Computational Intelligence

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Plan for Today

Lecture 06

- Fuzzy sets
  - Axioms of fuzzy complement, t- and s-norms
  - Generators
  - Dual tripels

Fuzzy Sets

Lecture 06

Considered so far:

Standard fuzzy operators

- $A^c(x) = 1 – A(x)$
- $(A \cap B)(x) = \min \{ A(x), B(x) \}$
- $(A \cup B)(x) = \max \{ A(x), B(x) \}$

$\Rightarrow$ Compatible with operators for crisp sets
with membership functions with values in $\mathbb{B} = \{ 0, 1 \}$

$\exists$ Non-standard operators? $\Rightarrow$ Yes! Innumerable many!

- Defined via axioms.
- Creation via generators.

Fuzzy Complement: Axioms

Lecture 06

Definition

A function $c: [0,1] \rightarrow [0,1]$ is a ***fuzzy complement*** iff

(A1) $c(0) = 1$ and $c(1) = 0$.

(A2) $\forall\, a, b \in [0,1]: a \le b \Rightarrow c(a) \ge c(b)$. monotone decreasing

"nice to have":

(A3) $c(\cdot)$ is continuous.

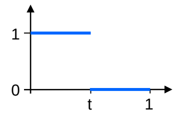
(A4) $\forall\, a \in [0,1]: c(c(a)) = a$ involutive

Examples:

a) standard fuzzy complement $c(a) = 1 – a$

ad (A1): $c(0) = 1 – 0 = 1$ and $c(1) = 1 – 1 = 0$ ad (A3): ☑
ad (A2): $c'(a) = –1 < 0$ (monotone decreasing) ad (A4): $1 – (1 – a) = a$

b) $c(a) = \begin{cases} 1 & \text{if } a \le t \\ 0 & \text{otherwise} \end{cases}$ for some $t \in (0, 1)$



ad (A1): $c(0) = 1$ since $0 < t$ and $c(1) = 0$ since $t < 1$.

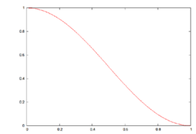
ad (A2): monotone (actually: constant) from 0 to t and t to 1, decreasing at t ☑

ad (A3): not valid → discontinuity at t

ad (A4): not valid → counter example

$c(c(¼)) = c(1) = 0 \neq ¼$ for $t = ½$

c) $c(a) = \dfrac{1 + \cos(\pi\, a)}{2}$



ad (A1): $c(0) = 1$ and $c(1) = 0$

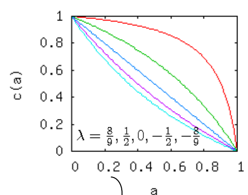
ad (A2): $c'(a) = -½\, \pi \sin(\pi\, a) < 0$ since $\sin(\pi\, a) > 0$ for $a \in (0,1)$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): not valid → counter example

$$c\left(c\left(\tfrac{1}{3}\right)\right) = c\left(\tfrac{3}{4}\right) = \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) \neq \tfrac{1}{3}$$

d) $c(a) = \dfrac{1-a}{1+\lambda\, a}$ for $\lambda > -1$ **Sugeno class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $c(a) \ge c(b) \Leftrightarrow \dfrac{1-a}{1+\lambda a} \ge \dfrac{1-b}{1+\lambda b} \Leftrightarrow$

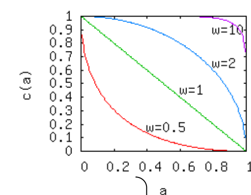
$(1-a)(1+\lambda b) \ge (1-b)(1+\lambda a) \Leftrightarrow$

$b(\lambda+1) \ge a(\lambda+1) \Leftrightarrow b \ge a$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left(\dfrac{1-a}{1+\lambda a}\right) = \dfrac{1-\frac{1-a}{1+\lambda a}}{1+\lambda\frac{1-a}{1+\lambda a}} = \dfrac{a(\lambda+1)}{\lambda+1} = a$ ☑

e) $c(a) = (1 - a^w)^{1/w}$ for $w > 0$ **Yager class**



ad (A1): $c(0) = 1$ and $c(1) = 0$

ad (A2): $(1 - a^w)^{1/w} \ge (1 - b^w)^{1/w} \Leftrightarrow 1 - a^w \ge 1 - b^w \Leftrightarrow$

$a^w \le b^w \Leftrightarrow a \le b$ ☑

ad (A3): is continuous as a composition of continuous functions

ad (A4): $c(c(a)) = c\left((1-a^w)^{\frac{1}{w}}\right) = \left(1 - \left[(1-a^w)^{\frac{1}{w}}\right]^w\right)^{\frac{1}{w}}$

$= (1 - (1 - a^w))^{\frac{1}{w}} = (a^w)^{\frac{1}{w}} = a$ ☑

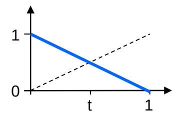
Fuzzy Complement: Fixed Points

Theorem

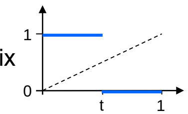
If function c:[0,1] → [0,1] satisfies axioms (A1) and (A2) of fuzzy complement then it has at most one fixed point a* with c(a*) = a*.

