C_m(y)) \}$

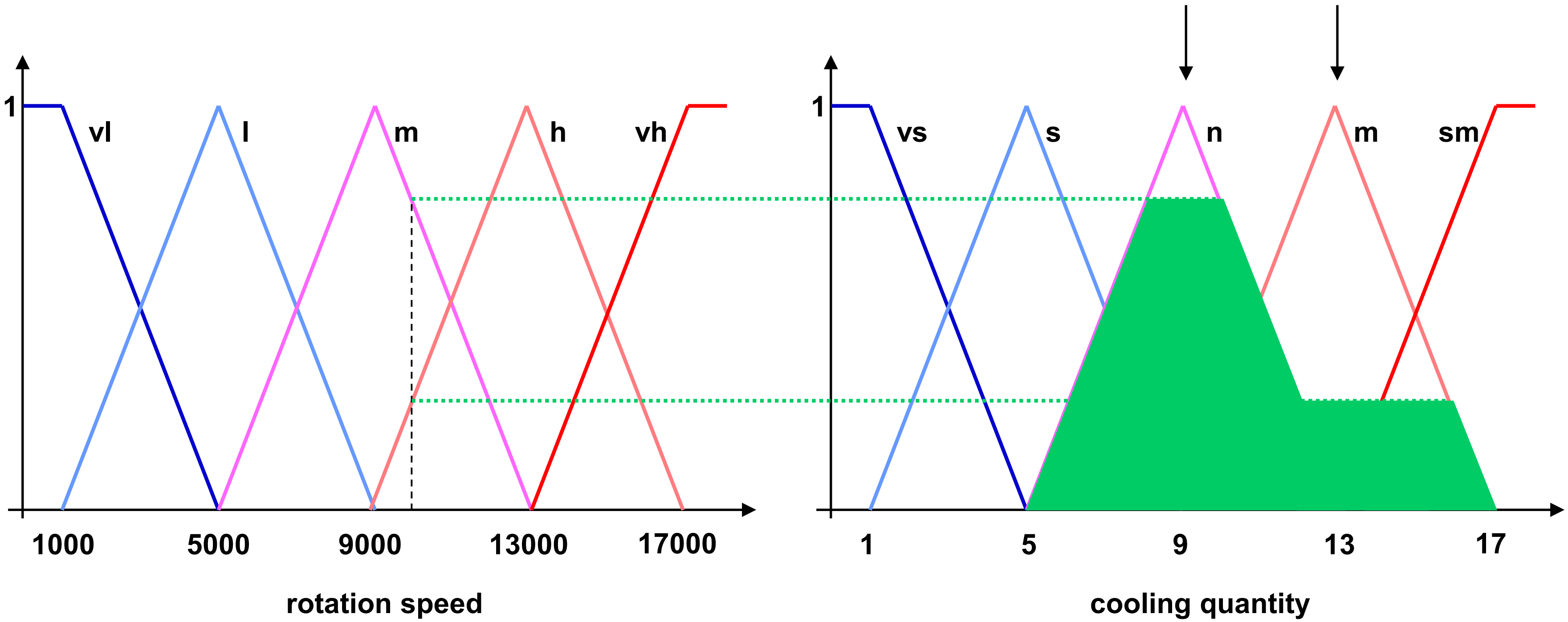
Remark:
This approach can be applied with every t-norm and max-aggregation
$\Rightarrow C'(y) = \max\{ t(\frac{3}{4}, C_n(y)), t(\frac{1}{4}, C_m(y)) \}$

