Computational Intelligence

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Plan for Today

- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Fuzzy Sets

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 - A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets

   with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Fuzzy Complement: Axioms

Definition

A function c: $[0,1] \rightarrow [0,1]$ is a **fuzzy complement** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall a, b \in [0,1]: a \leq b \Rightarrow c(a) \geq c(b)$.   monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

(A4) $\forall a \in [0,1]: c(c(a)) = a$ involutive
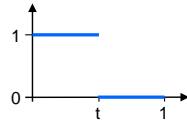
Examples:

a) standard fuzzy complement $c(a) = 1 - a$

   ad (A1): $c(0) = 1 - 0 = 1$ and $c(1) = 1 - 1 = 0$ ad (A3): ☑
   ad (A2): $c'(a) = -1 < 0$ (monotone decreasing)   ad (A4): $1 - (1 - a) = a$

b) $c(a) = \begin{cases} 1 & \text{if } a \leq t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$
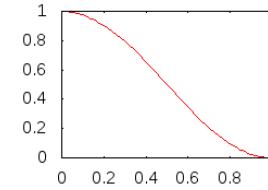
ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t ☑

ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(¼)) = c(1) = 0 \neq ¼$ for $t = ½$

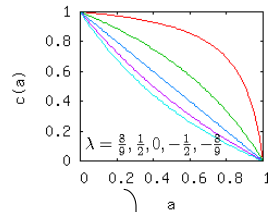
c) $c(a) = \dfrac{1 + \cos(\pi a)}{2}$

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c'(a) = -½\,\pi \sin(\pi a) < 0$ since $\sin(\pi a) > 0$ for $a \in (0,1)$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$c\left(c\left(\tfrac{1}{3}\right)\right) = c\left(\tfrac{3}{4}\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) \neq \tfrac{1}{3}$
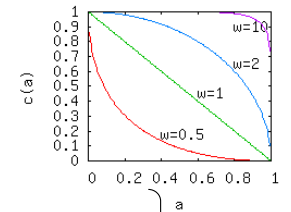
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d) $c(a) = \dfrac{1-a}{1+\lambda a}$ for $\lambda > -1$ **Sugeno class**

$\lambda = \tfrac{8}{9}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{8}{9}$

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c(a) \geq c(b) \Leftrightarrow \dfrac{1-a}{1+\lambda a} \geq \dfrac{1-b}{1+\lambda b} \Leftrightarrow$

$(1-a)(1+\lambda b) \geq (1-b)(1+\lambda a) \Leftrightarrow$

$b(\lambda+1) \geq a(\lambda+1) \Leftrightarrow b \geq a$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left(\tfrac{1-a}{1+\lambda a}\right) = \dfrac{1-\frac{1-a}{1+\lambda a}}{1+\lambda\frac{1-a}{1+\lambda a}} = \dfrac{a(\lambda+1)}{\lambda+1} = a$ ☑

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e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**

$w=10$, $w=2$, $w=1$, $w=0.5$

ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \geq (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \geq 1 - b^w \Leftrightarrow$
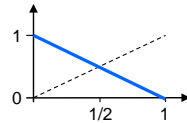
$a^w \leq b^w \Leftrightarrow a \leq b$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$

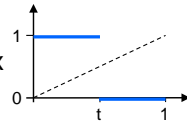$= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$ ☑

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix

no fixed point → see example (b) → no intersection with bisectrix

assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b*   (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2)   ∎

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a* with c(a*) = a*.

Proof:

Intermediate value theorem →

If c(·) continuous (A3) and c(0) ≥ c(1) (A1/A2)

then ∀ v ∈ [c(1), c(0)] = [0,1]: ∃ a ∈ [0,1]: c(a) = v.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points! ∎

Examples:

(a) c(a) = 1 – a   ⇒ a = 1 – a   ⇒ a* = ½

(b) c(a) = $(1 - a^w)^{1/w}$ ⇒ a = $(1 - a^w)^{1/w}$ ⇒ a* = $(½)^{1/w}$

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃ continuous function g: [0,1] → ℝ with

• g(0) = 0

• strictly monotone increasing

• ∀ a ∈ [0,1]: c(a) = $g^{(-1)}$(g(1) – g(a)).   ∎

defines an **increasing generator**

$g^{(-1)}(x)$ pseudo-inverse

Examples

a) g(x) = x   ⇒ $g^{-1}(x)$ = x   ⇒ c(a) = 1 – a   (Standard)

b) g(x) = x^w ⇒ $g^{-1}(x)$ = $x^{1/w}$ ⇒ c(a) = $(1 - a^w)^{1/w}$ (Yager class, w > 0)

c) g(x) = log(x+1) ⇒ $g^{-1}(x)$ = e^x – 1 ⇒ c(a) = exp(log(2) – log(a+1)) – 1

$$= \frac{1-a}{1+a}$$ (Sugeno class. λ = 1)

Examples

d) $g(a) = \frac{1}{\lambda} \log_e (1 + \lambda a)$ for $\lambda > -1$

• $g(0) = \log_e(1) = 0$

• strictly monotone increasing since $g'(a) = \frac{1}{1 + \lambda a} > 0$ for $a \in [0, 1]$

• inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda a) - 1}{\lambda}$, thus

$$c(a) = g^{-1}\left(\frac{\log(1 + \lambda)}{\lambda} - \frac{\log(1 + \lambda a)}{\lambda} \right)$$

$$= \frac{\exp(\log(1 + \lambda) - \log(1 + \lambda a)) - 1}{\lambda}$$

$$= \frac{1}{\lambda}\left(\frac{1 + \lambda}{1 + \lambda a} - 1 \right) = \frac{1 - a}{1 + \lambda a} \quad \text{(Sugeno Complement)}$$

Theorem

c: $[0,1] \to [0,1]$ is involutive fuzzy complement iff

$\exists$ continuous function f: $[0,1] \to \mathbb{R}$ with

• f(1) = 0

• strictly monotone decreasing

• $\forall\, a \in [0,1]$: c(a) = f$^{(-1)}$(f(0) − f(a)).　■

defines a
decreasing generator

f$^{(-1)}$(x) pseudo-inverse

Examples

a) f(x) = $k - k \cdot$ x $(k > 0)$　f$^{(-1)}$(x) = 1 − x/k　　c(a) = $1 - \frac{k-(k-ka)}{k} = 1 - a$

b) f(x) = 1 − x^w　　　　f$^{(-1)}$(x) = (1 − x)$^{1/\,w}$　c(a) = f^{-1}(a^w) = (1 − a^w)$^{1/w}$　(Yager)

Definition

A function t:$[0,1] \times [0,1] \to [0,1]$ is a **fuzzy intersection** or **t-norm** iff

(A1)　t(a, 1) = a

(A2)　b ≤ d $\Rightarrow$ t(a, b) ≤ t(a, d)　　　　　　(monotonicity)

(A3)　t(a,b) = t(b, a)　　　　　　　　　　　(commutative)

(A4)　t(a, t(b, d)) = t(t(a, b), d)　　　　　　(associative)　■

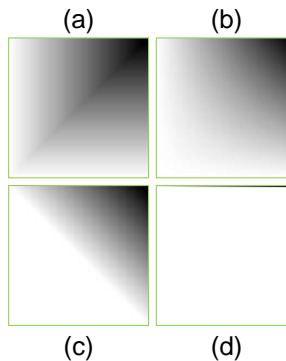**"nice to have"**

(A5)　t(a, b) is continuous　　　　　　　　(continuity)

(A6)　t(a, a) < a　　　　　　　　　　　　(subidempotent)

(A7)　$a_1 < a_2$ and $b_1 \leq b_2$ $\Rightarrow$　t(a_1, b_1) < t(a_2, b_2)　(strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

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Examples:

Name	Function
(a) Standard	t(a, b) = min { a, b }
(b) Algebraic Product	t(a, b) = a · b
(c) Bounded Difference	t(a, b) = max { 0, a + b − 1 }
(d) Drastic Product	t(a, b) = $\begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$

(a)　(b)

(c)　(d)

Is algebraic product a t-norm?　Check the 4 axioms!

ad (A1): t(a, 1) = a · 1 = a　　☑　　　ad (A3): t(a, b) = a · b = b · a = t(b, a)　☑

ad (A2): a · b ≤ a · d $\Leftrightarrow$ b ≤ d　☑　　　ad (A4): a · (b · d) = (a · b) · d　　　　☑

Theorem

Function t: $[0,1] \times [0,1] \to [0,1]$ is a t-norm $\Leftrightarrow$

$\exists$ decreasing generator f:$[0,1] \to \mathbb{R}$ with　t(a, b) = f$^{(-1)}$(f(a) + f(b)).　■

Example:

f(x) = 1/x − 1 is decreasing generator since

• f(x) is continuous　　　　　　　　　　☑

• f(1) = 1/1 − 1 = 0　　　　　　　　　　☑

• f'(x) = −1/x^2 < 0 (monotone decreasing)　☑

inverse function is f^{-1}(x) = $\dfrac{1}{x+1}$

$\Rightarrow$ t(a, b) = $f^{-1}\left(\dfrac{1}{a} + \dfrac{1}{b} - 2\right) = \dfrac{1}{\frac{1}{a}+\frac{1}{b}-1} = \dfrac{ab}{a+b-a\,b}$

Definition

A function s:[0,1] × [0,1] → [0,1] is a *fuzzy union* or *s-norm* or *t-conorm* iff

(A1) s(a, 0) = a

(A2) b ≤ d ⇒ s(a, b) ≤ s(a, d)　　　　　　　　　(monotonicity)

(A3) s(a, b) = s(b, a)　　　　　　　　　　　　(commutative)

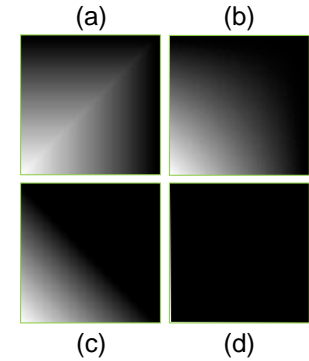(A4) s(a, s(b, d)) = s(s(a, b), d)　　　　　　　　(associative)　　■