Proof:

one fixed point → see example (a) → intersection with bisectrix



no fixed point → see example (b) → no intersection with bisectrix



assume ∃ n > 1 fixed points, for example a* and b* with a* < b*

⇒ c(a*) = a* and c(b*) = b* (fixed points)

⇒ c(a*) < c(b*) with a* < b* impossible if c(·) is monotone decreasing

⇒ contradiction to axiom (A2) ■

Fuzzy Complement: Fixed Points

Theorem

If function c:[0,1] → [0,1] satisfies axioms (A1) – (A3) of fuzzy complement then it has exactly one fixed point a* with c(a*) = a*.

Proof:

Intermediate value theorem →

If c(·) continuous (A3) and c(0) ≥ c(1) (A1/A2)

then ∀ v ∈ [c(1), c(0)] = [0,1]: ∃ a ∈ [0,1]: c(a) = v.

⇒ there must be an intersection with bisectrix

⇒ a fixed point exists and by previous theorem there are no other fixed points! ■

Examples:

(a) c(a) = 1 – a ⇒ a = 1 – a ⇒ a* = ½

(b) $c(a) = (1 – a^w)^{1/w}$ ⇒ $a = (1 – a^w)^{1/w}$ ⇒ $a^* = (½)^{1/w}$

Fuzzy Complement: 1st Characterization

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff

∃ continuous function g: [0,1] → ℝ with

- g(0) = 0
- strictly monotone increasing
- ∀ a ∈ [0,1]: $c(a) = g^{(-1)}(g(1) – g(a))$. ■

defines an **increasing generator**

$g^{(-1)}(x)$ pseudo-inverse

Examples

a) g(x) = x ⇒ $g^{-1}(x) = x$ ⇒ c(a) = 1 – a (Standard)

b) $g(x) = x^w$ ⇒ $g^{-1}(x) = x^{1/w}$ ⇒ $c(a) = (1 – a^w)^{1/w}$ (Yager class, w > 0)

c) g(x) = log(x+1) ⇒ $g^{-1}(x) = e^x – 1$ ⇒ c(a) = exp(log(2) – log(a+1)) – 1 $= \frac{1-a}{1+a}$ (Sugeno class. λ = 1)

Fuzzy Complement: 1st Characterization

Examples

d) $g(a) = \frac{1}{\lambda}\log_e(1+\lambda a)$ for $\lambda > -1$

- $g(0) = \log_e(1) = 0$
- strictly monotone increasing since $g'(a) = \frac{1}{1+\lambda a} > 0$ for $a \in [0,1]$
- inverse function on [0,1] is $g^{-1}(a) = \frac{\exp(\lambda a)-1}{\lambda}$, thus

$$\begin{aligned} c(a) &= g^{-1}\left(\frac{\log(1+\lambda)}{\lambda} - \frac{\log(1+\lambda a)}{\lambda}\right) \\ &= \frac{\exp(\log(1+\lambda) - \log(1+\lambda a)) - 1}{\lambda} \\ &= \frac{1}{\lambda}\left(\frac{1+\lambda}{1+\lambda a} - 1\right) = \frac{1-a}{1+\lambda a} \quad \text{(Sugeno Complement)} \end{aligned}$$

Fuzzy Complement: 2nd Characterization

Theorem

c: [0,1] → [0,1] is involutive fuzzy complement iff
∃ continuous function f: [0,1] → ℝ with
- f(1) = 0
- strictly monotone decreasing
- ∀ a ∈ [0,1]: $c(a) = f^{(-1)}(f(0) - f(a))$. ■

(f(1) = 0 and strictly monotone decreasing) defines a **decreasing generator**

$f^{(-1)}(x)$ pseudo-inverse

Examples

a) $f(x) = k - k \cdot x \;\; (k > 0)$ $\quad f^{(-1)}(x) = 1 - x/k$ $\quad c(a) = 1 - \frac{k-(k-ka)}{k} = 1 - a$

b) $f(x) = 1 - x^w$ $\quad f^{(-1)}(x) = (1 - x)^{1/w}$ $\quad c(a) = f^{-1}(a^w) = (1 - a^w)^{1/w}$ (Yager)

Fuzzy Intersection: t-norm

Definition

A function t:[0,1] × [0,1] → [0,1] is a ***fuzzy intersection*** or ***t-norm*** iff

(A1) t(a, 1) = a

(A2) b ≤ d ⇒ t(a, b) ≤ t(a, d) (monotonicity)

(A3) t(a,b) = t(b, a) (commutative)

