

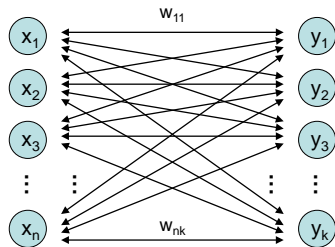
# Computational Intelligence

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- Bidirectional Associative Memory (BAM)
  - Fixed Points
  - Concept of Energy Function
  - Stable States = Minimizers of Energy Function
  
- Hopfield Network
  - Convergence
  - Application to Combinatorial Optimization

## Network Model



- fully connected
- bidirectional edges
- synchronized:
  - step t : data flow from x to y
  - step t + 1 : data flow from y to x

**start:**  $y^{(0)} = \text{sgn}(x^{(0)} W)$   
 $x^{(1)} = \text{sgn}(y^{(0)} W')$   
 $y^{(1)} = \text{sgn}(x^{(1)} W)$   
 $x^{(2)} = \text{sgn}(y^{(1)} W')$   
 ...

$x, y$  : row vectors  
 $W$  : weight matrix  
 $W'$  : transpose of  $W$   
 bipolar inputs  $\in \{-1, +1\}$

## Fixed Points

### Definition

$(x, y)$  is **fixed point** of BAM iff  $y = \text{sgn}(x W)$  and  $x' = \text{sgn}(W y')$ . □

Set  $W = x' y$ . (note:  $x$  is row vector)

$$y = \text{sgn}(x W) = \text{sgn}(x (x' y)) = \text{sgn}(\underbrace{(x x')}_> 0 y) = y$$

> 0 (does not alter sign)

$$x' = \text{sgn}(W y') = \text{sgn}((x' y) y') = \text{sgn}(x' \underbrace{(y y')}_{> 0}) = x'$$

> 0 (does not alter sign)

**Theorem:** If  $W = x' y$  then  $(x, y)$  is fixed point of BAM. □

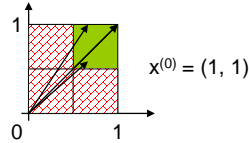
Concept of Energy Function

given: BAM with  $W = x'y$   $\Rightarrow (x,y)$  is stable state of BAM

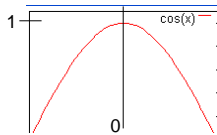
starting point  $x^{(0)}$   $\Rightarrow y^{(0)} = \text{sgn}(x^{(0)} W)$   
 $\Rightarrow$  excitation  $e' = W y^{(0)'$   
 $\Rightarrow$  if  $\text{sign}(e') = x^{(0)}$  then  $(x^{(0)}, y^{(0)})$  stable state

small angle between  $e'$  and  $x^{(0)}$

$\Leftarrow$  true if  $e'$  close to  $x^{(0)}$



recall:  $\frac{ab'}{\|a\| \cdot \|b\|} = \cos \angle(a, b)$



small angle  $\alpha \Rightarrow$  large  $\cos(\alpha)$

Concept of Energy Function

required:

small angle between  $e' = W y^{(0)'$  and  $x^{(0)}$

$\Rightarrow$  larger cosine of angle indicates greater similarity of vectors

$\Rightarrow \forall e'$  of equal size: try to maximize  $x^{(0)} e' = \underbrace{\|x^{(0)}\|}_{\text{fixed}} \cdot \underbrace{\|e'\|}_{\text{fixed}} \cdot \underbrace{\cos \angle(x^{(0)}, e')}_{\rightarrow \text{max!}}$

$\Rightarrow$  maximize  $x^{(0)} e' = x^{(0)} W y^{(0)'$

$\Rightarrow$  identical to minimize  $-x^{(0)} W y^{(0)'$

Definition

Energy function of BAM at iteration  $t$  is  $E(x^{(t)}, y^{(t)}) = -\frac{1}{2} x^{(t)} W y^{(t)'$   $\square$

Stable States

Theorem

An asynchronous BAM with arbitrary weight matrix  $W$  reaches steady state in a finite number of updates.

