technische universität dortmund

# Computational Intelligence

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## Plan for Today

Lecture 04

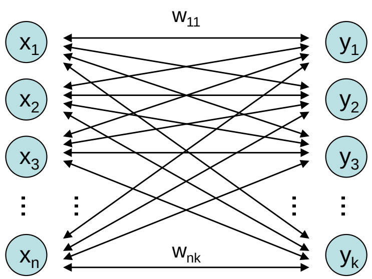
- Bidirectional Associative Memory (BAM)
  - Fixed Points
  - Concept of Energy Function
  - Stable States = Minimizers of Energy Function
- Hopfield Network
  - Convergence
  - Application to Combinatorial Optimization



## Bidirectional Associative Memory (BAM)

Lecture 04

**Network Model**




- fully connected
- bidirectional edges
- synchronized:

step t : data flow from x to y
step t + 1 : data flow from y to x

**start:** $y^{(0)} = \text{sgn}(x^{(0)} W)$

$x^{(1)} = \text{sgn}(y^{(0)} W')$

$y^{(1)} = \text{sgn}(x^{(1)} W)$

$x^{(2)} = \text{sgn}(y^{(1)} W')$

...

x, y : row vectors

W : weight matrix

W' : transpose of W

bipolar inputs $\in \{-1,+1\}$



## Bidirectional Associative Memory (BAM)

Lecture 04

**Fixed Points**

**Definition**

(x, y) is ***fixed point*** of BAM iff $y = \text{sgn}(x W)$ and $x' = \text{sgn}(W y')$. □

Set $W = x' y$. (note: x is row vector)

$y = \text{sgn}(x W) = \text{sgn}(x (x' y)) = \text{sgn}((x x') y) = \text{sgn}(\underbrace{\| x \|^2}_{> 0 \text{ (does not alter sign)}} y) = y$

$x' = \text{sgn}(W y') = \text{sgn}((x' y) y') = \text{sgn}(x' (y y')) = \text{sgn}(x' \underbrace{\| y \|^2}_{> 0 \text{ (does not alter sign)}}) = x'$

**Theorem:** If $W = x'y$ then (x,y) is fixed point of BAM. □

### Concept of Energy Function

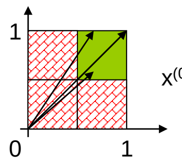
<u>given:</u> BAM with W = x'y $\Rightarrow$ (x,y) is stable state of BAM

starting point $x^{(0)}$ $\Rightarrow$ $y^{(0)}$ = sgn( $x^{(0)}$ W )

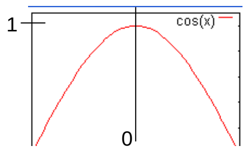
$\Rightarrow$ excitation e' = W $(y^{(0)})'$

$\Rightarrow$ if sign( e' ) = $x^{(0)}$ then ( $x^{(0)}$ , $y^{(0)}$ ) stable state

small angle between e' and $x^{(0)}$ $\Leftarrow$ true if e' *close* to $x^{(0)}$




$x^{(0)}$ = (1, 1)




**recall:** $\dfrac{ab'}{\|a\| \cdot \|b\|} = \cos \angle(a,b)$

small angle α $\Rightarrow$ large cos( α )





### Concept of Energy Function

<u>required:</u>

small angle between e' = W $y^{(0)}$ ' and $x^{(0)}$

$\Rightarrow$ larger cosine of angle indicates greater similarity of vectors

$\Rightarrow$ $\forall$e' of equal size: try to maximize $x^{(0)}$ e' = $\underbrace{\| x^{(0)} \|}_{\text{fixed}} \cdot \underbrace{\| e \|}_{\text{fixed}} \cdot \underbrace{\cos \angle (x^{(0)}, e)}_{\to \text{max!}}$

$\Rightarrow$ maximize $x^{(0)}$ e' = $x^{(0)}$ W $y^{(0)}$ '

$\Rightarrow$ identical to minimize $-x^{(0)}$ W $y^{(0)}$ '

**Definition**

Energy function of BAM at iteration t is E( $x^{(t)}$ , $y^{(t)}$ ) = $-\frac{1}{2}$ $x^{(t)}$ W $y^{(t)}$ ' □





### Stable States

**Theorem**

An asynchronous BAM with arbitrary weight matrix W reaches steady state in a finite number of updates.

