

Computational Intelligence

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- Fuzzy relations
- Fuzzy logic
 - Linguistic variables and terms
 - Inference from fuzzy statements

relations with conventional sets $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$:

$$R(\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n) \subseteq \mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$$

notice that cartesian product is a **set**!

⇒ all set operations remain valid!

crisp membership function (of x to relation R)

$$R(x_1, x_2, \dots, x_n) = \begin{cases} 1 & \text{if } (x_1, x_2, \dots, x_n) \in R \\ 0 & \text{otherwise} \end{cases}$$

Definition

Fuzzy relation = fuzzy set over crisp cartesian product $\mathcal{X}_1 \times \mathcal{X}_2 \times \dots \times \mathcal{X}_n$ ■

→ each tuple (x_1, \dots, x_n) has a degree of membership to relation

→ degree of membership expresses *strength of relationship* between elements of tuple

appropriate representation: n-dimensional membership matrix

example: Let $X = \{ \text{New York, Paris} \}$ and $Y = \{ \text{Beijing, New York, Dortmund} \}$.

relation $R = \text{"very far away"}$

membership matrix →

| relation R | New York | Paris |
|------------|----------|-------|
| Beijing | 1.0 | 0.9 |
| New York | 0.0 | 0.7 |
| Dortmund | 0.6 | 0.3 |

Definition

Let $R(X, Y)$ be a fuzzy relation with membership matrix R . The **inverse fuzzy relation** to $R(X, Y)$, denoted $R^{-1}(Y, X)$, is a relation on $Y \times X$ with membership matrix $R^{-1} = R'$. ■

Remark: R' is the transpose of membership matrix R .

Evidently: $(R^{-1})^{-1} = R$ since $(R')' = R$

Definition

Let $P(X, Y)$ and $Q(Y, Z)$ be fuzzy relations. The operation \circ on two relations, denoted $P(X, Y) \circ Q(Y, Z)$, is termed **max-min-composition** iff

$$R(x, z) = (P \circ Q)(x, z) = \max_{y \in Y} \min \{ P(x, y), Q(y, z) \}. \quad \blacksquare$$

Theorem

- max-min composition is associative.
- max-min composition is not commutative.
- $(P(X, Y) \circ Q(Y, Z))^{-1} = Q^{-1}(Z, Y) \circ P^{-1}(Y, X)$.

membership matrix of max-min composition
determinable via "fuzzy matrix multiplication": $R = P \circ Q$

$$\text{fuzzy matrix multiplication} \quad r_{ij} = \max_k \min \{ p_{ik}, q_{kj} \}$$

$$\text{crisp matrix multiplication} \quad r_{ij} = \sum_k p_{ik} \cdot q_{kj}$$

further methods for realizing compositions of relations:

max-prod composition

$$(P \odot Q)(x, z) = \max_{y \in \mathcal{Y}} \{ P(x, y) \cdot Q(y, z) \}$$

generalization: sup-t composition

$$(P \circ Q)(x, z) = \sup_{y \in \mathcal{Y}} \{ t(P(x, y), Q(y, z)) \}, \quad \text{where } t(\dots) \text{ is a t-norm}$$

e.g.: $t(a, b) = \min\{a, b\} \Rightarrow$ max-min-composition
 $t(a, b) = a \cdot b \Rightarrow$ max-prod-composition

Binary fuzzy relations on $\mathcal{X} \times \mathcal{X}$: properties

- reflexive** $\Leftrightarrow \forall x \in \mathcal{X}: R(x, x) = 1$
- irreflexive** $\Leftrightarrow \exists x \in \mathcal{X}: R(x, x) < 1$
- antireflexive** $\Leftrightarrow \forall x \in \mathcal{X}: R(x, x) < 1$
- symmetric** $\Leftrightarrow \forall (x, y) \in \mathcal{X} \times \mathcal{X}: R(x, y) = R(y, x)$
- asymmetric** $\Leftrightarrow \exists (x, y) \in \mathcal{X} \times \mathcal{X}: R(x, y) \neq R(y, x)$
- antisymmetric** $\Leftrightarrow \forall (x, y) \in \mathcal{X} \times \mathcal{X}: R(x, y) \neq R(y, x)$
- transitive** $\Leftrightarrow \forall (x, z) \in \mathcal{X} \times \mathcal{X}: R(x, z) \geq \max_{y \in \mathcal{Y}} \min \{ R(x, y), R(y, z) \}$
- intransitive** $\Leftrightarrow \exists (x, z) \in \mathcal{X} \times \mathcal{X}: R(x, z) < \max_{y \in \mathcal{Y}} \min \{ R(x, y), R(y, z) \}$
- antitransitive** $\Leftrightarrow \forall (x, z) \in \mathcal{X} \times \mathcal{X}: R(x, z) < \max_{y \in \mathcal{Y}} \min \{ R(x, y), R(y, z) \}$

actually, here: max-min-transitivity (\rightarrow in general: sup-t-transitivity)

binary fuzzy relation on $\mathcal{X} \times \mathcal{X}$: example

Let \mathcal{X} be the set of all cities in Germany.

