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Computational Intelligence

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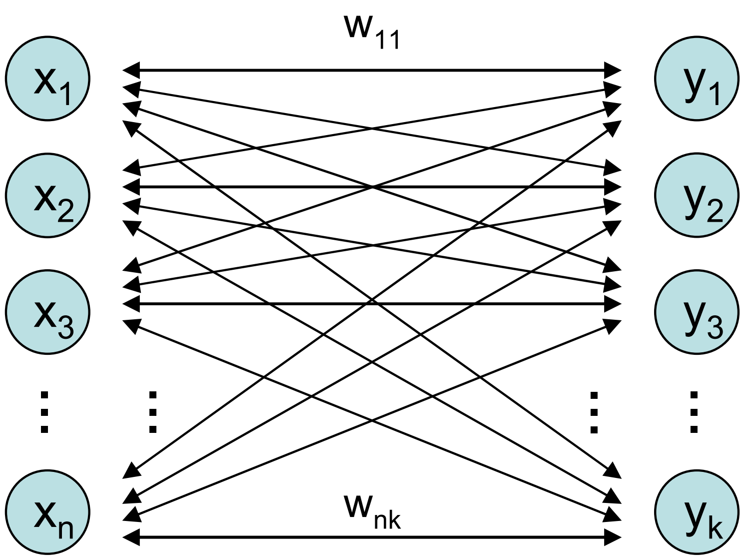
Plan for Today

- Bidirectional Associative Memory (BAM)
  - Fixed Points
  - Concept of Energy Function
  - Stable States = Minimizers of Energy Function
- Hopfield Network
  - Convergence
  - Application to Combinatorial Optimization

Network Model



- fully connected
- bidirectional edges
- synchonized:

step t : data flow from x to y
step t + 1 : data flow from y to x

x, y : row vectors

W : weight matrix

W' : transpose of W

bipolar inputs $\in \{-1,+1\}$

start: $y^{(0)} = \text{sgn}(x^{(0)} W)$

$x^{(1)} = \text{sgn}(y^{(0)} W')$

$y^{(1)} = \text{sgn}(x^{(1)} W)$

$x^{(2)} = \text{sgn}(y^{(1)} W')$

...

Fixed Points

Definition

(x, y) is ***fixed point*** of BAM iff $y = \text{sgn}(x\,W)$ and $x' = \text{sgn}(W\,y')$. □

Set $W = x'\,y$. (note: x is row vector)

$$y = \text{sgn}(\,x\,W\,) = \text{sgn}(\,x\,(x'\,y)\,) = \text{sgn}(\,(x\,x')\,y) = \text{sgn}(\,\underbrace{\|\,x\,\|^2}_{>0 \text{ (does not alter sign)}}\,y) = y$$

$$x' = \text{sgn}(\,W\,y') = \text{sgn}(\,(x'y)\,y'\,) = \text{sgn}(\,x'\,(y\,y')\,) = \text{sgn}(\,x'\,\underbrace{\|\,y\,\|^2}_{>0 \text{ (does not alter sign)}}\,) = x'$$

Theorem: If $W = x'y$ then (x,y) is fixed point of BAM. □

Concept of Energy Function

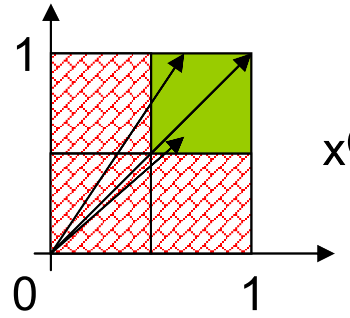
given: BAM with W = x'y $\Rightarrow$ (x,y) is stable state of BAM

