Appendix A

Catalogue of Problems

The catalogue is divided into three groups of test problems corresponding to the three divisions of the numerical strategy comparison. The optimization problems are all formulated as minimum problems with a specified objective function $F(x)$ and solution x^*. For the second set of problems, the initial conditions $x^{(0)}$ are also given. Occasionally, further local minima and other stationary points of the objective function are also indicated. Inequality constraints are formulated such that the constraint functions $G_j(x)$ are all greater than zero within the allowed or feasible region. If a solution lies on the edge of the feasible region, then the active constraints are mentioned. The values of these constraint functions must be just equal to zero at the optimum. Where possible the structure of the minimum problem is depicted geometrically by means of a two dimensional contour diagram with lines $F(x_1, x_2) = const.$ and as a three dimensional picture in which values of $F(x_1, x_2)$ are plotted as elevation over the (x_1, x_2) plane. Additionally, the values of the objective function on the contour lines are specified. Constraints are shown as bold lines in the contour diagrams. In the 3D plots the objective function is mostly floored to minimal values within non-feasible regions. In some cases there is a brief mention of any especially characteristic behavior shown by individual strategies during their iterative search for the minimum.

A.1 Test Problems for the First Part of the Strategy Comparison

Problem 1.1 (sphere model)

Objective function:

$$F(x) = \sum_{i=1}^{n} x_i^2$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

For $n = 2$ a contour diagram as well as a 3D plot are sketched under Problem 2.17. For this, the simplest of all quadratic problems, none of the strategies fails.

Problem 1.2

Objective function:

$$F(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{i} x_j \right)^2$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

A contour diagram as well as a 3D plot for $n = 2$ are given under Problem 2.9. The objective function of this true quadratic minimum problem can be written in matrix notation as:

$$F(x) = x^T A\, x$$

The $n \times n$ matrix of coefficients A is symmetric and positive-definite. According to Schwarz, Rutishauser, and Stiefel (1968) its condition number K is a measure of the numerical difficulty of the problem. Among other definitions, that of Todd (1949) is useful, namely:

$$K = \frac{\lambda_{max}}{\lambda_{min}} = \frac{a_{max}^2}{a_{min}^2}$$

where

$$\lambda_{max} = \max_i \{|\lambda_i|\,, i = 1(1)n\}$$

and similarly for λ_{min}. The λ_i are the eigenvalues of the matrix A, and the a_i are the lengths of the semi-axes of an n-dimensional elliptic contour surface $F(x) = const.$ Condition numbers for the present matrix

$$A = (a_{ij}) = \begin{bmatrix} n & & n-1 & & n-2 & \dots & n-j+1 & \dots & 1 \\ n-1 & & n-1 & & n-2 & \dots & n-j+1 & \dots & 1 \\ n-2 & & n-2 & & n-2 & \dots & n-j+1 & \dots & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \\ n-i+1 & \dots & n-i+1 & \dots & n-i+1 & \dots & & \dots & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \\ 1 & & 1 & & 1 & \dots & 1 & \dots & 1 \end{bmatrix}$$

were calculated for various values of n by means of an algorithm of Greenstadt (1967b), which uses the Jacobi method of diagonalization. As can be seen from the following table, K increases with the number of variables as $O(n^2)$.

n	K	K/n^2
1	1	1
2	6.85	1.71
3	16.4	1.82
6	64.9	1.80
10	175	1.75
20	678	1.69
30	1500	1.67
60	5930	1.65
100	16400	1.64

Not all the search methods achieved the required accuracy. For many variables the coordinate strategies and the complex method of Box terminated the search prematurely. Powell's method of conjugate gradients even got stuck without the termination criterion taking effect.

A.2 Test Problems for the Second Part of the Strategy Comparison

Problem 2.1 after Beale (1958)

Objective function:

$$F(x) = [1.5 - x_1\,(1 - x_2)]^2 + \left[2.25 - x_1\,(1 - x_2^2)\right]^2 + \left[2.625 - x_1\,(1 - x_2^3)\right]^2$$

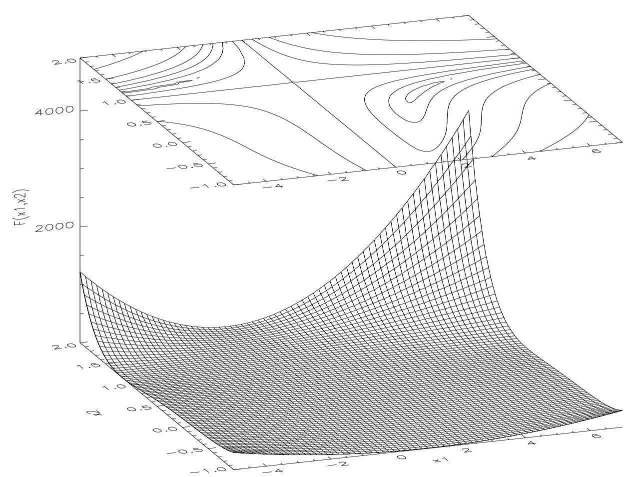

Figure A.1: Graphical representation of Problem 2.1
$F(x) = /0.1, 1, 4, \simeq 14.20, 36, 100/$

Minimum:

$$x^* = (3, 0.5)\,, \qquad F(x^*) = 0$$

Besides the strong minimum x^* there is a weak minimum at infinity:

$$x' \to (-\infty, 1)\,, \qquad F(x') \to 0$$

Saddle point:

$$x'' = (0, 1)\,, \qquad F(x'') \simeq 14.20$$

Start:

$$x^{(0)} = (0, 0)\,, \qquad F(x^{(0)}) \simeq 14.20$$

For very large initial step lengths the (1+1) evolution strategy converged once to the weak minimum x'.

Problem 2.2

As Problem 2.1, but with:

Start:

$$x^{(0)} = (0.1, 0.1)\,, \qquad F(x^{(0)}) \simeq 12.99$$

Problem 2.3

Objective function:

$$F(x) = -|x \sin(\sqrt{|x|})|$$

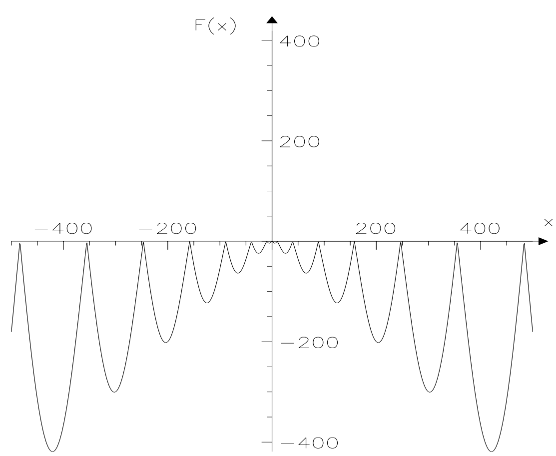

Figure A.2: Diagram $F(x)$ for Problem 2.3

There are infinitely many local minima, the position of which can be specified by a transcendental equation:

$$\sqrt{|x^*|} = 2 \tan\left(\sqrt{|x^*|}\right)$$

For $|x^*| \gg 1$ we have approximately

$$x^* \simeq \pm(\pi\,(0.5+k))^2\,, \qquad \text{for } k = 1, 2, 3, \ldots, \text{integer}$$

and

$$F(x^*) \simeq |x^*|$$

Whereas in reality none of the finite local minima is at the same time a global minimum, the finite word length of the digital computer used together with the system-specific method of evaluating the sine function give rise to an apparent global minimum at

$$x^* = \pm 4.44487453 \cdot 10^{16}$$

$$F(x^*) = -4.44487453 \cdot 10^{16}$$

Counting from the origin it is the $67,108,864$th local minimum in each direction. If x is increased above this value, the objective function value is always set to zero. (Note that this behavior is machine dependent.)

Start:

$$x^{(0)} = 0\,, \qquad F(x^{(0)}) = 0$$

Most strategies located the first or highest local minimum left or right of the starting point (the origin). Depending on the sequence of random numbers, the two membered evolution method found (for example) the 2nd, 9th, and 34th local minimum. Only the (10, 100) evolution strategy almost always reached the apparent global minimum.