Fuzzy relation R is intended to represent the concept of „very close to“.

- $R(x,x) = 1$, since every city is certainly very close to itself.
⇒ **reflexive**
- $R(x,y) = R(y,x)$: if city x is very close to city y, then also vice versa.
⇒ **symmetric**
- $R(\text{Dortmund, Essen}) = 0.8$ (DU) (E) (DO)
 $R(\text{Essen, Duisburg}) = 0.7$
 $R(\text{Dortmund, Duisburg}) = 0.5$
 $R(\text{Dortmund, Hagen}) = 0.9$ (HA)
 ⇒ **intransitive**

crisp:

relation R is equivalence relation \Leftrightarrow R reflexive, symmetric, transitive

fuzzy:

relation R is similarity relation \Leftrightarrow R reflexive, symmetric, (max-min-) transitive

Bsp:

| | a | b | c | d | e | f | g |
|---|-----|-----|-----|-----|-----|-----|-----|
| a | 1,0 | 0,8 | 0,0 | 0,4 | 0,0 | 0,0 | 0,0 |
| b | 0,8 | 1,0 | 0,0 | 0,4 | 0,0 | 0,0 | 0,0 |
| c | 0,0 | 0,0 | 1,0 | 0,0 | 1,0 | 0,9 | 0,5 |
| d | 0,4 | 0,4 | 0,0 | 1,0 | 0,0 | 0,0 | 0,0 |
| e | 0,0 | 0,0 | 1,0 | 0,0 | 1,0 | 0,9 | 0,5 |
| f | 0,0 | 0,0 | 0,9 | 0,0 | 0,9 | 1,0 | 0,5 |
| g | 0,0 | 0,0 | 0,5 | 0,0 | 0,5 | 0,5 | 1,0 |

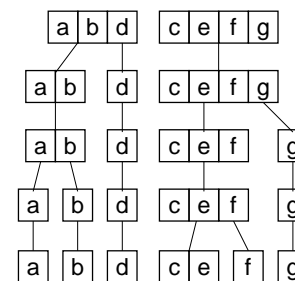
$\alpha = 0,4$

$\alpha = 0,5$

$\alpha = 0,8$

$\alpha = 0,9$

$\alpha = 1,0$



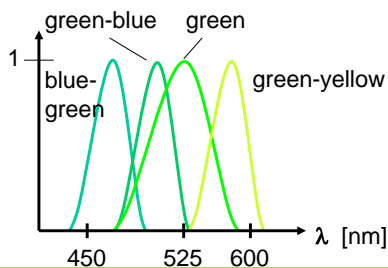
linguistic variable:

variable that can attain several values of linguistic / verbal nature

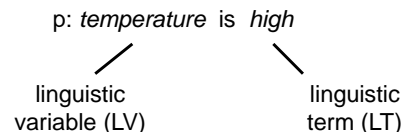
e.g.: **color** can attain values **red, green, blue, yellow, ...**

values (red, green, ...) of linguistic variable are called **linguistic terms**

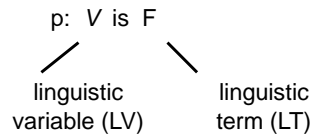
linguistic terms are associated with fuzzy sets



fuzzy proposition



- LV may be associated with several LT: *high, medium, low, ...*
- *high, medium, low* temperature are fuzzy sets over numerical scale of crisp temperatures
- trueness of fuzzy proposition „temperature is high“ for a given **concrete crisp** temperature value v is interpreted as equal to the degree of membership *high(v)* of the fuzzy set *high*

fuzzy proposition

establishes connection between *degree of membership* of a fuzzy set and the *degree of trueness* of a fuzzy proposition

actually:

p: V is F(v)

and

$T(p) = F(v)$ for a concrete crisp value v

trueness(p)

fuzzy proposition

p: IF *heating* is *hot*, THEN *energy consumption* is *high*

```

    graph TD
      A["p: IF heating is hot, THEN energy consumption is high"]
      B["LV"]
      C["LT"]
      D["LV"]
      E["LT"]
      A --- B
      A --- C
      A --- D
      A --- E
  
```

expresses relation between

- temperature of heating and
- quantity of energy consumption

p: (*heating, energy consumption*) $\in R$ relation

fuzzy proposition

p: IF X is A, THEN Y is B

```

    graph TD
      A["p: IF X is A, THEN Y is B"]
      B["LV"]
      C["LT"]
      D["LV"]
      E["LT"]
      A --- B
      A --- C
      A --- D
      A --- E
  
```

How can we determine / express degree of trueness $T(p)$?