→ graphical illustration

example: (continued)

industrial drill machine → control of cooling supply

$C'(y) = \max \{ \min \{ \frac{3}{4}, C_n(y) \}, \min \{ \frac{1}{4}, C_m(y) \} \}, x_0 = 10\,000$ [rpm]



open and closed loop control:

affect the dynamical behavior of a system
in a desired manner

- **open loop control**

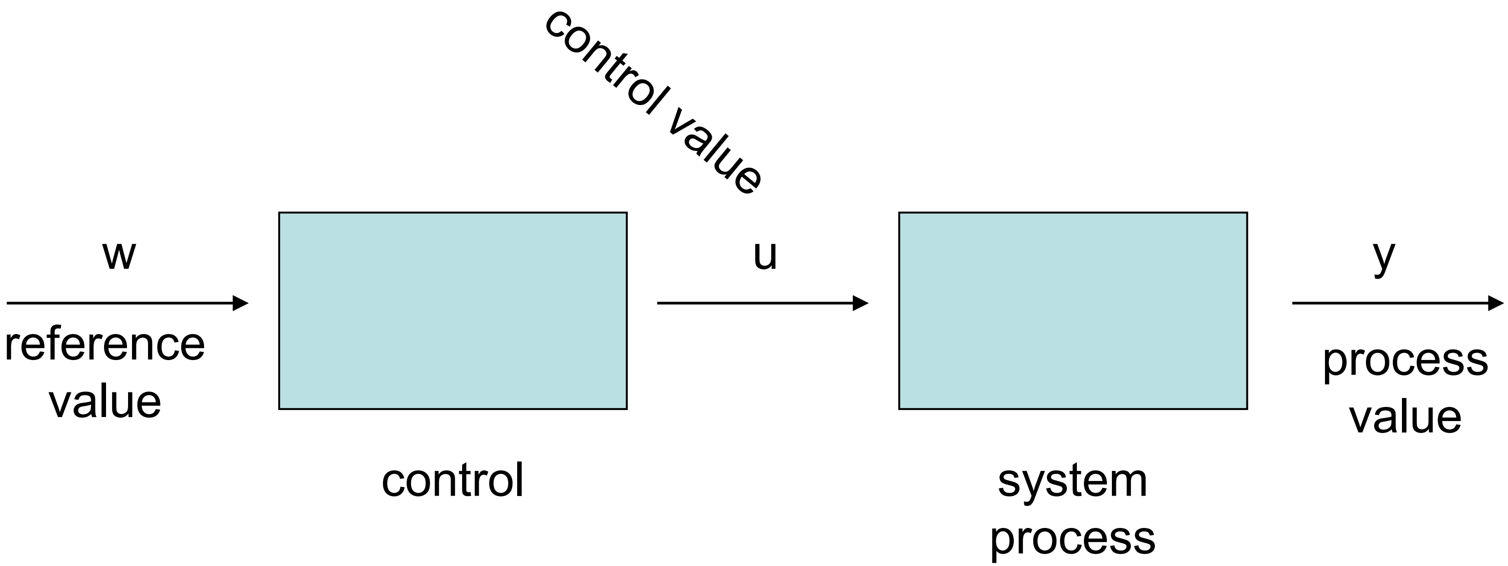
  control is aware of reference values and has a model of the system
  $\Rightarrow$ control values can be adjusted,
  such that process value of system is equal to reference value