Problem 2.4

Objective function:

$$F(x) = \sum_{i=1}^{n} \left[\left(x_1 - x_i^2\right)^2 + (x_i - 1]^2\right)\,, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 1, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 40905$$

Problem 2.5 after Booth (1949)

Objective function:

$$F(x) = (x_1 + 2\,x_2 - 7)^2 + (2\,x_1 + x_2 - 5)^2$$

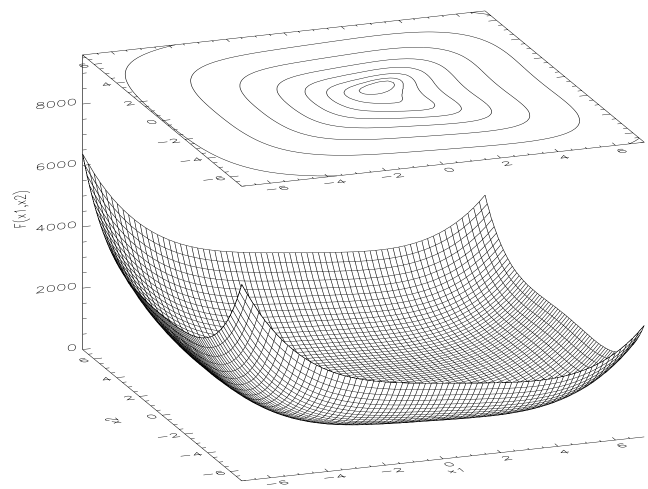

Figure A.3: Graphical representation of Problem 2.4 for $n = 2$
$F(x) = /10^0, 10^1, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7/$

This minimum problem is equivalent to solving the following pair of linear equations:

$$x_1 + 2\,x_2 = 7$$

$$2\,x_1 + x_2 = 5$$

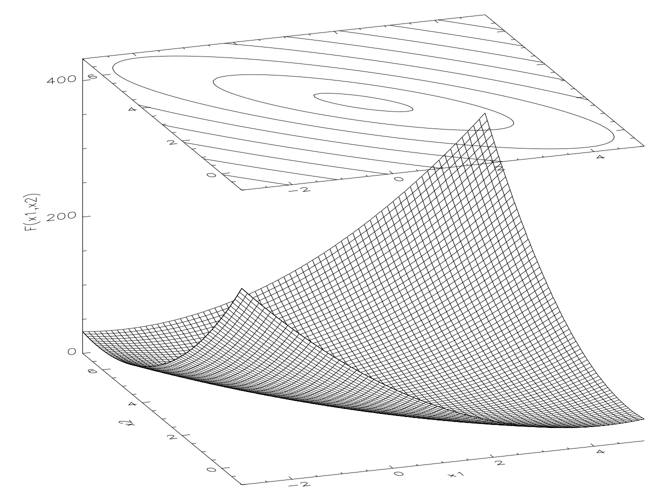

Figure A.4: Graphical representation of Problem 2.5
$F(x) = /1, 9, 25, 49, 81, 121, 169, 225/$

An approach to the latter problem is to determine those values of x_1 and x_2 that minimize the error in the equations. The error is defined here in the sense of a Gaussian approximation as the sum of the squares of the components of the residual vector.

Minimum:

$$x^* = (1,3)\,, \qquad F(x^*) = 0$$

Start:

$$x^{(0)} = (0,0)\,, \qquad F(x^{(0)}) = 74$$

Problem 2.6

Objective function:

$$F(x) = \max\{|x_1 + 2\,x_2 - 7|, |2\,x_1 + x_2 - 5|\}$$

This represents an attempt to solve the previous system of linear equations of Problem 2.5 in the sense of a Tchebycheff approximation. Accordingly, the error is defined as the absolute maximum component of the residual vector.

Minimum:

$$x^* = (1,3)\,, \qquad F(x^*) = 0$$

Start:

$$x^{(0)} = (0,0)\,, \qquad F(x^{(0)}) = 7$$

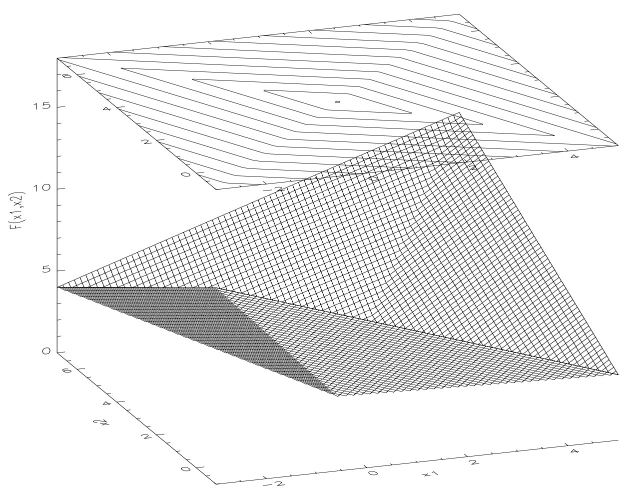

Figure A.5: Graphical representation of Problem 2.6
$F(x) = /1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11/$

Several of the search procedures were unable to find the minimum. They converge to a point on the line $x_1+x_2=4$, which joins together the sharpest corners of the rhombohedral contours. The partial derivatives of the objective function are discontinuous there; in the unit vector directions, parallel to the coordinate axes, no improvement can be made. Besides the coordinate strategies, the methods of Hooke and Jeeves and of Powell are thwarted by this property.

Problem 2.7 after Box (1966)

Objective function:

$$F(x) = \sum_{j=1}^{10} \left(\exp\left(-0.1\, j\, x_1\right) - \exp\left(-0.1\, j\, x_2\right) - x_3 \left[\exp\left(-0.1\, j\right) - \exp\left(-j\right)\right]\right)^2$$

Minima:

$$x^* = (1, 10, 1)\,, \qquad F(x^*) = 0$$

$$x^* = (10, 1, -1)\,, \qquad F(x^*) = 0$$

Besides these two equivalent, strong minima there is a weak minimum along the line

$$x'_1 = x'_2\,, \qquad x'_3 = 0\,, \qquad F(x') = 0$$

Because of the finite computational accuracy the weak minimum is actually broadened into a region:

$$x''_1 \simeq x''_2\,, \qquad x''_3 \simeq 0\,, \qquad F(x'') = 0\,, \qquad \text{if } x_1 \gg 1$$

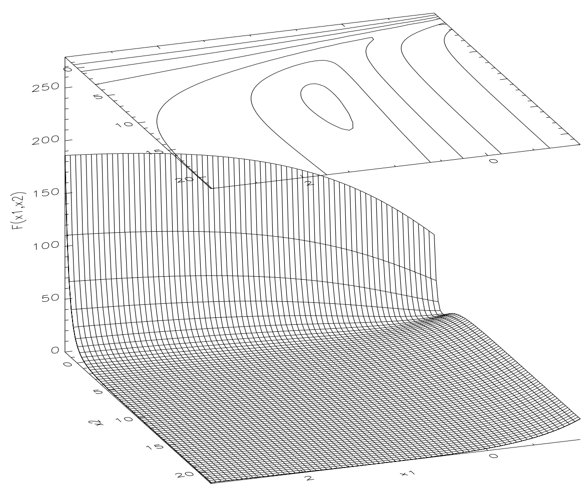

Figure A.6: Graphical representation of Problem 2.7 on the plane $x_3 = 1$, $F(x) = /0.03, 0.3, 1, \simeq 3.064, 10, 30/$

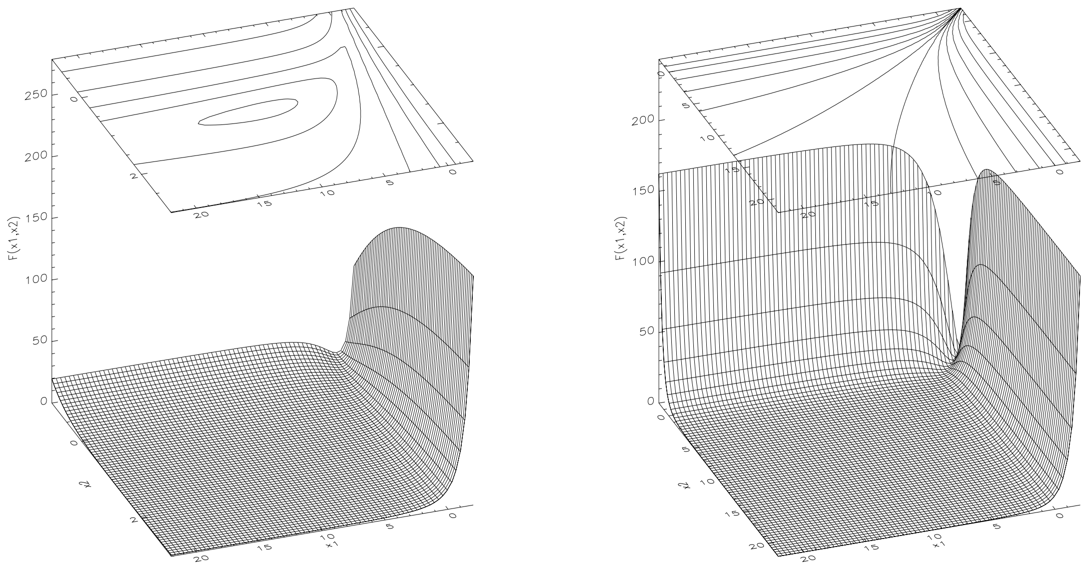

Figure A.7: Graphical representation of Problem 2.7 on the planes
left: $x_3 = 0$, right: $x_3 = -1$,
$F(x) = /0.03, 0.3, 1, \simeq 3.064, 10, 30/$

Start:

$$x^{(0)} = (0, 20, 20)\,, \qquad F(x^{(0)}) \simeq 1022$$

Many strategies only roughly located the first of the strong minima defined above. The evolution strategies tended to converge to the weak minimum, since the minima are at equal values of the objective function. The second strong minimum, which is never referred to in the relevant literature, was sometimes found by the multimembered evolution strategy.