- For crisp, given values x, y we know $A(x)$ and $B(y)$
- $A(x)$ and $B(y)$ must be processed to single value via relation R
- $R(x, y) = \text{function}(A(x), B(y))$ is fuzzy set over $X \times Y$
- as before: interpret $T(p)$ as degree of membership $R(x, y)$

fuzzy proposition

p: IF X is A, THEN Y is B

A is fuzzy set over X

B is fuzzy set over Y

R is fuzzy set over $X \times Y$

$\forall (x, y) \in X \times Y: R(x, y) = \text{Imp}(A(x), B(y))$

What is $\text{Imp}(\cdot, \cdot)$?

\Rightarrow „appropriate“ fuzzy implication $[0, 1] \times [0, 1] \rightarrow [0, 1]$

assumption: we know an „appropriate“ $\text{Imp}(a,b)$.

How can we determine the degree of trueness $T(p)$?

example:

let $\text{Imp}(a, b) = \min\{ 1, 1 - a + b \}$ and consider fuzzy sets

A:

| | | |
|-------|-------|-------|
| x_1 | x_2 | x_3 |
| 0.1 | 0.8 | 1.0 |

B:

| | |
|-------|-------|
| y_1 | y_2 |
| 0.5 | 1.0 |

⇒

| | | | |
|----------|-------|-------|-------|
| R | x_1 | x_2 | x_3 |
| y_1 | 1.0 | 0.7 | 0.5 |
| y_2 | 1.0 | 1.0 | 1.0 |

z.B.

$$R(x_2, y_1) = \text{Imp}(A(x_2), B(y_1)) = \text{Imp}(0.8, 0.5) = \min\{1.0, 0.7\} = 0.7$$

and $T(p)$ for (x_2, y_1) is $R(x_2, y_1) = 0.7$ ■

inference from fuzzy statements

- let $\forall x, y: y = f(x)$.

IF $X = x$ THEN $Y = f(x)$

- IF $X \in A$ THEN $Y \in B = \{ y \in Y: y = f(x), x \in A \}$

inference from fuzzy statements

- Let relationship between x and y be a relation R on $X \times Y$

IF $X = x$ THEN $Y \in B = \{ y \in Y: (x, y) \in R \}$

- IF $X \in A$ THEN $Y \in B = \{ y \in Y: (x, y) \in R, x \in A \}$

inference from fuzzy statements

IF $X \in A$ THEN $Y \in B = \{ y \in Y: (x, y) \in R, x \in A \}$

also expressible via characteristic functions of sets A, B, R :

$$\forall y \in Y: B(y) = \sup_{x \in X} \min \{ A(x), R(x, y) \}$$

Now: A' , B' fuzzy set over X resp. Y

Assume R and A' are given:

$$\forall y \in Y: B'(y) = \sup_{x \in X} \min \{ A'(x), R(x, y) \}$$

composition rule of inference (in matrix form): $B' = A' \circ R$

inference from fuzzy statements

- conventional: modus ponens

$$\begin{array}{l} a \Rightarrow b \\ \underline{a} \\ b \end{array}$$

- fuzzy: generalized modus ponens (GMP)

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ \underline{X \text{ is } A'} \\ Y \text{ is } B' \end{array}$$

e.g.: IF heating is hot, THEN energy consumption is high
heating is warm
energy consumption is normal

example: GMP

consider

A:

| | | |
|-------|-------|-------|
| x_1 | x_2 | x_3 |
| 0.5 | 1.0 | 0.6 |

B:

| | |
|-------|-------|
| y_1 | y_2 |
| 1.0 | 0.4 |

with the rule: IF X is A THEN Y is B

given fact

A':

| | | |
|-------|-------|-------|
| x_1 | x_2 | x_3 |
| 0.6 | 0.9 | 0.7 |

\Rightarrow

| R | x_1 | x_2 | x_3 |
|-------|-------|-------|-------|
| y_1 | 1.0 | 1.0 | 1.0 |
| y_2 | 0.9 | 0.4 | 0.8 |