starting point $x^{(0)}$ $\Rightarrow$ $y^{(0)} = \text{sgn}(x^{(0)} W)$

$\Rightarrow$ excitation $e' = W (y^{(0)})'$

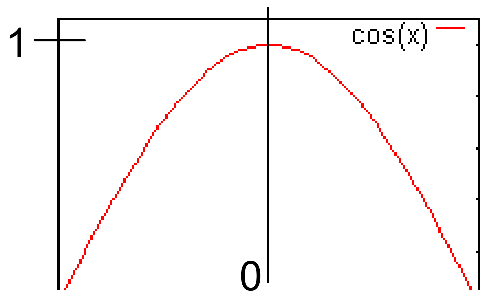
$\Rightarrow$ if $\text{sign}(e') = x^{(0)}$ then $(x^{(0)}, y^{(0)})$ stable state

small angle between e' and $x^{(0)}$ $\Leftarrow$ true if e' *close* to $x^{(0)}$



$x^{(0)} = (1, 1)$

recall: $$\frac{ab'}{||a|| \cdot ||b||} = \cos \angle(a, b)$$



small angle $\alpha \Rightarrow$ large $\cos(\alpha)$

Concept of Energy Function

<u>required:</u>

small angle between $e' = W\, y^{(0)}{}'$ and $x^{(0)}$

$\Rightarrow$ larger cosine of angle indicates greater similarity of vectors

$\Rightarrow$ $\forall e'$ of equal size: try to maximize $x^{(0)} e' = \underbrace{\| x^{(0)} \|}_{\text{fixed}} \cdot \underbrace{\| e \|}_{\text{fixed}} \cdot \underbrace{\cos \angle (x^{(0)}, e)}_{\to \text{ max!}}$

$\Rightarrow$ maximize $x^{(0)} e' = x^{(0)}\, W\, y^{(0)}{}'$

$\Rightarrow$ identical to minimize $-x^{(0)}\, W\, y^{(0)}{}'$

Definition

Energy function of BAM at iteration t is $E(x^{(t)} , y^{(t)}) = -\frac{1}{2}\, x^{(t)}\, W\, y^{(t)}{}'$ □

Stable States

Theorem

An asynchronous BAM with arbitrary weight matrix W reaches steady state in a finite number of updates.

Proof:

$$E(x,y) = -\frac{1}{2} x W y' = \begin{cases} -\frac{1}{2} x (W y') = -\frac{1}{2} x b' = -\frac{1}{2} \sum_{i=1}^{n} b_i \, x_i \\ -\frac{1}{2} (x W) y' = -\frac{1}{2} a y' = -\frac{1}{2} \sum_{i=1}^{k} a_i \, y_i \end{cases}$$

excitations (b_i, a_i)

BAM asynchronous ⇒ select neuron at random from left or right layer, compute its excitation and change state if necessary (states of other neurons not affected) neuron i of left layer has changed $\Rightarrow$ $\text{sgn}(x_i) \neq \text{sgn}(b_i)$

$\Rightarrow$ x_i was updated to $\tilde{x}_i = -x_i$

$$E(x,y) - E(\tilde{x},y) = -\frac{1}{2}\, \underbrace{b_i\,(x_i - \tilde{x}_i)}_{< 0} > 0$$

x_i	b_i	$x_i - \tilde{x}_i$
-1	> 0	< 0
+1	< 0	> 0

use analogous argumentation if neuron of right layer has changed

$\Rightarrow$ every update (change of state) decreases energy function

$\Rightarrow$ since number of different bipolar vectors is finite
update stops after finite #updates

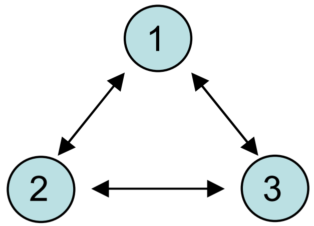
remark: dynamics of BAM get stable in local minimum of energy function!