Problem 2.8

As Problem 2.7, but with

Start:

$$x^{(0)} = (0, 10, 20)\,, \qquad F(x^{(0)}) \simeq 1031$$

Problem 2.9

Objective function:

$$F(x) = \sum_{i=1}^{n} (\sum_{j=1}^{i} x_j)^2\,, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Start:

$$x_i^{(0)} = 10\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 5500$$

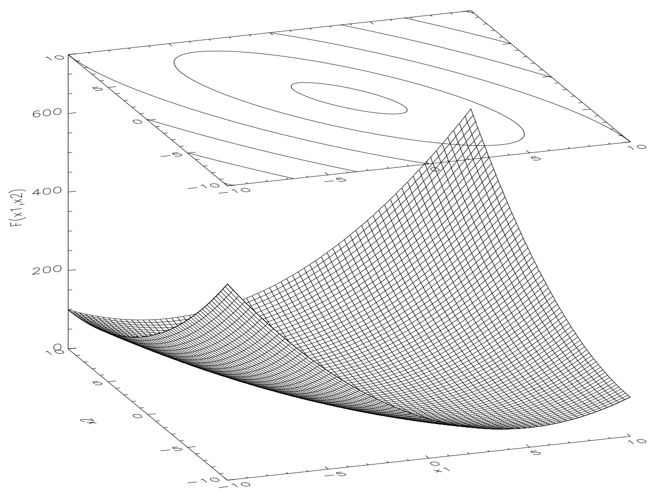

Figure A.8: Graphical representation of Problem 2.9 for $n = 2$
$F(x) = /4, 36, 100, 196, 324, 484/$

Problem 2.10 after Kowalik (1967; see also Kowalik and Morrison, 1968)

Objective function:

$$F(x) = \sum_{i=1}^{11} \left(a_i - \frac{x_1\,(b_i^2 + b_i\,x_2)}{b_i^2 + b_i\,x_3 + x_4} \right)^2$$

Numerical values of the constants a_i and b_i for $i = 1(1)11$ can be taken from the following table:

i	a_i	b_i^{-1}
1	0.1957	0.25
2	0.1947	0.5
3	0.1735	1
4	0.1600	2
5	0.0844	4
6	0.0627	6
7	0.0456	8
8	0.0342	10
9	0.0323	12
10	0.0235	14
11	0.0246	16

In this non-linear fitting problem, formulated as a minimum problem, the free parameters

$a_j, j = 1(1)4$ of a function

$$y(z) = \frac{\alpha_1\,(z^2 + \alpha_2\,z)}{z^2 + \alpha_3\,z + \alpha_4}$$

have to be determined with reference to eleven data points $\{y_i, z_i\}$ such that the error, as measured by the Euclidean norm, is minimized (Gaussian or least squares approximation).
Minimum:

$$x^* \simeq (0.1928, 0.1908, 0.1231, 0.1358)\,, \qquad F(x^*) \simeq 0.0003075$$

Start:

$$x^{(0)} = (0, 0, 0, 0)\,, \qquad F(x^{(0)}) \simeq 0.1484$$

Near the optimum, if the variables are changed in the last decimal place (with respect to the machine accuracy), rounding errors cause the objective function to behave almost stochastically. The multimembered evolution strategy with recombination yields the best solution. It deviates significantly from the optimum solution as defined by Kowalik and Osborne (1968). Since this best value has a quasi-singular nature, it is repeatedly lost by the population of a (10 , 100) evolution strategy, with the result that the termination criterion of the search sometimes only takes effect after a long time if at all.

Problem 2.11

As Problem 2.10, but with:

Start:

$$x^{(0)} = (0.25, 0.39, 0.415, 0.39)\,, \qquad F(x^{(0)}) \simeq 0.005316$$

Problem 2.12

As Problem 2.10, but with:

Start:

$$x^{(0)} = (0.25, 0.40, 0.40, 0.40)\,, \qquad F(x^{(0)}) \simeq 0.005566$$

Problem 2.13 after Fletcher and Powell (1963)

Objective function:

$$F(x) = \sum_{i=1}^{n} (A_i - B_i(x))^2\,, \qquad \text{for } n = 5$$

where

$$\left.\begin{aligned} A_i &= \sum_{j=1}^{n} (a_{ij}\sin\alpha_j + b_{ij}\cos\alpha_j) \\ B_i(x) &= \sum_{j=1}^{n} (a_{ij}\sin x_j + b_{ij}\cos x_j) \end{aligned}\right\} \text{ for } i = 1(1)n$$

a_{ij} and b_{ij} are integer random numbers in the range $[-100, 100]$, and α_i are random numbers in the range $[-\pi, \pi]$. A minimum of this problem is simultaneously a solution of the equivalent system of n simultaneous non-linear (transcendental) equations:

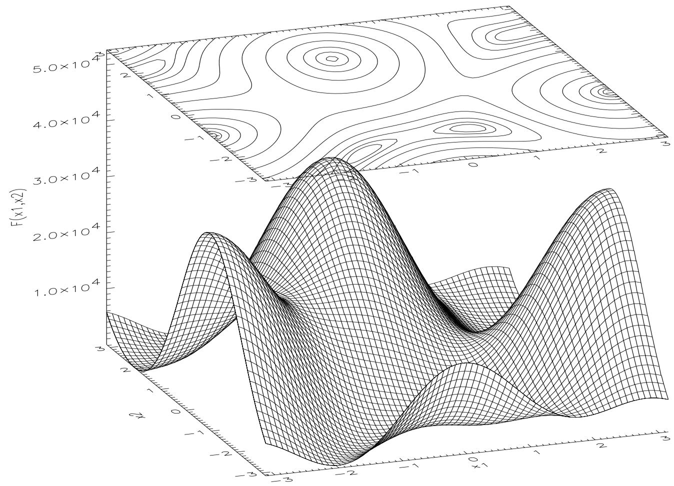

Figure A.9: Graphical representation of Problem 2.13 for $n = 2$.
$a_{11} = -2, \ a_{12} = 27, \ a_{21} = -70, \ a_{22} = -48$
$b_{11} = -76, \ b_{12} = -51, \ b_{21} = 63, \ b_{22} = -50$
$\alpha_1 = -3.0882, \ \alpha_2 = 2.0559$
$F(x) = /238.864, 581.372, 1403.11, 3283.14, 7153.45,$
$13635.3, 21479.6, 27961.4, 31831.7, 33711.8, 34533.5/$

$$\sum_{j=1}^{n} (a_{ij} \sin x_j + b_{ij} \cos x_j) = A_i \,, \qquad \text{for } i = 1(1)n$$

The solution is again approximated in the least squares sense.

Minimum:

$$x_i^* = \alpha_i, \qquad \text{for } i = 1(1)n \,, \qquad F(x^*) = 0$$

Because the trigonometric functions are multivalued there are infinitely many equivalent minima (real solutions of the system of equations), of which up to 2^n lie in the interval

$$\alpha_i - \pi \le x_i \le \alpha_i + \pi \,, \qquad \text{for } i = 1(1)n$$

Start:

$$x_i^{(0)} = \alpha_i + \delta_i \,, \qquad \text{for } i = 1(1)n$$

where δ_i are random numbers in the range $[-\pi/10, \ \pi/10]$. To provide the same conditions for all the search methods the same sequence of random numbers was used in each case, and hence

$$F(x^{(0)}) \simeq 1182$$

Because of the proximity of the starting point to the one solution, $x_i^* = \alpha_i$ for $i = 1(1)n$, all the strategies approached this minimum only.

Problem 2.14 after Powell (1962)

Objective function:

$$F(x) = (x_1 + 10\,x_2)^2 + 5\,(x_3 - x_4)^2 + (x_2 - 2\,x_3)^4 + 10\,(x_1 - x_4)^4$$

Minimum:

$$x^* = (0,0,0,0)\,, \qquad F(x^*) = 0$$

Start:

$$x^{(0)} = (3,-1,0,1)\,, \qquad F(x^{(0)}) = 215$$

The matrix of second partial derivatives of the objective function goes singular at the minimum. Thus it is not surprising that a quasi-Newton method like the variable metric method of Davidon, Fletcher, and Powell (applied here in Stewart's derivative-free form) got stuck a long way from the minimum. Geometrically speaking, there is a valley which becomes extremely narrow as it approaches the minimum. The evolution strategies therefore ended up by converging very slowly with a minimum step length, and the search had to be terminated for reasons of time.

Problem 2.15

As Problem 2.14, except:

Start:

$$x^{(0)} = (1,2,3,4)\,, \qquad F(x^{(0)}) = 1512$$

Problem 2.16 after Leon (1966a)

Objective function:

$$F(x) = 100\,(x_2 - x_1^3)^2 + (x_1 - 1)^2$$

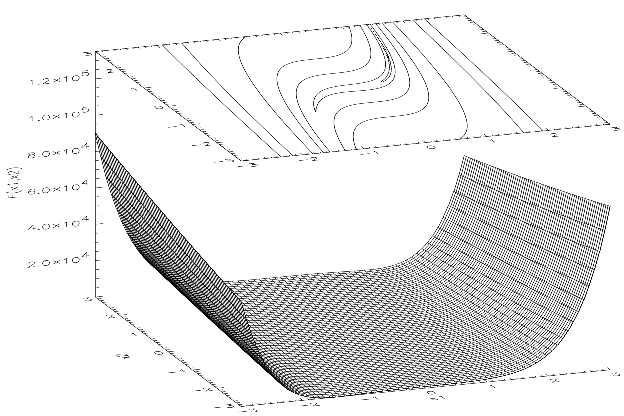

Figure A.10: Graphical representation of Problem 2.16
$F(x) = /0.25, 4, 64, 250, 1000, 5000, 10000/$

Minimum:

$$x^* = (1,1)\,, \qquad F(x^*) = 0$$

Start:

$$x^{(0)} = (-1.2,1)\,, \qquad F(x^{(0)}) \simeq 749.$$

Problem 2.17 (sphere model)

Objective function:

$$F(x) = \sum_{i=1}^{n} x_i^2, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 0, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 500$$

Problem 2.18 after Matyas (1965)