with $\text{Imp}(a,b) = \min\{1, 1-a+b\}$

thus: $A' \circ R = B'$ $(0.6 \ 0.9 \ 0.7) \circ \begin{pmatrix} 1.0 & 0.9 \\ 1.0 & 0.4 \\ 1.0 & 0.8 \end{pmatrix} = (0.9 \ 0.7)$
 with max-min-composition

inference from fuzzy statements

- conventional: modus tollens

$$\begin{array}{l} a \Rightarrow b \\ \underline{\bar{b}} \\ \bar{a} \end{array}$$

- fuzzy: generalized modus tollens (GMT)

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ \underline{Y \text{ is } B'} \\ X \text{ is } A' \end{array}$$

e.g.: IF heating is hot, THEN energy consumption is high
energy consumption is normal
heating is warm

example: GMT

consider

A:

| | | |
|-------|-------|-------|
| x_1 | x_2 | x_3 |
| 0.5 | 1.0 | 0.6 |

B:

| | |
|-------|-------|
| y_1 | y_2 |
| 1.0 | 0.4 |

with the rule: IF X is A THEN Y is B

given fact

B':

| | |
|-------|-------|
| y_1 | y_2 |
| 0.9 | 0.7 |

\Rightarrow

| R | x_1 | x_2 | x_3 |
|-------|-------|-------|-------|
| y_1 | 1.0 | 1.0 | 1.0 |
| y_2 | 0.9 | 0.4 | 0.8 |

with $\text{Imp}(a,b) = \min\{1, 1-a+b\}$

thus: $B' \circ R^{-1} = A'$ $(0.9 \ 0.7) \circ \begin{pmatrix} 1.0 & 1.0 & 1.0 \\ 0.9 & 0.4 & 0.8 \end{pmatrix} = (0.9 \ 0.9 \ 0.9)$
 with max-min-composition

inference from fuzzy statements

- conventional:
hypothetic syllogism

$$\begin{array}{l} a \Rightarrow b \\ b \Rightarrow c \\ \hline a \Rightarrow c \end{array}$$

- fuzzy:
generalized HS

$$\begin{array}{l} \text{IF } X \text{ is } A, \text{ THEN } Y \text{ is } B \\ \text{IF } Y \text{ is } B, \text{ THEN } Z \text{ is } C \\ \hline \text{IF } X \text{ is } A, \text{ THEN } Z \text{ is } C \end{array}$$

e.g.: IF heating is hot, THEN energy consumption is high
IF energy consumption is high, THEN living is expensive

IF heating is hot, THEN living is expensive

example: GHS

let fuzzy sets A(x), B(x), C(x) be given

⇒ determine the three relations

$$\begin{array}{l} R_1(x,y) = \text{Imp}(A(x),B(y)) \\ R_2(y,z) = \text{Imp}(B(y),C(z)) \\ R_3(x,z) = \text{Imp}(A(x),C(z)) \end{array}$$

and express them as matrices R₁, R₂, R₃

We say:

GHS is valid if R₁ ◦ R₂ = R₃

So, ... what makes sense for Imp(·,·) ?

Imp(a,b) ought to express fuzzy version of implication (a ⇒ b)

conventional: a ⇒ b identical to $\bar{a} \vee b$

But how can we calculate with fuzzy “boolean” expressions?

request: must be compatible to crisp version (and more) for a,b ∈ { 0, 1 }

| a | b | a ∧ b | t(a,b) |
|---|---|-------|--------|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 |

| a | b | a ∨ b | s(a,b) |
|---|---|-------|--------|
| 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 |

| a | \bar{a} | c(a) |
|---|-----------|------|
| 0 | 1 | 1 |
| 1 | 0 | 0 |

So, ... what makes sense for Imp(·,·) ?