***Proof:***

$$E(x,y) = -\frac{1}{2} x W y' = \begin{cases} -\frac{1}{2} x (W y') = -\frac{1}{2} x b' = -\frac{1}{2} \sum_{i=1}^{n} b_i \, x_i \\ -\frac{1}{2} (x W) y' = -\frac{1}{2} a y' = -\frac{1}{2} \sum_{i=1}^{k} a_i \, y_i \end{cases}$$

excitations ($b_i$, $a_i$)

BAM asynchronous $\Rightarrow$ select neuron at random from left or right layer, compute its excitation and change state if necessary (states of other neurons not affected)





neuron i of left layer has changed $\Rightarrow$ sgn($x_i$) ≠ sgn($b_i$)

$\Rightarrow$ $x_i$ was updated to $\tilde{x}_i = -x_i$

$$E(x,y) - E(\tilde{x},y) = -\frac{1}{2} b_i \underbrace{(x_i - \tilde{x}_i)}_{<\,0} > 0$$

| $x_i$ | $b_i$ | $x_i - \tilde{x}_i$ |
|---|---|---|
| -1 | > 0 | < 0 |
| +1 | < 0 | > 0 |

use analogous argumentation if neuron of right layer has changed

$\Rightarrow$ every update (change of state) decreases energy function

$\Rightarrow$ since number of different bipolar vectors is finite update stops after finite #updates

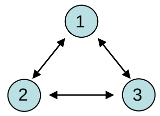
remark: dynamics of BAM get stable in local minimum of energy function!

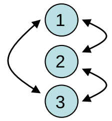
q.e.d.

special case of BAM but proposed earlier (1982)

**characterization:**

- neurons preserve state until selected at random for update
- n neurons fully connected
- symmetric weight matrix
- no self-loops (→ zero main diagonal entries)
- thresholds $\theta$ , neuron i fires if excitations larger than $\theta_i$




**transition**: select index k at random, new state is $\tilde{x} = \text{sgn}(\, x\, W - \theta\, )$

where $\tilde{x} = (x_1, \ldots, x_{k-1}, \tilde{x}_k, x_{k+1}, \ldots, x_n)$

energy of state x is $E(x) \;=\; -\frac{1}{2}\, x W x' + \theta\, x'$





**Theorem:**

Hopfield network converges to local minimum of energy function after a finite number of updates. □

***Proof:*** assume that $x_k$ has been updated $\Rightarrow \tilde{x}_k = -x_k$ and $\tilde{x}_i = x_i$ for $i \neq k$

$$E(x) - E(\tilde{x}) \;=\; -\tfrac{1}{2}\, x W x' + \theta\, x' + \tfrac{1}{2}\, \tilde{x} W \tilde{x}' - \theta\, \tilde{x}'$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, x_i\, x_j + \sum_{i=1}^{n} \theta_i\, x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, \tilde{x}_i\, \tilde{x}_j - \sum_{i=1}^{n} \theta_i\, \tilde{x}_i$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, (x_i\, x_j - \tilde{x}_i\, \tilde{x}_j) + \sum_{i=1}^{n} \theta_i\, \underbrace{(x_i - \tilde{x}_i)}_{=0 \text{ if } i \neq k}$$

$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{j=1}^{n} w_{ij}\, (x_i\, x_j - \underset{\substack{\| \\ x_i}}{\tilde{x}_i}\, \tilde{x}_j) - \frac{1}{2} \sum_{j=1}^{n} \underset{\substack{\| \\ 0 \text{ if } j = k}}{w_{kj}}\, (x_k\, x_j - \tilde{x}_k\, \underset{\substack{\| \\ x_j \text{ if } j \neq k}}{\tilde{x}_j}) + \theta_k\, (x_k - \tilde{x}_k)$$





$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{j=1}^{n} w_{ij}\, x_i\, \underbrace{(x_j - \tilde{x}_j)}_{=0 \text{ if } j \neq k} - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^{n} w_{kj}\, x_j\, (x_k - \tilde{x}_k) + \theta_k\, (x_k - \tilde{x}_k)$$

$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^{n} w_{ik}\, x_i\, (x_k - \tilde{x}_k) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^{n} w_{kj}\, x_j\, (x_k - \tilde{x}_k) + \theta_k\, (x_k - \tilde{x}_k)$$