  problem: noise! $\Rightarrow$ deviation from reference value not detected

- **closed loop control**

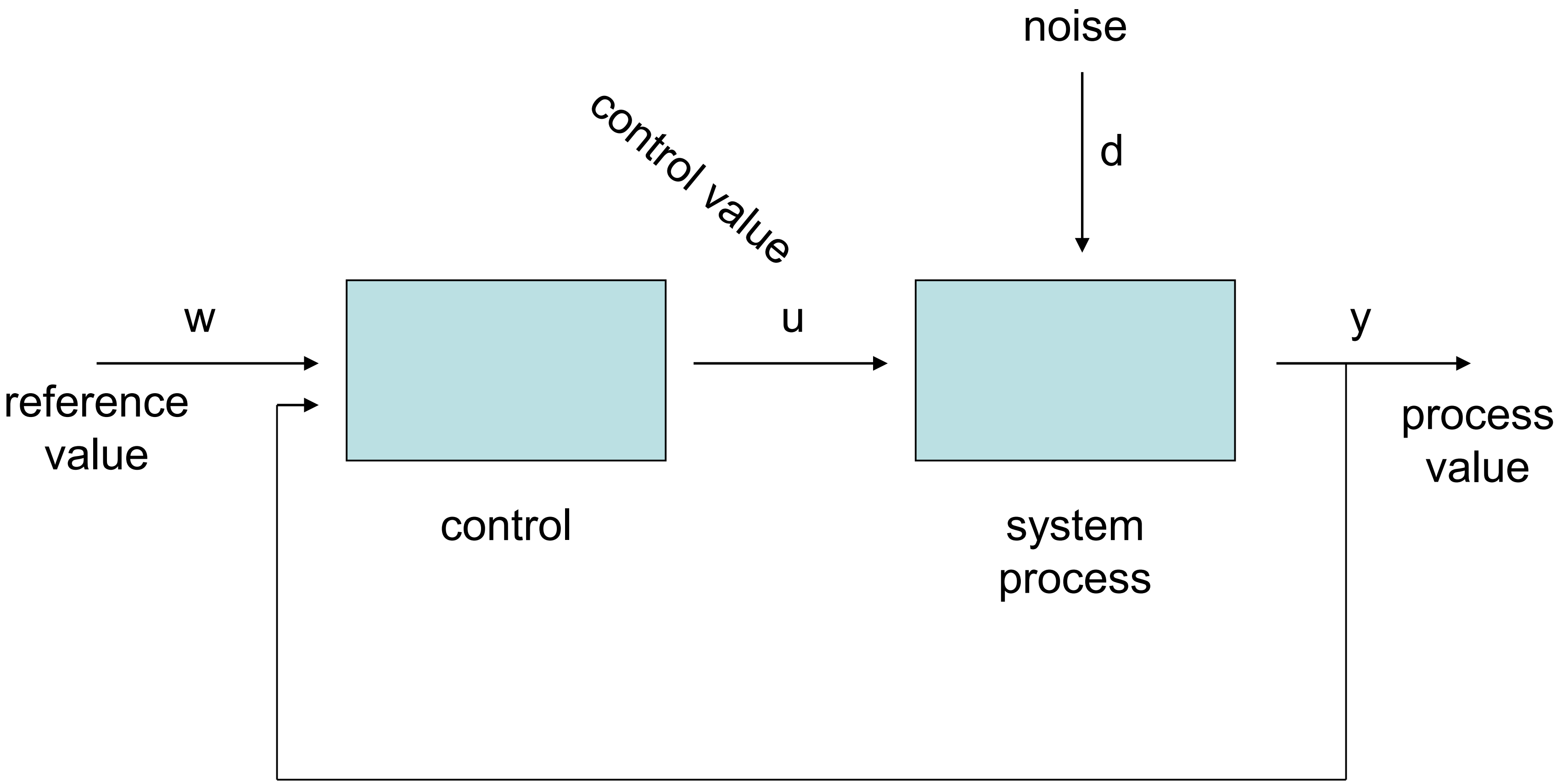
  now: detection of deviations from reference value possible
  (by means of measurements / sensors)
  and new control values can take into account the amount of deviation

open loop control



assumption: undisturbed operation $\Rightarrow$ process value = reference value

closed loop control



control deviation = reference value – process value

required:

model of system / process

→ as differential equations or difference equations (DEs)

→ well developed theory available

so, why fuzzy control?