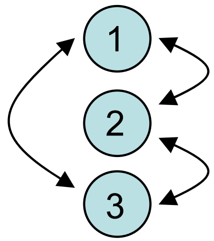
q.e.d.

special case of BAM but proposed earlier (1982)

characterization:

- neurons preserve state until selected at random for update
- n neurons fully connected
- symmetric weight matrix
- no self-loops (→ zero main diagonal entries)
- thresholds θ , neuron i fires if excitations larger than θ_i





transition: select index k at random, new state is $\tilde{x} = \text{sgn}(\, x\, W - \theta\,)$

where $\tilde{x} = (x_1, \ldots, x_{k-1}, \tilde{x}_k, x_{k+1}, \ldots, x_n)$

energy of state x is $E(x) \;=\; -\frac{1}{2}\, xWx' + \theta\, x'$

Theorem:

Hopfield network converges to local minimum of energy function after a finite number of updates. □

Proof: assume that x_k has been updated $\Rightarrow \tilde{x}_k = -x_k$ and $\tilde{x}_i = x_i$ for $i \neq k$

$$E(x) - E(\tilde{x}) = -\tfrac{1}{2} x W x' + \theta\, x' + \tfrac{1}{2} \tilde{x} W \tilde{x}' - \theta\, \tilde{x}'$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, x_i\, x_j + \sum_{i=1}^{n} \theta_i\, x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, \tilde{x}_i\, \tilde{x}_j - \sum_{i=1}^{n} \theta_i\, \tilde{x}_i$$

$$= -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij}\, (x_i\, x_j - \tilde{x}_i\, \tilde{x}_j) + \sum_{i=1}^{n} \theta_i \underbrace{(x_i - \tilde{x}_i)}_{=0 \text{ if } i \neq k}$$

$$= -\frac{1}{2} \sum_{\substack{i=1 \\ i \neq k}}^{n} \sum_{j=1}^{n} w_{ij}\, (x_i\, x_j - \underbrace{\tilde{x}_i}_{x_i}\, \tilde{x}_j) - \frac{1}{2} \sum_{j=1}^{n} \underbrace{w_{kj}}_{0 \text{ if } j = k} (x_k\, x_j - \tilde{x}_k\, \underbrace{\tilde{x}_j}_{x_j \text{ if } j \neq k}) + \theta_k\, (x_k - \tilde{x}_k)$$

$$= -\frac{1}{2}\sum_{\substack{i=1\\i\neq k}}^{n}\sum_{j=1}^{n} w_{ij}\,x_i\,\underbrace{(x_j - \tilde{x}_j)}_{=0 \text{ if } j \neq k} - \frac{1}{2}\sum_{\substack{j=1\\j\neq k}}^{n} w_{kj}\,x_j\,(x_k - \tilde{x}_k) + \theta_k\,(x_k - \tilde{x}_k)$$

$$= -\frac{1}{2}\sum_{\substack{i=1\\i\neq k}}^{n} w_{ik}\,x_i\,(x_k - \tilde{x}_k) - \frac{1}{2}\sum_{\substack{j=1\\j\neq k}}^{n} w_{kj}\,x_j\,(x_k - \tilde{x}_k) + \theta_k\,(x_k - \tilde{x}_k)$$

(rename j to i, recall W = W‘, $w_{kk} = 0$)

$$= -\sum_{i=1}^{n} w_{ik}\,x_i\,(x_k - \tilde{x}_k) + \theta_k\,(x_k - \tilde{x}_k)$$

$$= \underbrace{-(x_k - \tilde{x}_k)\left[\underbrace{\sum_{i=1}^{n} w_{ik}\,x_i}_{\text{excitation } e_k} - \theta_k\right]}_{> 0 \text{ if } x_k < 0 \text{ and vice versa}} > 0$$

since:

x_k	$x_k - \tilde{x}_k$	$e_k - \theta_k$	ΔE
$+1$	> 0	< 0	> 0
-1	< 0	> 0	> 0

q.e.d.

Application to Combinatorial Optimization

Idea:

- transform combinatorial optimization problem as objective function with $x \in \{-1,+1\}^n$
- rearrange objective function to look like a Hopfield energy function
- extract weights W and thresholds θ from this energy function
- initialize a Hopfield net with these parameters W and θ
- run the Hopfield net until reaching stable state (= local minimizer of energy function)
- stable state is local minimizer of combinatorial optimization problem