(A4) t(a, t(b, d)) = t(t(a, b), d) (associative) ■

"nice to have"

(A5) t(a, b) is continuous (continuity)

(A6) t(a, a) < a (subidempotent)

(A7) $a_1 \le a_2$ and $b_1 \le b_2$ ⇒ $t(a_1, b_1) \le t(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent t-norm is the standard fuzzy intersection

Fuzzy Intersection: t-norm

Examples:

Name	Function
Standard	t(a, b) = min { a, b }
Algebraic Product	t(a, b) = a · b
Bounded Difference	t(a, b) = max { 0, a + b – 1 }
Drastic Product	$t(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$

Is algebraic product a t-norm? Check the 4 axioms!

ad (A1): t(a, 1) = a · 1 = a ☑

ad (A2): a · b ≤ a · d ⇔ b ≤ d ☑

ad (A3): t(a, b) = a · b = b · a = t(b, a) ☑

ad (A4): a · (b · d) = (a · b) · d ☑

Fuzzy Intersection: Characterization

Theorem

Function t: [0,1] × [0,1] → [0,1] is a t-norm ⇔
∃ decreasing generator f:[0,1] → ℝ with $t(a, b) = f^{(-1)}(f(a) + f(b))$. ■

Example:

f(x) = 1/x – 1 is decreasing generator since
- f(x) is continuous ☑
- f(1) = 1/1 – 1 = 0 ☑
- $f'(x) = -1/x^2 < 0$ (monotone decreasing) ☑

inverse function is $f^{-1}(x) = \frac{1}{x+1}$

$$\Rightarrow t(a,b) = f^{-1}\left(\frac{1}{a} + \frac{1}{b} - 2\right) = \frac{1}{\frac{1}{a}+\frac{1}{b}-1} = \frac{ab}{a+b-a\,b}$$

Definition

A function s:[0,1] × [0,1] → [0,1] is a ***fuzzy union*** or ***s-norm*** or ***t-conorm*** iff

(A1) s(a, 0) = a

(A2) b ≤ d ⇒ s(a, b) ≤ s(a, d) (monotonicity)

(A3) s(a, b) = s(b, a) (commutative)

(A4) s(a, s(b, d)) = s(s(a, b), d) (associative) ■

"nice to have"

(A5) s(a, b) is continuous (continuity)

(A6) s(a, a) > a (superidempotent)

(A7) $a_1 \le a_2$ and $b_1 \le b_2$ ⇒ $s(a_1, b_1) \le s(a_2, b_2)$ (strict monotonicity)

Note: the only idempotent s-norm is the standard fuzzy union

Examples:

Name	Function
Standard	s(a, b) = max { a, b }
Algebraic Sum	s(a, b) = a + b – a · b
Bounded Sum	s(a, b) = min { 1, a + b }
Drastic Union	s(a, b) = { a if b = 0; b if a = 0; 1 otherwise

Is algebraic sum a t-norm? Check the 4 axioms!

ad (A1): s(a, 0) = a + 0 – a · 0 = a ☑

ad (A2): a + b – a · b ≤ a + d – a · d ⇔ b (1 – a) ≤ d (1 – a) ⇔ b ≤ d ☑

ad (A3): ☑

ad (A4): ☑

Theorem

Function s: [0,1] × [0,1] → [0,1] is a s-norm ⇔

∃ increasing generator g:[0,1] → ℝ with $s(a, b) = g^{(-1)}(g(a) + g(b))$. ■

Example:

g(x) = –log(1 – a) is decreasing generator since

- g(x) is continuous ☑
- g(0) = –log(1 – 0) = 0 ☑
- g'(x) = 1/(1-a) > 0 (monotone increasing) ☑

inverse function is $g^{-1}(x) = 1 - \exp(-a)$

$$\Rightarrow s(a,b) = g^{-1}(-\log(1-a) - \log(1-b))$$
$$= 1 - \exp(\log(1-a) + \log(1-b))$$
$$= 1 - (1-a)(1-b) = a + b - ab \quad \text{(algebraic sum)}$$

Background from classical set theory:

∩ and ∪ operations are dual w.r.t. complement since they obey DeMorgan's laws

Definition

A pair of t-norm t(·, ·) and s-norm s(·, ·) is said to be ***dual with regard to the fuzzy complement*** c(·) iff

- c(t(a, b)) = s(c(a), c(b))
- c(s(a, b)) = t(c(a), c(b))

for all a, b ∈ [0,1]. ■

Definition

Let (c, s, t) be a tripel of fuzzy complement c(·), s- and t-norm.

If t and s are dual to c then the tripel (c,s, t) is called a ***dual tripel***. ■

Examples of dual tripels

t-norm	s-norm	complement
min { a, b }	max { a, b }	1 – a
a · b	a + b – a · b	1 – a
max { 0, a + b – 1 }	min { 1, a + b }	1 – a