- if there exists no process model in form of DEs etc.
  (operator/human being has realized control by hand)
- if process with high-dimensional nonlinearities → no classic methods available
- if control goals are vaguely formulated („soft“ changing gears in cars)

fuzzy description of control behavior

IF X is A_1, THEN Y is B_1
IF X is A_2, THEN Y is B_2
IF X is A_3, THEN Y is B_3
…
IF X is A_n, THEN Y is B_n
X is A‘

Y is B‘

similar to approximative reasoning

but fact A‘ is not a fuzzy set but a crisp input

→ actually, it is the current process value

fuzzy controller executes inference step

→ yields fuzzy output set B‘(y)

but crisp control value required for the process / system

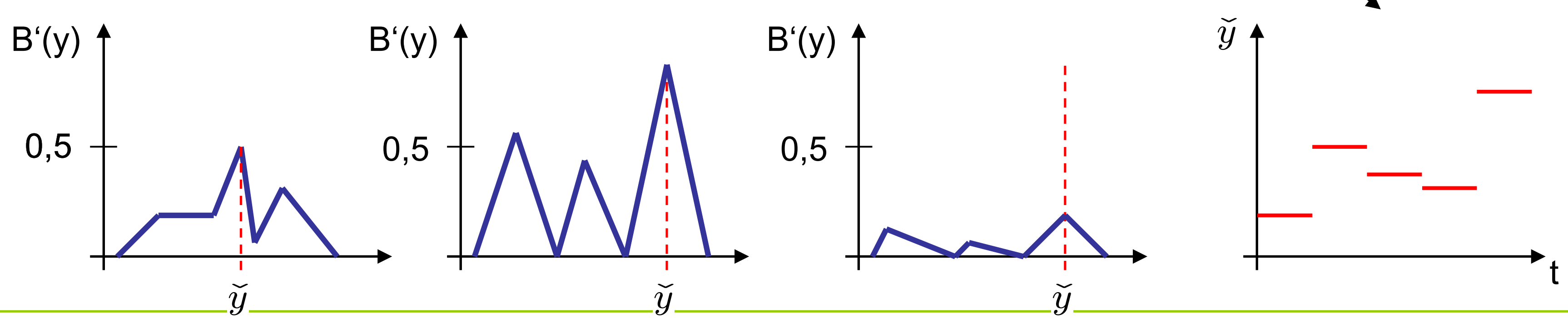
→ defuzzification (= “condense” fuzzy set to crisp value)

defuzzification

Def: rule k active ⇔ $A_k(x_0) > 0$

- <u>maximum method</u>
  - only active rule with largest activation level is taken into account
    - → suitable for pattern recognition / classification
    - → decision for a single alternative among finitely many alternatives
  - selection independent from activation level of rule (0.05 vs. 0.95)
  - if used for control: discontinuous curve of output values (leaps)

$$\boxed{\check{y} = \text{argmax } B'(y)}$$



defuzzification

$Y^* = \{ y \in Y: B'(y) = hgt(B') \}$

- <u>maximum mean value method</u>

  - all active rules with largest activation level are taken into account

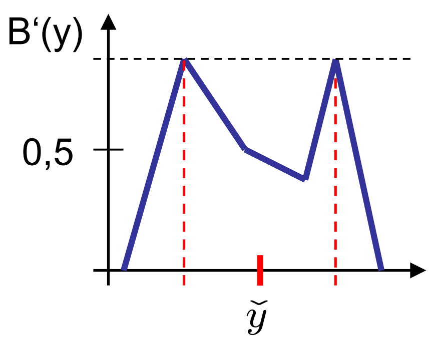
    → interpolations possible, but need not be useful

    → obviously, only useful for neighboring rules with max. activation

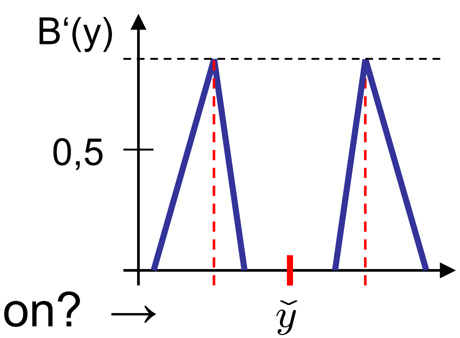
  - selection independent from activation level of rule (0.05 vs. 0.95)