Proof:

$$E(x, y) = -\frac{1}{2} x W y' = \begin{cases} -\frac{1}{2} x (W y') = -\frac{1}{2} x b' = -\frac{1}{2} \sum_{i=1}^n b_i x_i \\ -\frac{1}{2} (x W) y' = -\frac{1}{2} a y' = -\frac{1}{2} \sum_{i=1}^k a_i y_i \end{cases}$$

$\swarrow \searrow$  excitations

BAM asynchronous  $\Rightarrow$  select neuron at random from left or right layer, compute its excitation and change state if necessary (states of other neurons not affected)

neuron  $i$  of left layer has changed  $\Rightarrow \text{sgn}(x_i) \neq \text{sgn}(b_i)$   
 $\Rightarrow x_i$  was updated to  $\tilde{x}_i = -x_i$

$E(x, y) - E(\tilde{x}, y) = -\frac{1}{2} b_i (x_i - \tilde{x}_i) > 0$

$< 0$

$x_i$	$b_i$	$x_i - \tilde{x}_i$
-1	> 0	< 0
+1	< 0	> 0

use analogous argumentation if neuron of right layer has changed  
 $\Rightarrow$  every update (change of state) decreases energy function  
 $\Rightarrow$  since number of different bipolar vectors is finite update stops after finite #updates

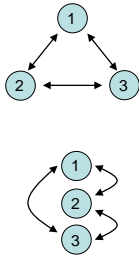
remark: dynamics of BAM get stable in local minimum of energy function!

q.e.d.

special case of BAM but proposed earlier (1982)

**characterization:**

- neurons preserve state until selected at random for update
- n neurons fully connected
- symmetric weight matrix
- no self-loops ( $\rightarrow$  zero main diagonal entries)
- thresholds  $\theta$ , neuron i fires if excitations larger than  $\theta_i$



**transition:** select index k at random, new state is  $\tilde{x} = \text{sgn}(xW - \theta)$

where  $\tilde{x} = (x_1, \dots, x_{k-1}, \tilde{x}_k, x_{k+1}, \dots, x_n)$

energy of state x is  $E(x) = -\frac{1}{2}xWx' + \theta x'$

**Theorem:**

Hopfield network converges to local minimum of energy function after a finite number of updates.  $\square$

**Proof:** assume that  $x_k$  has been updated  $\Rightarrow \tilde{x}_k = -x_k$  and  $\tilde{x}_i = x_i$  for  $i \neq k$

$$\begin{aligned} E(x) - E(\tilde{x}) &= -\frac{1}{2}xWx' + \theta x' + \frac{1}{2}\tilde{x}W\tilde{x}' - \theta \tilde{x}' \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n w_{ij} x_i x_j + \sum_{i=1}^n \theta_i x_i + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n w_{ij} \tilde{x}_i \tilde{x}_j - \sum_{i=1}^n \theta_i \tilde{x}_i \\ &= -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n w_{ij} (x_i x_j - \tilde{x}_i \tilde{x}_j) + \sum_{i=1}^n \theta_i \underbrace{(x_i - \tilde{x}_i)}_{=0 \text{ if } i \neq k} \\ &= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^n \sum_{j=1}^n w_{ij} \underbrace{(x_i x_j - \tilde{x}_i \tilde{x}_j)}_{\substack{\parallel \\ x_i}} - \frac{1}{2} \sum_{j=1}^n w_{kj} \underbrace{(x_k x_j - \tilde{x}_k \tilde{x}_j)}_{\substack{\parallel \\ 0 \text{ if } j = k \\ x_j \text{ if } j \neq k}} + \theta_k (x_k - \tilde{x}_k) \end{aligned}$$

$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^n \sum_{j=1}^n w_{ij} x_i \underbrace{(x_j - \tilde{x}_j)}_{=0 \text{ if } j \neq k} - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^n w_{kj} x_j (x_k - \tilde{x}_k) + \theta_k (x_k - \tilde{x}_k)$$

$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^n w_{ik} x_i (x_k - \tilde{x}_k) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^n w_{kj} x_j (x_k - \tilde{x}_k) + \theta_k (x_k - \tilde{x}_k) \quad (\text{rename } j \text{ to } i, \text{ recall } W = W', w_{kk} = 0)$$

$$= -\sum_{i=1}^n w_{ik} x_i (x_k - \tilde{x}_k) + \theta_k (x_k - \tilde{x}_k)$$

$$= -(x_k - \tilde{x}_k) \left[ \underbrace{\sum_{i=1}^n w_{ik} x_i}_{\text{excitation } e_k} - \theta_k \right] > 0 \quad \text{since:}$$