1st approach: S implications

conventional: a ⇒ b identical to $\bar{a} \vee b$

fuzzy: Imp(a, b) = s(c(a), b)

2nd approach: R implications

conventional: a ⇒ b identical to $\max\{ x \in \mathbb{B} : a \wedge x \leq b \}$

fuzzy: Imp(a, b) = $\max\{ x \in [0,1] : t(a, x) \leq b \}$

3rd approach: QL implications

conventional: a ⇒ b identical to $\bar{a} \vee b \equiv \bar{a} \vee (a \wedge b)$ law of absorption

fuzzy: Imp(a, b) = s(c(a), t(a, b)) (dual tripel ?)

example: S implication $\text{Imp}(a, b) = s(c_s(a), b)$ (c_s : std. complement)

1. Kleene-Dienes implication

$$s(a, b) = \max\{a, b\} \quad (\text{standard}) \quad \text{Imp}(a, b) = \max\{1-a, b\}$$

2. Reichenbach implication

$$s(a, b) = a + b - ab \quad (\text{algebraic sum}) \quad \text{Imp}(a, b) = 1 - a + ab$$

3. Łukasiewicz implication

$$s(a, b) = \min\{1, a + b\} \quad (\text{bounded sum}) \quad \text{Imp}(a, b) = \min\{1, 1 - a + b\}$$

example: R implicationen $\text{Imp}(a, b) = \max\{x \in [0, 1] : t(a, x) \leq b\}$

1. Gödel implication

$$t(a, b) = \min\{a, b\} \quad (\text{std.}) \quad \text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ b & , \text{ else} \end{cases}$$

2. Goguen implication

$$t(a, b) = ab \quad (\text{algeb. product}) \quad \text{Imp}(a, b) = \begin{cases} 1 & , \text{ if } a \leq b \\ \frac{b}{a} & , \text{ else} \end{cases}$$

3. Łukasiewicz implication

$$t(a, b) = \max\{0, a + b - 1\} \quad (\text{bounded diff.}) \quad \text{Imp}(a, b) = \min\{1, 1 - a + b\}$$

example: QL implication $\text{Imp}(a, b) = s(c(a), t(a, b))$

1. Zadeh implication

$$t(a, b) = \min\{a, b\} \quad (\text{std.}) \quad \text{Imp}(a, b) = \max\{1 - a, \min\{a, b\}\}$$

$$s(a, b) = \max\{a, b\} \quad (\text{std.})$$

2. „NN“ implication © (Klir/Yuan 1994)

$$t(a, b) = ab \quad (\text{algebr. prd.}) \quad \text{Imp}(a, b) = 1 - a + a^2b$$

$$s(a, b) = a + b - ab \quad (\text{algebr. sum})$$

3. Kleene-Dienes implication

$$t(a, b) = \max\{0, a + b - 1\} \quad (\text{bounded diff.}) \quad \text{Imp}(a, b) = \max\{1 - a, b\}$$

$$s(a, b) = \min\{1, a + b\} \quad (\text{bounded sum})$$

axioms for fuzzy implications

1. $a \leq b$ implies $\text{Imp}(a, x) \geq \text{Imp}(b, x)$ monotone in 1st argument
2. $a \leq b$ implies $\text{Imp}(x, a) \leq \text{Imp}(x, b)$ monotone in 2nd argument
3. $\text{Imp}(0, a) = 1$ dominance of falseness
4. $\text{Imp}(1, b) = b$ neutrality of trueness
5. $\text{Imp}(a, a) = 1$ identity
6. $\text{Imp}(a, \text{Imp}(b, x)) = \text{Imp}(b, \text{Imp}(a, x))$ exchange property
7. $\text{Imp}(a, b) = 1$ iff $a \leq b$ boundary condition
8. $\text{Imp}(a, b) = \text{Imp}(c(b), c(a))$ contraposition
9. $\text{Imp}(\cdot, \cdot)$ is continuous continuity

characterization of fuzzy implication

Theorem:

Imp: $[0,1] \times [0,1] \rightarrow [0,1]$ satisfies axioms 1-9 for fuzzy implications for a certain fuzzy complement $c(\cdot) \Leftrightarrow$

\exists strictly monotone increasing, continuous function $f: [0,1] \rightarrow [0, \infty)$ with

- $f(0) = 0$
- $\forall a, b \in [0,1]: \text{Imp}(a, b) = f^{-1}(f(1) - f(a) + f(b))$
- $\forall a \in [0,1]: c(a) = f^{-1}(f(1) - f(a))$

Proof: Smets & Magrez (1987). ■

examples: (in tutorial)

choosing an „appropriate“ fuzzy implication ...

apt quotation: (Klir & Yuan 1995, p. 312)

„To select an appropriate fuzzy implication for approximate reasoning under each particular situation is a difficult problem.“

guideline:

GMP, GMT, GHS should be compatible with MP, MT, HS

for fuzzy implication in calculations with relations:

$$B(y) = \sup \{ t(A(x), \text{Imp}(A(x), B(y))) : x \in \mathcal{X} \}$$

example:

Gödel implication for t-norm = bounded difference