  - if used in control: incontinuous curve of output values (leaps)

$$\boxed{\breve{y} = \frac{1}{|Y^*|} \sum_{y^* \in Y^*} y^*}$$



useful solution? →

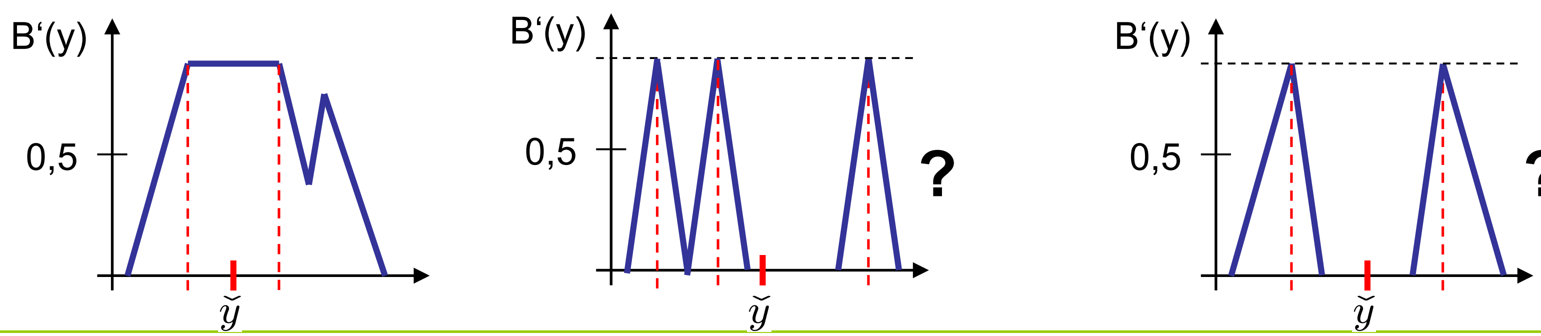


defuzzification

$Y^* = \{ y \in Y: B'(y) = hgt(B') \}$

- center-of-maxima method (COM)
  - only **extreme** active rules with largest activation level are taken into account
    - → interpolations possible, but need not be useful
    - → obviously, only useful for neighboring rules with max. activation level
  - selection indepependent from activation level of rule (0.05 vs. 0.95)
  - in case of control: incontinuous curve of output values (leaps)

$$\boxed{\check{y} = \frac{\inf Y^* + \sup Y^*}{2}}$$



defuzzification

- Center of Gravity (COG)

    - all active rules are taken into account

        → but numerically expensive … …only valid for HW solution, today!

        → borders cannot appear in output (∃ work-around)

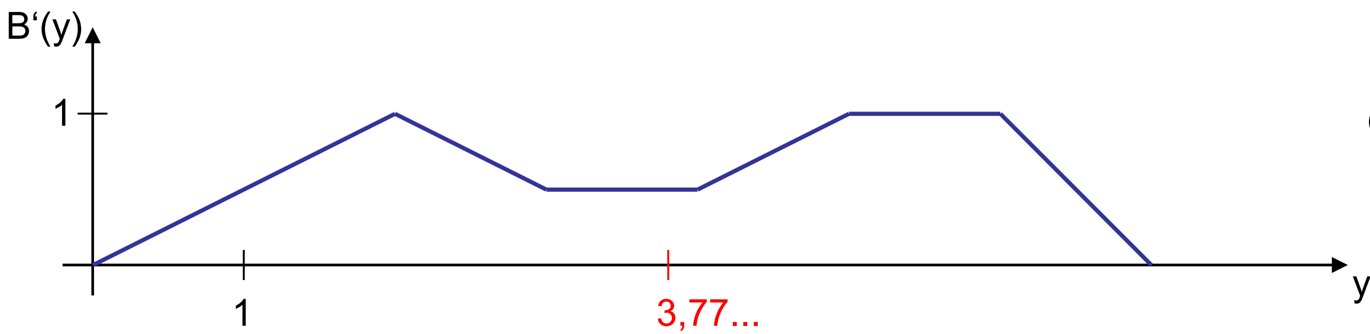
    - if only single active rule: independent from activation level