Objective function:

$$F(x) = 0.26\,(x_1^2 + x_2^2) - 0.48\,x_1\,x_2$$

Minimum:

$$x^* = (0,0)\,, \qquad F(x^*) = 0$$

Start:

$$x^{(0)} = (15,30)\,, \qquad F(x^{(0)}) = 76.5$$

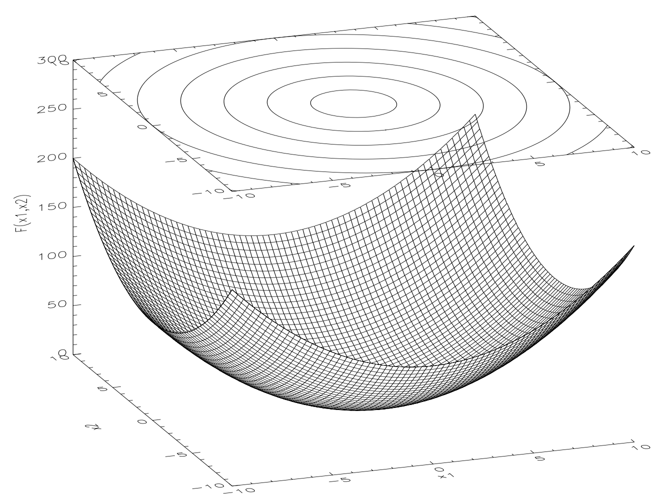

Figure A.11: Graphical representation of Problem 2.17 for $n = 2$
$F(x) = /4, 16, 36, 64, 100, 144, 196/$

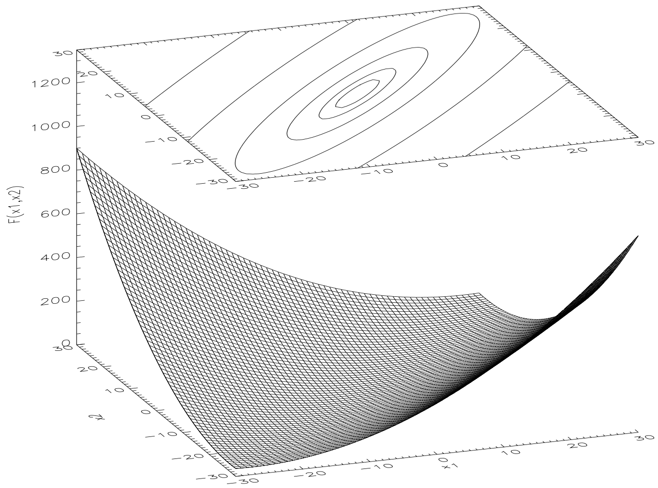

Figure A.12: Graphical representation of Problem 2.18
$F(x) = /1, 3, 10, 30, 100, 300/$

The coordinate strategies terminated the search prematurely because of the lower bounds on the step lengths (as determined by the machine), which precluded making any more successful line searches in the coordinate directions.

Problem 2.19 by Wood (after Colville, 1968)

Objective function:

$$F(x) = 100\,\left(x_1 - x_2^2\right)^2 + (x_2 - 1)^2 + 90\,\left(x_3 - x_4^2\right)^2 + (x_4 - 1)^2$$

$$+\,10.1\,\left[(x_1 - 1)^2 + (x_3 - 1)^2\right] + 19.8\,(x_1 - 1)(x_3 - 1)$$

Minimum:

$$x^* = (1, 1, 1, 1)\,, \qquad F(x^*) = 0$$

There is another stationary point near

$$x' \simeq (1, -1, 1, -1)\,, \qquad F(x') \simeq 8$$

According to Himmelblau (1972a,b) there are still further local minima.

Start:

$$x^{(0)} = (-1, -3, -1, -3)\,, \qquad F(x^{(0)}) = 19192$$

A very narrow valley appears to run from the stationary point x' to the minimum. All the coordinate strategies together with the methods of Hooke and Jeeves and of Powell ended the search in this region.

Problem 2.20

Objective function:

$$F(x) = \sum_{i=1}^{n} |x_i|, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 0, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 50$$

Problem 2.21

Objective function:

$$F(x) = \max_i \{|x_i|,\, i = 1(1)n\}, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 0, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 10$$

Since the starting point is at a corner of the cubic contour surface, none of the coordinate strategies could find a point with a lower value of the objective function. The method of

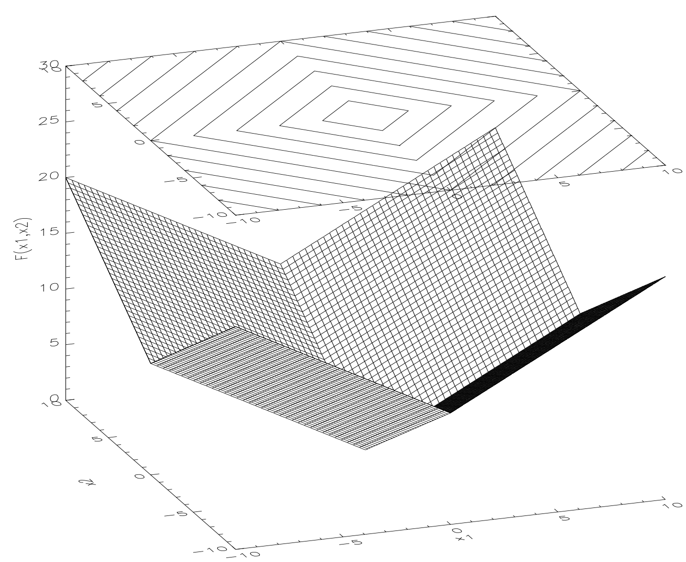

Figure A.13: Graphical representation of Problem 2.20 for $n = 2$
$F(x) = /2, 4, 6, 8, 10, 12, 14, 16, 18, 20/$

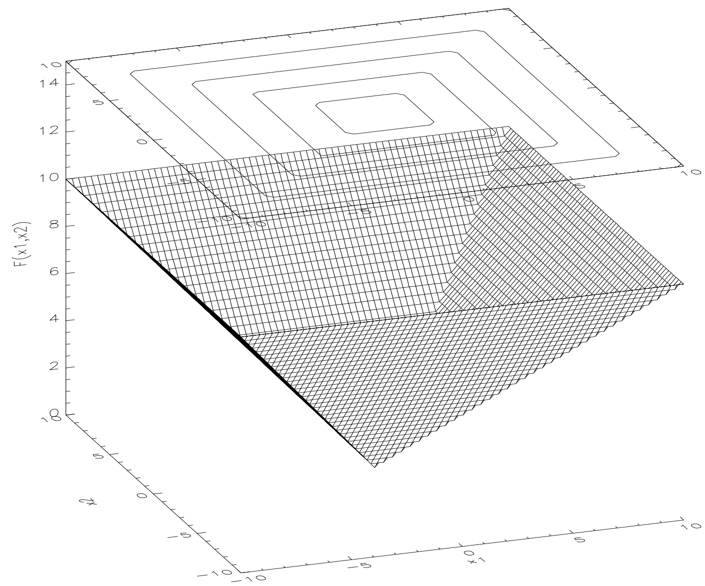

Figure A.14: Graphical representation of Problem 2.21 for $n = 2$
$F(x) = /2, 4, 6, 8, 10/$

Powell also ended the search without making any significant improvement on the initial condition. Both the simplex method of Nelder and Mead and the complex method of Box also had trouble in the minimum search; in their cases the initially constructed simplex or complex collapsed long before reaching the minimum, again near one of the corners.

Problem 2.22

Objective function:

$$F(x) = \sum_{i=1}^{n} |x_i| + \prod_{i=1}^{n} |x_i|, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 0, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 100050$$

The simplex and complex methods did not find the minimum. As in the previous Problem 2.21, this is due to the sharply pointed corners of the contours. The variable metric strategy also finally got stuck at one of these corners and converged no further. In this case the discontinuity in the partial derivatives of the objective function at the corners is to blame for its failure.

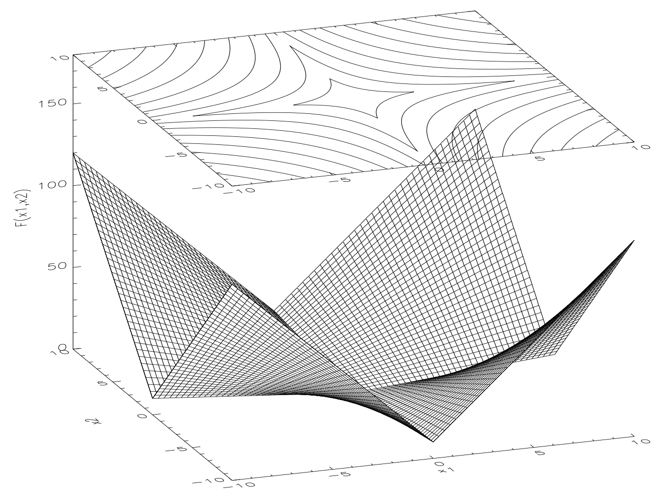

Figure A.15: Graphical representation of Problem 2.22 for $n = 2$
$F(x) = /3, 8, 15, 24, 35, 48, 63, 80, 99/$

Problem 2.23

Objective function:

$$F(x) = \sum_{i=1}^{n} x_i^{10}, \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 0, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

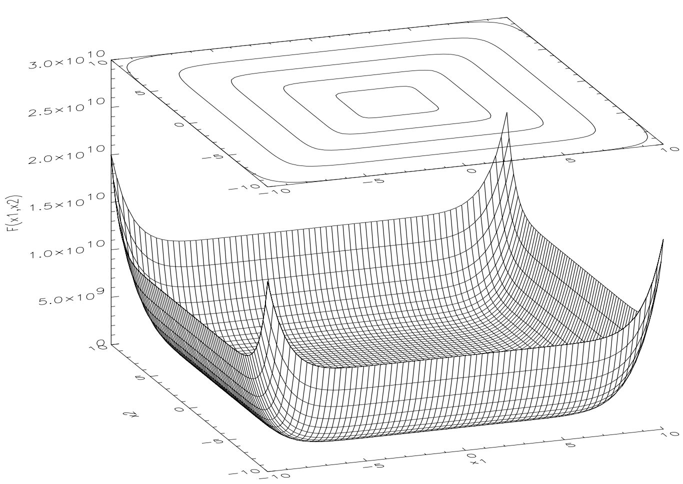

Figure A.16: Graphical representation of Problem 2.23 for $n = 2$
$F(x) = /2^{10}, 4^{10}, 6^{10}, 8^{10}, 10^{10}/$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 5 \cdot 10^{10}$$

Only the two strategies that have a quadratic internal model of the objective function, namely the variable metric and conjugate directions methods, failed to converge, because the function $F(x)$ is of much higher (10th) order.