$x_k$	$x_k - \tilde{x}_k$	$e_k - \theta_k$	$\Delta E$
+1	> 0	< 0	> 0
-1	< 0	> 0	> 0

$> 0$  if  $x_k < 0$  and vice versa **q.e.d.**

**Application to Combinatorial Optimization**Idea:

- transform combinatorial optimization problem as objective function with  $x \in \{-1, +1\}^n$
- rearrange objective function to look like a Hopfield energy function
- extract weights W and thresholds  $\theta$  from this energy function
- initialize a Hopfield net with these parameters W and  $\theta$
- run the Hopfield net until reaching stable state (= local minimizer of energy function)
- stable state is local minimizer of combinatorial optimization problem

## Example I: Linear Functions

$$f(x) = \sum_{i=1}^n c_i x_i \rightarrow \min! \quad (x_i \in \{-1, +1\})$$

Evidently:  $E(x) = f(x)$  with  $W = 0$  and  $\theta = c$

⇓

choose  $x^{(0)} \in \{-1, +1\}^n$   
set iteration counter  $t = 0$

repeat

choose index  $k$  at random

$$x_k^{(t+1)} = \text{sgn}(x^{(t)} \cdot W_{\cdot,k} - \theta_k) = \text{sgn}(x^{(t)} \cdot 0 - c_k) = -\text{sgn}(c_k) = \begin{cases} -1 & \text{if } c_k > 0 \\ +1 & \text{if } c_k < 0 \end{cases}$$

increment  $t$

until reaching fixed point

⇒ fixed point reached after  $\Theta(n \log n)$  iterations on average

## Example II: MAXCUT

given: graph with  $n$  nodes and symmetric weights  $\omega_{ij} = \omega_{ji}$ ,  $\omega_{ii} = 0$ , on edges

task: find a partition  $V = (V_0, V_1)$  of the nodes such that the weighted sum of edges with one endpoint in  $V_0$  and one endpoint in  $V_1$  becomes maximal

encoding:  $\forall i=1, \dots, n: \quad y_i = 0 \Leftrightarrow \text{node } i \text{ in set } V_0; \quad y_i = 1 \Leftrightarrow \text{node } i \text{ in set } V_1$

objective function:  $f(y) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \omega_{ij} [y_i(1-y_j) + y_j(1-y_i)] \rightarrow \max!$

## preparations for applying Hopfield network

step 1: conversion to minimization problem

step 2: transformation of variables

step 3: transformation to "Hopfield normal form"

step 4: extract coefficients as weights and thresholds of Hopfield net

## Example II: MAXCUT (continued)

step 1: conversion to minimization problem

$$\Rightarrow \text{multiply function with } -1 \Rightarrow E(y) = -f(y) \rightarrow \min!$$

step 2: transformation of variables

$$\Rightarrow y_i = (x_i + 1) / 2$$

$$\Rightarrow f(x) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \omega_{ij} \left[ \frac{x_i + 1}{2} \left( 1 - \frac{x_j + 1}{2} \right) + \frac{x_j + 1}{2} \left( 1 - \frac{x_i + 1}{2} \right) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \omega_{ij} [1 - x_i x_j]$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \omega_{ij} - \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \omega_{ij} x_i x_j$$

*constant value* (does not affect location of optimal solution)

## Example II: MAXCUT (continued)

step 3: transformation to "Hopfield normal form"

$$\begin{aligned} E(x) &= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^n \omega_{ij} x_i x_j = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \underbrace{\left( -\frac{1}{2} \omega_{ij} \right)}_{w_{ij}} x_i x_j \\ &= -\frac{1}{2} x' W x + \theta' x \\ &\quad \downarrow \\ &\quad 0' \end{aligned}$$

step 4: extract coefficients as weights and thresholds of Hopfield net

$$w_{ij} = -\frac{\omega_{ij}}{2} \text{ for } i \neq j, \quad w_{ii} = 0, \quad \theta_i = 0$$

**remark**:  $\omega_{ij}$ : weights in graph —  $w_{ij}$ : weights in Hopfield net