"nice to have"

(A5) s(a, b) is continuous　　　　　　　　　　(continuity)

(A6) s(a, a) > a　　　　　　　　　　　　　(superidempotent)

(A7) $a_1 < a_2$ and $b_1 ≤ b_2$ ⇒ $s(a_1, b_1) < s(a_2, b_2)$　　(strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	s(a, b) = max { a, b }
Algebraic Sum	s(a, b) = a + b − a · b
Bounded Sum	s(a, b) = min { 1, a + b }
Drastic Union	$s(a, b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$

(a)　　　(b)

(c)　　　(d)

Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): s(a, 0) = a + 0 − a · 0 = a   ☑　　　　　　　ad (A3): ☑

ad (A2): a + b − a · b ≤ a + d − a · d ⇔ b (1 − a) ≤ d (1 − a) ⇔ b ≤ d   ☑　　ad (A4): ☑

Theorem

Function s: [0,1] × [0,1] → [0,1] is a s-norm ⇔

∃ increasing generator g:[0,1] → ℝ with s(a, b) = g$^{(-1)}$(g(a) + g(b)).　　■

Example:

g(x) = −log(1 − x) is increasing generator since

- g(x) is continuous　　　　　　　　　　　☑
- g(0) = −log(1 − 0) = 0　　　　　　　　　☑
- g'(x) = 1/(1 − x) > 0 (monotone increasing) ☑

inverse function is g^{-1}(x) = 1 − exp(−x)

$$⇒ s(a, b) = g^{-1}(-\log(1-a) - \log(1-b))$$
$$= 1 - \exp(\log(1-a) + \log(1-b))$$
$$= 1 - (1-a)(1-b) = a + b - a\,b \quad \text{(algebraic sum)}$$

Background from classical set theory:

∩ and ∪ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition

A pair of t-norm t(·, ·) and s-norm s(·, ·) is said to be *dual with regard to the fuzzy complement* c(·) iff

• c(t(a, b)) = s(c(a), c(b))

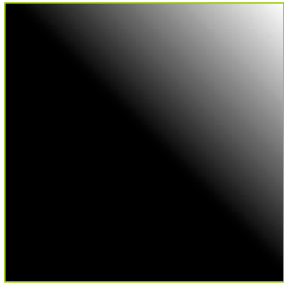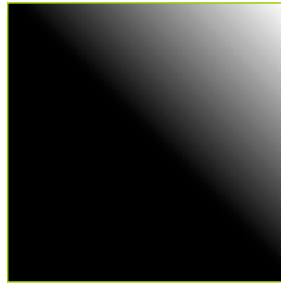• c(s(a, b)) =   t(c(a), c(b))

for all a, b ∈ [0,1].　　　　　　■

Definition

Let (c, s, t) be a tripel of fuzzy complement c(·), s- and t-norm.

If t and s are dual to c then the tripel (c,s, t) is called a *dual tripel*.　■

Examples of dual tripels

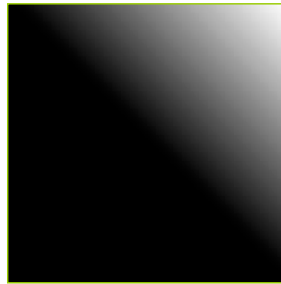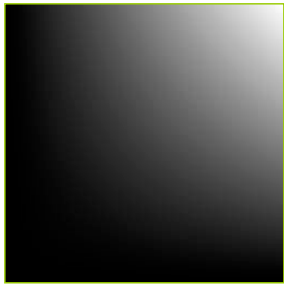| t-norm	s-norm	complement
min { a, b }	max { a, b }	1 − a
a · b	a + b − a · b	1 − a
max { 0, a + b − 1 }	min { 1, a + b }	1 − a

c(t(a, b))   s(c(a), c(b))

Dual Triple:

- bounded difference

- bounded sum

- standard complement

$\Rightarrow$ left image = right image

Non-Dual Triple:

- algebraic product

- bounded sum

- standard complement

$\Rightarrow$ left image $\neq$ right image

Why are dual triples so important?

$\Rightarrow$ allow equivalence transformations of fuzzy set expressions

$\Rightarrow$ required to transform into some equivalent normal form (standardized input)

$\Rightarrow$ e.g. two stages: intersection of unions $\quad \displaystyle\bigcap_{i=1}^{n} (A_i \cup B_i)$

or union of intersections $\quad \displaystyle\bigcup_{i=1}^{n} (A_i \cap B_i)$

Example:

$A \cup (B \cap (C \cap D)^c) \;=\;$ ← not in normal form

$A \cup (B \cap (C^c \cup D^c)) \;=\;$ ← equivalent if DeMorgan's law valid (dual triples!)

$A \cup (B \cap C^c) \cup (B \cap D^c)$ ← equivalent (distributive lattice!)