Problem 2.24 after Rosenbrock (1960)

Objective function:

$$F(x) = 100\,(x_2 - x_1^2)^2 + (x_1 - 1)^2$$

Minimum:

$$x^* = (1, 1)\,, \qquad F(x^*) = 0$$

Start:

$$x^{(0)} = (-1.2, 1)\,, \qquad F(x^{(0)}) = 24.2$$

Problem 2.25

Objective function:

$$F(x) = \sum_{i=2}^{n} \left[\left(x_1 - x_i^2\right)^2 + (x_i - 1)^2 \right], \qquad \text{for } n = 5$$

Minimum:

$$x_i^* = 1, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

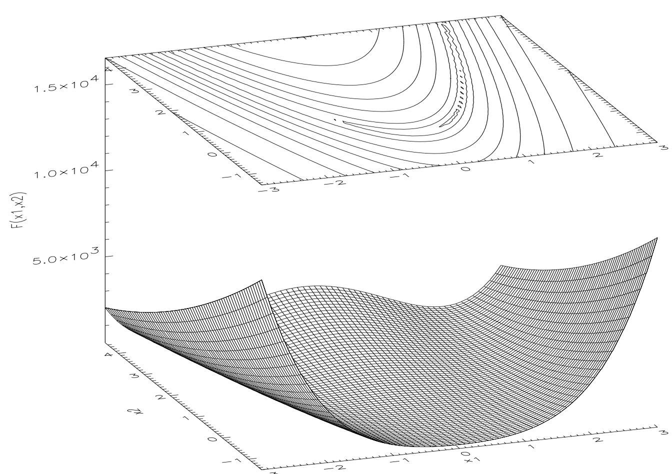

Figure A.17: Graphical representation of Problem 2.24
$F(x) = /0.5, 4, 20, 100, 250, 500, 1000, 2000, 5000/$

Start:

$$x_i^{(0)} = 10, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 32724$$

For $n = 2$ this becomes nearly the same as Problem 2.24.

Problem 2.26

Objective function:

$$F(x) = -x \sin(\sqrt{|x|})$$

This problem is the same as Problem 2.3 except for the modulus. The difference has the effect that the neighboring minima are further apart here. The positions of the local minima and maxima are described under Problem 2.3.

Start:

$$x^{(0)} = 0\,, \qquad F(x^{(0)}) = 0$$

Again, only the multimembered evolution strategy converged to the apparent global minimum; all the other methods only converged to the first (nearest) local minimum.

Problem 2.27 after Zettl (1970)

Objective function:

$$F(x) = (x_1^2 + x_2^2 - 2\,x_1)^2 + 0.25\,x_1$$

Minimum:

$$x^* \simeq (-0.02990, 0)\,, \qquad F(x^*) \simeq -0.003791$$

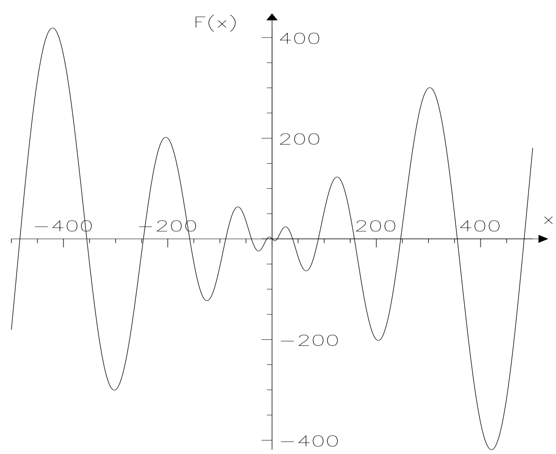

Figure A.18: Diagram $F(x)$ of Problem 2.26

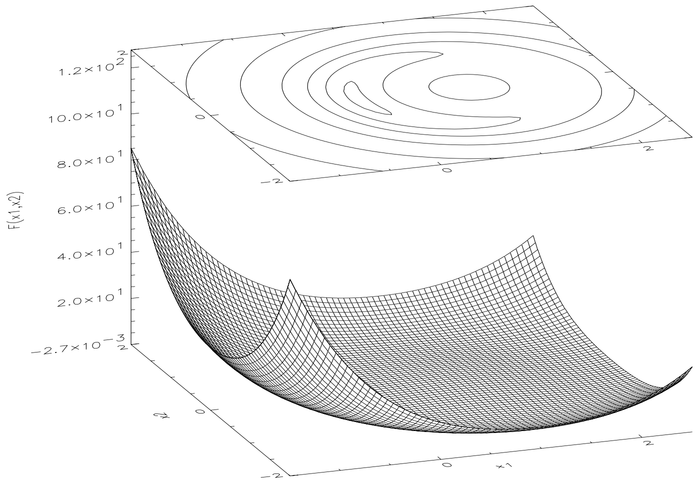

Figure A.19: Graphical representation of Problem 2.27
$F(x) = /0.03, 0.3, 1, 3, 10, 30/$

Because of rounding errors this same objective function value is reached for various pairs of values of x_1, x_2.

Local maximum:

$$x' \simeq (1.063, 0)\,, \qquad F(x') \simeq 1.258$$

Saddle point:

$$x'' \simeq (1.967, 0)\,, \qquad F(x'') \simeq 0.4962$$

Start:

$$x^{(0)} = (2, 0)\,, \qquad F(x^{(0)}) = 0.5$$

Problem 2.28 of Watson (after Kowalik and Osborne, 1968)

Objective function:

$$F(x) = \sum_{i=1}^{30} \left(\sum_{j=1}^{5} \left(j\, a_i^{j-1}\, x_{j+1} \right) - \left[\sum_{j=1}^{6} \left(a_i^{j-1}\, x_j \right) \right]^2 - 1 \right)^2 + x_1^2$$

where

$$a_i = \frac{i-1}{29}$$

The origin of this problem is the approximate solution of the ordinary differential equation

$$\frac{dz}{dy} - z^2 = 1$$

on the interval $0 \leq y \leq 1$ with the boundary condition $z(y=0)=0$. The function sought, $z(y)$, is to be approximated by a polynomial

$$\tilde{z}(c,y)=\sum_{j=1}^{n} c_j\, y^{j-1}$$

In the present case only the first six terms are considered. Suitable values of the polynomial coefficients c_j, $j=1(1)6$, are to be determined. The deviation from the exact solution of the differential equation is measured in the Gaussian sense as the sum of the squares of the errors at $m=30$ argument values y_i, uniformly distributed in the range [0,1]

$$F_1(c)=\sum_{i=1}^{m}\left(\left.\frac{\partial \tilde{z}(c,y)}{\partial y}\right|_{y_i} - \left.\tilde{z}^2(c,y)\right|_{y_i} - 1\right)^2$$

The boundary condition is treated as a second simultaneous equation by means of a similarly constructed term:

$$F_2(c)=\left.\tilde{z}^2(c,y)\right|_{y=0}$$

By inserting the polynomial and redefining the parameters c_i as variables x_i we obtain the objective function $F(x)=F_1(x)+F_2(x)$, the minimum of which is an approximate solution of the parameterized functional problem.

Minimum:

$$x^* \simeq (-0.0158, 1.012, -0.2329, 1.260, -1.513, 0.9928), \qquad F(x^*) \simeq 0.002288$$

Start:

$$x^{(0)}=(0,0,0,0,0,0), \qquad F(x^{(0)})=30$$

Judging by the number of objective function evaluations all the search methods found this a difficult problem to solve. The best solution was provided by the complex strategy.

Problem 2.29 after Beale (1967)

Objective function:

$$F(x)=2\,x_1^2+2\,x_2^2+x_3^2+2\,x_1\,x_2+2\,x_1\,x_3-8\,x_1-6\,x_2-4\,x_3+9$$

Constraints:

$$G_j(x)=x_j \geq 0, \qquad \text{for } j=1(1)3$$

$$G_4(x)=-x_1-x_2-2\,x_3+3 \geq 0$$

Minimum:

$$x^*=\left(\frac{4}{3},\frac{7}{9},\frac{4}{9}\right), \qquad F(x^*)=\frac{1}{9}, \qquad \text{only } G_4 \text{ active, i.e., } G_4(x^*)=0$$

Start:

$$x^{(0)}=(0.1,0.1,0.1), \qquad F(x^{(0)})=7.29$$

Problem 2.30

As Problem 2.3, but with the constraints

$$G_1(x) = -x + 300 \geq 0\,, \qquad G_2(x) = x + 300 \geq 0$$

The introduction of constraints gives rise to two equivalent, global minima at the edge of the feasible region:

Minima:

$$x^* = \pm 300\,, \qquad F(x^*) \simeq -299.7\,, \qquad G_1 \text{ or } G_2 \text{ active}$$

In addition there are five local minima within the feasible region. Here too, the absolute minima were only located by the multimembered evolution strategy.