(rename j to i, recall W = W‘, $w_{kk}$ = 0)

$$= -\sum_{i=1}^{n} w_{ik}\, x_i\, (x_k - \tilde{x}_k) \;+\; \theta_k\, (x_k - \tilde{x}_k)$$

$$\underbrace{= -(x_k - \tilde{x}_k) \left[ \underbrace{\sum_{i=1}^{n} w_{ik}\, x_i}_{\text{excitation } e_k} - \theta_k \right]}_{> 0 \text{ if } x_k < 0 \text{ and vice versa}} > 0$$

since:

| $x_k$ | $x_k - \tilde{x}_k$ | $e_k - \theta_k$ | $\Delta E$ |
|---|---|---|---|
| $+1$ | $> 0$ | $< 0$ | $> 0$ |
| $-1$ | $< 0$ | $> 0$ | $> 0$ |

**q.e.d.**





**Application to Combinatorial Optimization**

Idea:

- transform combinatorial optimization problem as objective function with $x \in \{-1,+1\}^n$
- rearrange objective function to look like a Hopfield energy function
- extract weights W and thresholds $\theta$ from this energy function
- initialize a Hopfield net with these parameters W and $\theta$
- run the Hopfield net until reaching stable state (= local minimizer of energy function)
- stable state is local minimizer of combinatorial optimization problem

### Example I: Linear Functions

$$f(x) = \sum_{i=1}^{n} c_i\, x_i \quad \to \min! \qquad (\, x_i \in \{-1, +1\}\,)$$

Evidently: $E(x) = f(x)$ with $W = 0$ and $\theta = c$

$\Downarrow$

choose $x^{(0)} \in \{-1, +1\}^n$
set iteration counter $t = 0$
**repeat**
  choose index $k$ at random

$$x_k^{(t+1)} = \text{sgn}(x^{(t)} \cdot W_{\cdot,k} - \theta_k) = \text{sgn}(x^{(t)} \cdot 0 - c_k) = -\text{sgn}(c_k) = \begin{cases} -1 & \text{if } c_k > 0 \\ +1 & \text{if } c_k < 0 \end{cases}$$

  increment $t$
**until** reaching fixed point

$\Rightarrow$ fixed point reached after $\Theta$(n log n) iterations on average

[ proof: $\to$ black board ]





### Example II: MAXCUT

given: graph with n nodes and symmetric weights $\omega_{ij} = \omega_{ji}$ , $\omega_{ii} = 0$, on edges

task: find a partition V = ($V_0$, $V_1$) of the nodes such that the weighted sum of edges with one endpoint in $V_0$ and one endpoint in $V_1$ becomes maximal

encoding: $\forall$ i=1,...,n: $y_i = 0 \Leftrightarrow$ node i in set $V_0$; $y_i = 1 \Leftrightarrow$ node i in set $V_1$

objective function: $f(y) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij}\, [y_i\,(1 - y_j) + y_j\,(1 - y_i)] \quad \to$ max!

**preparations for applying Hopfield network**

step 1: conversion to minimization problem

step 2: transformation of variables

step 3: transformation to “Hopfield normal form“

step 4: extract coefficients as weights and thresholds of Hopfield net





### Example II: MAXCUT (continued)

step 1: conversion to minimization problem

$\Rightarrow$ multiply function with -1 $\Rightarrow$ E(y) = -f(y) $\to$ min!

step 2: transformation of variables

$\Rightarrow y_i = (x_i+1) / 2$

$$\Rightarrow f(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij} \left[ \frac{x_i+1}{2} \left(1 - \frac{x_j+1}{2}\right) + \frac{x_j+1}{2} \left(1 - \frac{x_i+1}{2}\right) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij} \left[1 - x_i\, x_j\right]$$

$$= \underbrace{\frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij}}_{\text{constant value}} - \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij}\, x_i\, x_j$$

*constant value* (does not affect location of optimal solution)





### Example II: MAXCUT (continued)

step 3: transformation to “Hopfield normal form“

$$E(x) = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \omega_{ij}\, x_i\, x_j \quad = -\frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} \underbrace{\left(-\frac{1}{2}\omega_{ij}\right)}_{w_{ij}} x_i\, x_j$$

$$= -\frac{1}{2} x' W x + \underbrace{\theta'}_{0'} x$$

step 4: extract coefficients as weights and thresholds of Hopfield net

$$w_{ij} = -\frac{\omega_{ij}}{2} \text{ for } i \neq j, \quad w_{ii} = 0, \quad \theta_i = 0$$

remark: $\omega_{ij}$ : weights in graph — $w_{ij}$ : weights in Hopfield net