    - continuous curve for output values

$$\breve{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$

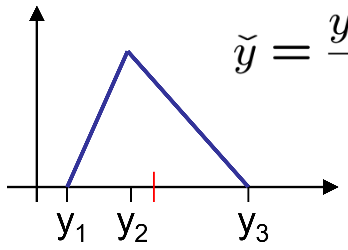
Excursion: COG

$$\breve{y} = \frac{\int y \cdot B'(y)\, dy}{\int B'(y)\, dy}$$



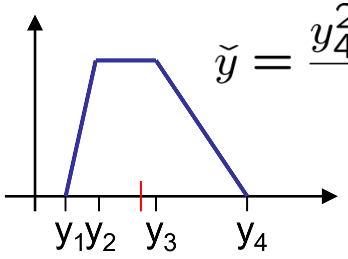
pendant in probability theory: expectation value

triangle:

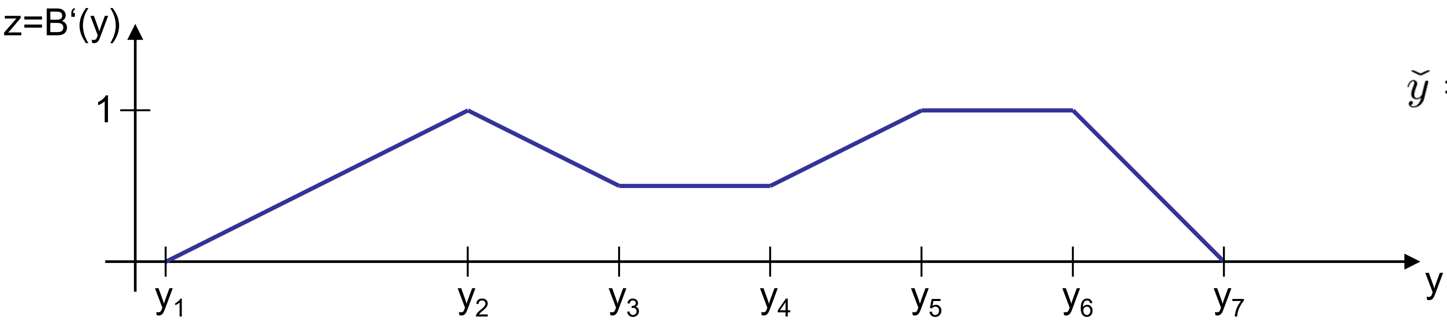


$$\breve{y} = \frac{y_1 + y_2 + y_3}{3}$$

trapezoid:



$$\breve{y} = \frac{y_4^2 + y_3^2 - y_2^2 - y_1^2 + y_3 y_4 - y_1 y_2}{3\,(y_4 + y_3 - y_2 - y_1)}$$



$$\breve{y} = \frac{\int y \cdot B'(y)\,dy}{\int B'(y)\,dy}$$

<u>assumption:</u> fuzzy membership functions piecewise linear

output set B‘(y) represented by sequence of points $(y_1, z_1), (y_2, z_2), \ldots, (y_n, z_n)$

⇒ area under B‘(y) and weighted area can be determined additively piece by piece

⇒ linear equation z = m y + b → insert (y_i, z_i) and (y_{i+1}, z_{i+1})