Problem 2.31

As Problem 2.4, but with constraints:

$$G_j(x) = x_j - 1 \geq 0\,, \qquad \text{for } j = 1(1)n\,, \qquad n = 5$$

Minimum:

$$x_i^* = 1\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0\,, \qquad \text{all } G_j \text{ active}$$

Start:

$$x_i^{(0)} = -10\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 61105$$

The starting point is located outside of the feasible region.

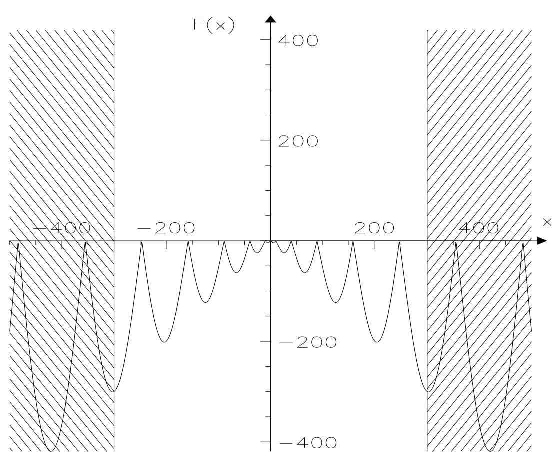

Figure A.20: Diagram $F(x)$ for Problem 2.30

Problem 2.32 after Bracken and McCormick (1970)

Objective function:

$$F(x) = -x_1^2 - x_2^2$$

Constraints:

$$G_j(x) = x_j \geq 0\,, \qquad \text{for } j = 1, 2$$

$$G_3(x) = -x_1 + 1 \geq 0\,, \qquad G_4(x) = -x_1 - 4\,x_2 + 5 \geq 0$$

Minimum:

$$x^* = (1, 1)\,, \qquad F(x^*) = -2\,, \qquad G_3 \text{ and } G_4 \text{ active}$$

Besides this global minimum there is another local one:

$$x' = \left(0, \frac{5}{4}\right)\,, \qquad F(x') = -\frac{25}{16}\,, \qquad G_1 \text{ and } G_4 \text{ active}$$

Start:

$$x^{(0)} = (0, 0)\,, \qquad F(x^{(0)}) = 0$$

All the search methods converged to the global minimum.

Problem 2.33 after Zettl (1970)

As Problems 2.14 and 2.15, but with the constraints:

$$G_j(x) = x_{j+2} - 2 \geq 0\,, \qquad \text{for } j = 1, 2$$

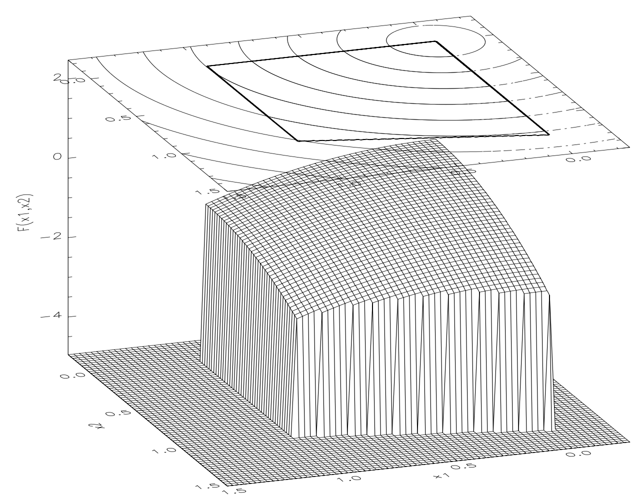

Figure A.21: Graphical representation of Problem 2.32
$F(x) = /0.04, 0.16, 0.36, 0.64, 1.0, 1.44, 1.96, 2.56, 3.24, 4/$

Minimum:

$$x^* = (1.275, 0.6348, 2.0, 2.0)\,, \qquad F(x^*) \simeq 189.1\,, \qquad \text{all } G_j \text{ active}$$

Start:

$$x^{(0)} = (1, 2, 3, 4)\,, \qquad F(x^{(0)}) = 1512$$

The (1+1) evolution strategy only solved the problem very inaccurately. Due to the 1/5 success rule the mutation variances vanish prematurely.

Problem 2.34 after Fletcher and Powell (1963)

Objective function:

$$F(x) = 100\,\left[(x_3 - 10\,\theta)^2 + (R-1)^2\right] + x_3^2$$

where

$$x_1 = R\,\cos\,(2\,\pi\,\theta)$$

$$x_2 = R\,\sin\,(2\,\pi\,\theta)$$

or

$$R = \sqrt{x_1^2 + x_2^2}$$

$$\theta = \begin{cases} \frac{1}{2\,\pi}\,\arctan\,\frac{x_2}{x_1}\,, & \text{if } x_2 \neq 0 \text{ and } x_1 > 0 \\ \frac{1}{2}\,, & \text{if } x_2 = 0 \\ \frac{1}{2\,\pi}\left(\pi + \arctan\,\frac{x_2}{x_1}\right), & \text{if } x_2 \neq 0 \text{ and } x_1 < 0 \end{cases}$$

Constraints:

$$G_1(x) = -x_3 + 7.5 \geq 0\,, \qquad G_2(x) = x_3 + 2.5 \geq 0$$

Minimum:

$$x^* = (1, \simeq 0, 0), \qquad F(x^*) = 0\,, \qquad \text{no constraint is active}$$

The objective function itself has a discontinuity at $x_2 = 0$, right at the minimum sought. Thus x_2 should only be allowed to approach closely to zero. Because of the multivalued trigonometric functions there are infinitely many solutions to the problem, of which only one, however, lies within the feasible region.

Start:

$$x^{(0)} = (-1, 0, 0)\,, \qquad F(x^{(0)}) = 2500$$

Problem 2.35 after Rosenbrock (1960)

Objective function:

$$F(x) = -x_1\,x_2\,x_3$$

Constraints:

$$G_j(x) = x_j \geq 0\,, \qquad \text{for } j = 1(1)3$$

$$G_4(x) = -x_1 - 2\,x_2 - 2\,x_3 + 72 \geq 0$$

The underlying question here was: What dimension should a parcel of maximum volume have, if the sum of its length and transverse circumference is bounded?

Minimum:

$$x^* = (24, 12, 12)\,, \qquad F(x^*) = -3456\,, \qquad G_4 \text{ active}$$

Start:

$$x^{(0)} = (0, 0, 0)\,, \qquad F(x^{(0)}) = 0$$

All variants of the evolution strategy converged only to within the neighborhood of the minimum sought, because in the end only a fraction of all trials were feasible.

Problem 2.36

This is derived from Problem 2.35 by treating the constraint G_4, which is active at the minimum, as an equation, and thereby eliminating one of the free variables. With

$$x_1' + 2\,x_2' + 2\,x_3' = 72$$

we obtain

$$F'(x) = -(72 - 2\,x_2' - 2\,x_3')\,x_2\,x_3$$

or by renumbering of the variables a new objective function:

$$F(x) = -x_1\,x_2\,(72 - 2\,x_1 - 2\,x_2)$$

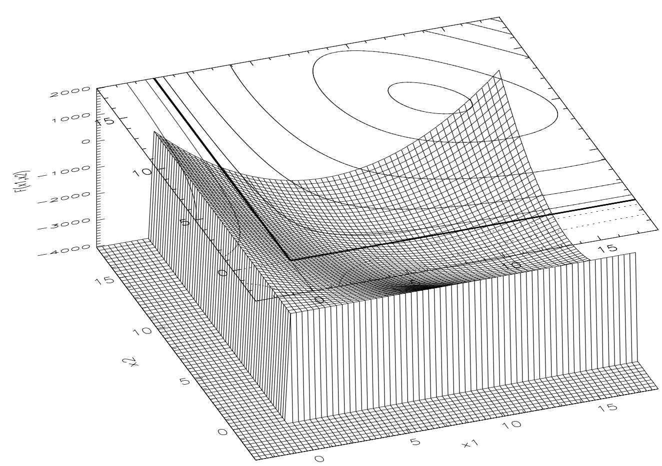

Figure A.22: Graphical representation of Problem 2.36
$F(x) = / -3400, -3000, -2000, -1000, -300, 300, 1000/$

Constraints:

$$G_j(x) = x_j \geq 0\,, \qquad \text{for } j = 1, 2$$

Minimum:

$$x^* = (12, 12)\,, \qquad F(x^*) = -3456\,, \qquad \text{no constraints are active}$$

Start:

$$x^{(0)} = (1, 1)\,, \qquad F(x^{(0)}) = -68$$

Problem 2.37 (corridor model)

Objective function:

$$F(x) = -\sum_{i=1}^{n} x_i\,, \qquad \text{for } n = 3$$

Constraints:

$$G_j(x) = \begin{cases} -x_j + 100 \geq 0\,, & \text{for } j = 1(1)n \\ x_{j-n+1} - \frac{1}{j-n} \sum\limits_{i=1}^{j-n} x_i + \sqrt{\frac{j-n+1}{j-n}} \geq 0\,, & \text{for } n+1 \leq j \leq 2\,n-1 \\ -x_{j-2\,n+2} + \frac{1}{j-2\,n+1} \sum\limits_{i=1}^{j-2\,n+1} x_i + \sqrt{\frac{j-2\,n+2}{j-2\,n+1}} \geq 0\,, & \text{for } 2\,n \leq j \leq 3\,n-2 \end{cases}$$

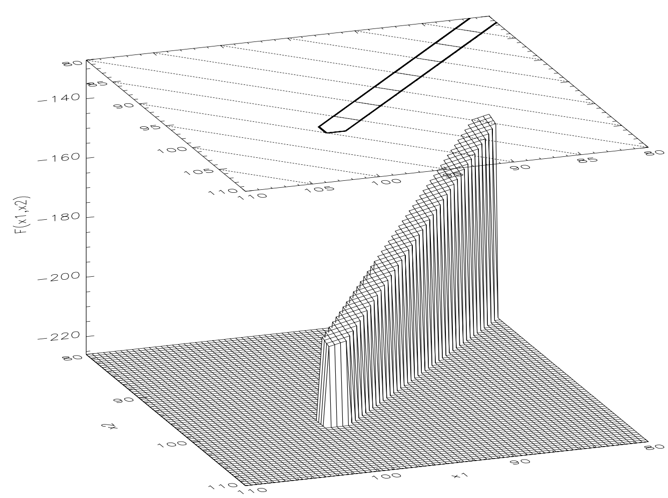

Figure A.23: Graphical representation of Problem 2.37 for $n = 2$
$F(x) = / -220, -215, -210, -205, -200, -195, -190, -185, -180, -175, -170, -165, -160/$

The constraints form a feasible region, which could be described as a corridor with a square cross section (three dimensionally speaking). The axis of the corridor runs along the diagonal in the space

$$x_1 = x_2 = x_3 = \ldots = x_n$$

The contours of the linear objective function run perpendicular to this axis. In order to obtain a finite minimum further constraints were added, whereby a kind of pencil point is placed on the end of the corridor. In the absence of these additional constraints the problem corresponds to the corridor model used by Rechenberg (1973), for which he derived theoretically the rate of progress (a measure of the convergence rate) of the two membered evolution strategy.

Minimum:

$$x_i^* = 100\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = -300\,, \qquad G_1 \text{ to } G_n \text{ active}$$

Start:

$$x_i^{(0)} = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 0$$

Problem 2.38

As Problem 2.25, but with the additional constraints:

$$G_j(x) = x_j - 1 \geq 0\,, \qquad \text{for } j = 1(1)n\,, \qquad n = 5$$

Minimum:

$$x_i^* = 1\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0\,, \qquad \text{all } G_j \text{ active}$$

Start:

$$x_i^{(0)} = -10\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 48884$$

The starting point is not in the feasible region.

Problem 2.39 after Rosen and Suzuki (1965)

Objective function:

$$F(x) = x_1^2 + x_2^2 + 2\,x_3^2 + x_4^2 - 5\,x_1 - 5\,x_2 - 21\,x_3 + 7\,x_4$$

Constraints:

$$\begin{aligned} G_1(x) &= -2\,x_1^2 - x_2^2 - x_3^2 - 2\,x_1 + x_2 + x_4 + 5 \geq 0 \\ G_2(x) &= -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8 \geq 0 \\ G_3(x) &= -x_1^2 - 2\,x_2^2 - x_3^2 - 2\,x_4^2 + x_1 + x_4 + 10 \geq 0 \end{aligned}$$

Minimum:

$$x^* = (0, 1, 2, -1)\,, \qquad F(x^*) = -44\,, \qquad G_1 \text{ active}$$

Start:

$$x^{(0)} = (0,0,0,0)\,, \qquad F(x^{(0)}) = 0$$

None of the search methods that operate directly with constraints, i.e., without reformulating the objective functions, managed to solve the problem to satisfactory accuracy.

Problem 2.40

Objective function:

$$F(x) = -\sum_{i=1}^{5} x_i$$

Constraints:

$$G_j(x) = \begin{cases} x_j \geq 0\,, & \text{for } j = 1(1)5 \\ -\sum\limits_{i=1}^{5} (9+i)\, x_i + 50000 \geq 0\,, & \text{for } j = 6 \end{cases}$$

This is a simple linear programming problem. The solution is in a corner of the allowed region defined by the constraints (simplex).

Minimum:

$$x^* = (5000,0,0,0,0)\,, \qquad F(x^*) = -5000\,, \qquad G_2 \text{ to } G_6 \text{ active}$$

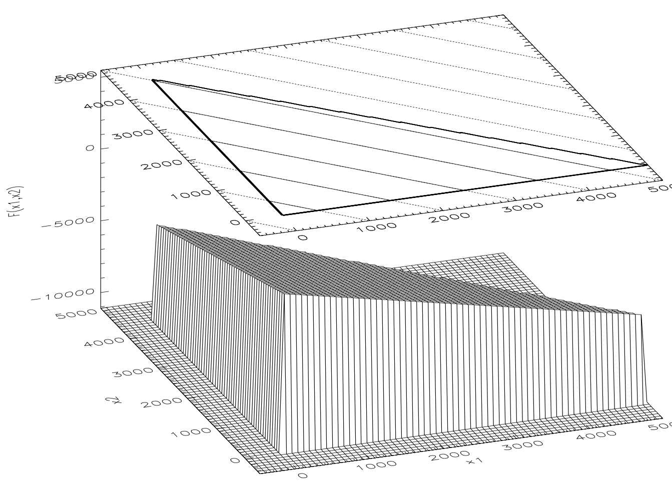

Figure A.24: Graphical representation of Problem 2.40 on the plane
$x_3 = x_4 = x_5 = 0$
$F(x) = / -10500, -9500, -8500, -7500, -6500,$
$-5500, -4500, -3500, -2500, -1500, -500, 500/$

Start:

$$x^{(0)} = (250, 250, 250, 250, 250), \qquad F(x^{(0)}) = -1250$$

In terms of the values of the variables, none of the strategies tested achieved accuracies better than 10^{-2}. The two variants of the $(10, 100)$ evolution strategy came closest to the exact solution.

Problem 2.41

Objective function:

$$F(x) = -\sum_{i=1}^{5} (i\, x_i)$$

Constraints:

as for Problem 2.40

Minimum:

$$x^* = \left(0, 0, 0, 0, \frac{50000}{14}\right), \qquad F(x^*) = \frac{-250000}{14}$$

$$G_j \text{ active for } j = 1, 2, 3, 4, 6$$

Start:

$$x^{(0)} = (250, 250, 250, 250, 250), \qquad F(x^{(0)}) = -3750$$

This problem differs from the previous one only in the numerical values; regarding the accuracies achieved, the same remarks apply as for Problem 2.40.

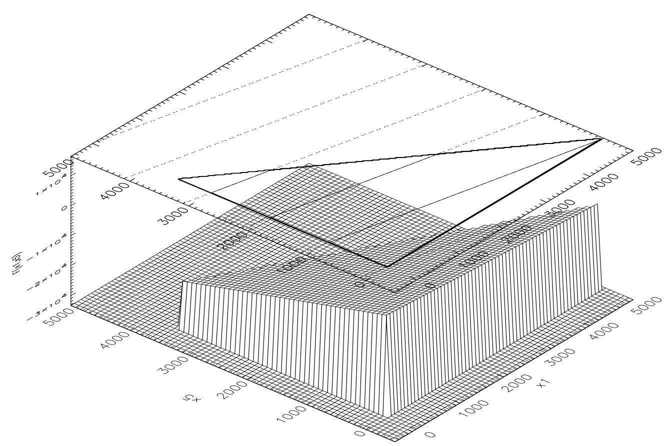

Figure A.25: Graphical representation of Problem 2.41 on the plane
$x_2 = x_3 = x_4 = 0$
$F(x) = / - 30000, -25000, -20000$
$-15000, -10000, -5000, 0/$

Problem 2.42

Objective function:

$$F(x) = \sum_{i=1}^{5} (i\, x_i)$$

Constraints:

as for Problems 2.40 and 2.41

Minimum:

$$x^* = (0, 0, 0, 0, 0)\,, \qquad F(x^*) = 0\,, \qquad G_1 \text{ to } G_5 \text{ active}$$

Start:

$$x^{(0)} = (250, 250, 250, 250, 250)\,, \qquad F(x^{(0)}) = 3750$$

The minimum is at the origin of coordinates. The evolution strategies were thus better able to approach the solution by adjusting the individual step lengths. The multimembered strategy with recombinations yielded an exact solution with variable values less than 10^{-38}.

Problem 2.43

As Problem 2.42, except:

Start:

$$x^{(0)} = (-250, -250, -250, -250, -250)\,, \qquad F(x^{(0)}) = -3750$$

The starting point is not in the feasible region.
The solutions are the same as in Problem 2.42.

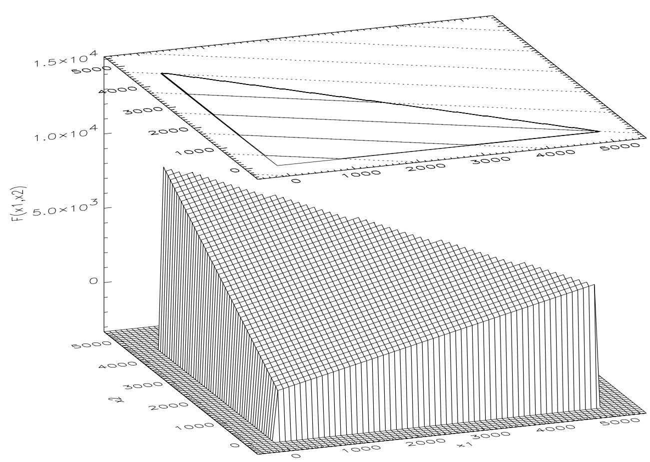

Figure A.26: Graphical representation of Problem 2.42 on the plane
$x_3 = x_4 = x_5 = 0$
$F(x) = / -1000, 1000, 3000, 5000,$
$7000, 9000, 11000, 13000, 15000/$

Problem 2.44

As Problem 2.26, but with additional constraints:

$$G_1(x) = -x + 300 \geq 0\,, \qquad G_2(x) = x + 300 \geq 0$$

Minimum:

$$x^* = -300\,, \qquad F(x^*) \simeq -299.7\,, \qquad G_2 \text{ active}$$

Besides this global minimum there are five more local minima within the feasible region.