⇒ yields m and b for each of the n-1 linear sections

$$\Rightarrow F_i = \int_{y_i}^{y_{i+1}} (m\,y + b)\,dy = \frac{m}{2}(y_{i+1}^2 - y_i^2) + b(y_{i+1} - y_i)$$

$$\Rightarrow G_i = \int_{y_i}^{y_{i+1}} y\,(m\,y + b)\,dy = \frac{m}{3}(y_{i+1}^3 - y_i^3) + \frac{b}{2}(y_{i+1}^2 - y_i^2)$$

$$\breve{y} = \frac{\sum_i G_i}{\sum_i F_i}$$

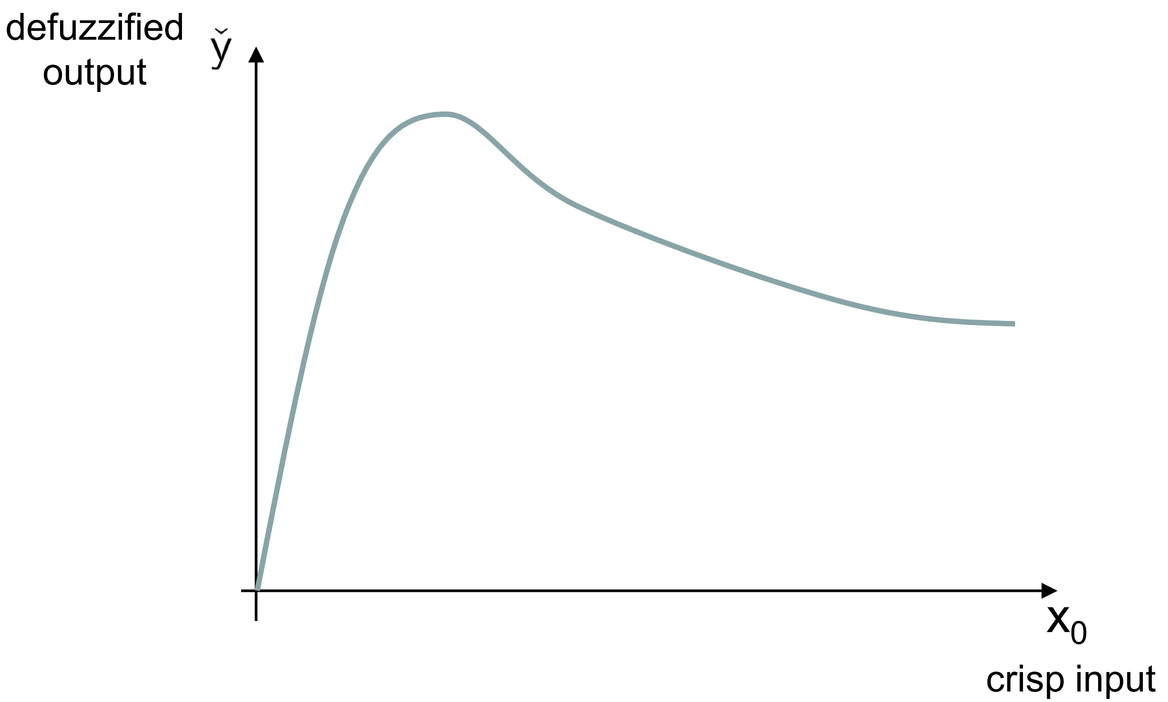
Defuzzification

- Center of Area (COA)
  - developed as an approximation of COG
  - let $\hat{y}_k$ be the COGs of the output sets $B'_k(y)$:

$$\check{y} = \frac{\sum_k A_k(x_0) \cdot \widehat{y}_k}{\sum_k A_k(x_0)}$$

how to:
assume that fuzzy sets $A_k(x)$ and $B_k(x)$ are triangles or trapezoids
let x_0 be the crisp input value
for each fuzzy rule "IF A_k is X THEN B_k is Y"
determine $B'_k(y) = R(A_k(x_0), B_k(y))$, where R(.,.) is the relation
find $\hat{y}_k$ as center of gravity of $B'_k(y)$

Putting all together:



→ map controller (german: Kennfeldregler)