Start:

$$x^{(0)} = 0\,, \qquad F(x^{(0)}) = 0$$

The global minimum could only be located by multimembered evolution. All the other search strategies converged to the local minimum nearest to the starting point.

Problem 2.45 of Smith and Rudd (after Leon, 1966a)

Objective function:

$$F(x) = \sum_{i=1}^{n} x_i^i\, e^{-x_i}\,, \qquad \text{for } n = 5$$

Constraints:

$$G_j = \begin{cases} x_j \geq 0\,, & \text{for } j = 1(1)n \\ 2 - x_{j-n} \geq 0\,, & \text{for } j = n+1(1)2\,n \end{cases}$$

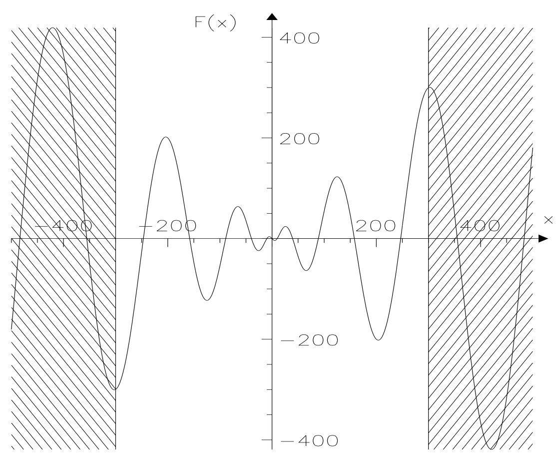

Figure A.27: Diagram $F(x)$ for Problem 2.44

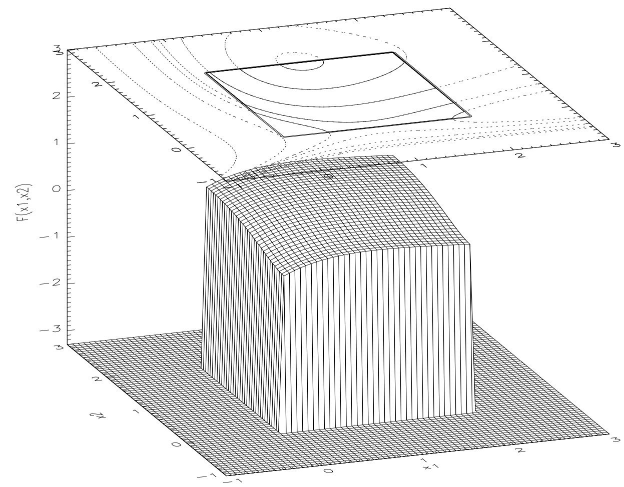

Figure A.28: Graphical representation of Problem 2.45 for n = 2
$F(x) = / -1.0, 0.0, 0.3, 0.4, 0.6, 0.8, 0.9/$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0\,, \qquad \text{all } G_1 \text{ to } G_n \text{ active}$$

Besides this global minimum there is another local one:

$$x_i' = (2, 0, \ldots, 0)\,, \qquad F(x') = 2\,e^{-2}\,, \qquad G_2 \text{ to } G_{n+1} \text{ active}$$

Start:

$$x_i^{(0)} = 1\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) \simeq 1.84$$

In the neighborhood of the minimum sought, the rate of convergence of a search strategy depends strongly on its ability to make widely different individual adjustments to the step lengths for the changes in the variables. The multimembered evolution solved this problem best when working with recombination. Rosenbrock's method converged to the local minimum, as did the complex method and the simple evolution strategies.

Problem 2.46

Objective function:

$$F(x) = x_1^2 + x_2^2$$

Constraints:

$$G_1(x) = x_1 + 2\,x_2 - 2 \geq 0$$

Minimum:

$$x^* = (0.4, 0.8)\,, \qquad F(x^*) = 0.8\,, \qquad G_1 \text{ active}$$

Start:

$$x^{(0)} = (10, 10)\,, \qquad F(x^{(0)}) = 200$$

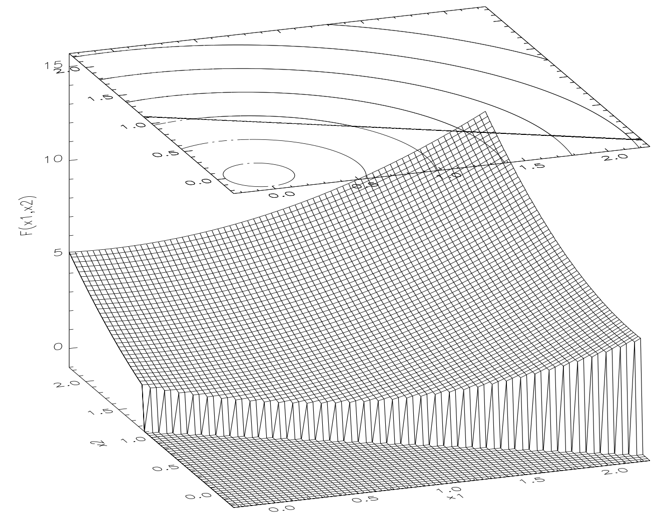

Figure A.29: Graphical representation of Problem 2.46
$F(x) = /0.04, 0.36, 1.00, 1.96, 3.24, 4.84, 6.76/$

Problem 2.47 after Ueing (1971)

Objective function:

$$F(x) = -x_1^2 - x_2^2$$

Constraints:

$$\begin{aligned}
G_j(x) &= x_j \geq 0\,, \qquad \text{for } j = 1, 2 \\
G_3(x) &= x_1^2 + x_2^2 - 17\,x_1 - 5\,x_2 + 66 \geq 0 \\
G_4(x) &= x_1^2 + x_2^2 - 10\,x_1 - 10\,x_2 + 41 \geq 0 \\
G_5(x) &= x_1^2 + x_2^2 - 4\,x_1 - 14\,x_2 + 45 \geq 0 \\
G_6(x) &= -x_1 + 7 \geq 0 \\
G_7(x) &= -x_2 + 7 \geq 0
\end{aligned}$$

Minimum:

$$x^* = (6, 0)\,, \qquad F(x^*) = -36\,, \qquad G_2 \text{ and } G_3 \text{ active}$$

Besides the global minimum x^* there are three other local minima:

$$\begin{aligned}
x' &\simeq (2.116, 4.174)\,, & F(x') &\simeq -21.90 \\
x'' &= (0, 5)\,, & F(x'') &= -25 \\
x''' &= (5, 2)\,, & F(x''') &= -29
\end{aligned}$$

Start:

$$x^{(0)} = (0, 0)\,, \qquad F(x^{(0)}) = 0$$

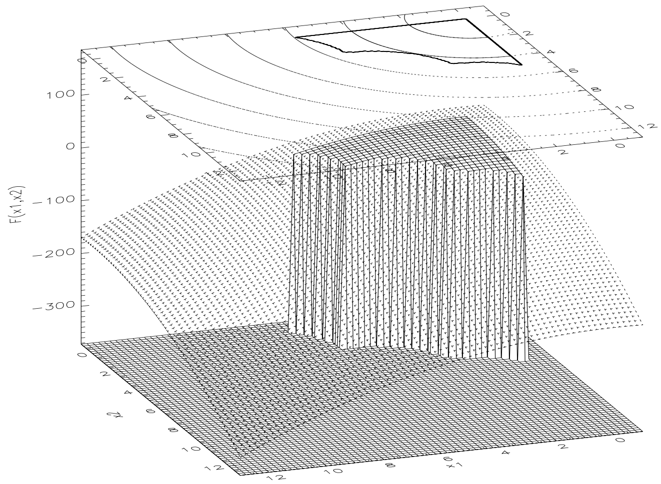

Figure A.30: Graphical representation of Problem 2.47
$F(x) = / -4, -16, -36, -64, -100, -144, -196, -256/$

To the original problem have been added the two constraints G_6 and G_7. Without them there are two separate feasible regions and the global minimum is at infinity, in the external, open region. Depending on the initial step lengths, the evolution strategies were sometimes able to go out from the starting point within the inner, closed region into the external region. After adding G_6 and G_7, the multimembered strategies converged to the global minimum, all other search methods located other local minima; which of these was located by the two membered evolution strategy, depended on the sequence of random numbers.

Problem 2.48 after Ueing (1971)

Objective function:

$$F(x) = -x_1^2 - x_2^2$$

Constraints:

$$\begin{aligned}
G_j(x) &= x_j \geq 0\,, \qquad \text{for } j = 1,2 \\
G_3(x) &= -x_1 + x_2 + 4 \geq 0 \\
G_4(x) &= \frac{x_1}{3} - x_2 + 4 \geq 0 \\
G_5(x) &= x_1^2 + x_2^2 - 10\,x_1 - 10\,x_2 + 41 \geq 0
\end{aligned}$$

Minimum:

$$x^* = (12, 8)\,, \qquad F(x^*) = -208\,, \qquad G_3 \text{ and } G_4 \text{ active}$$

Besides this global minimum there are two more local minima:

$$x' \simeq (2.018, 4.673)\,, \qquad F(x') \simeq -25.91$$

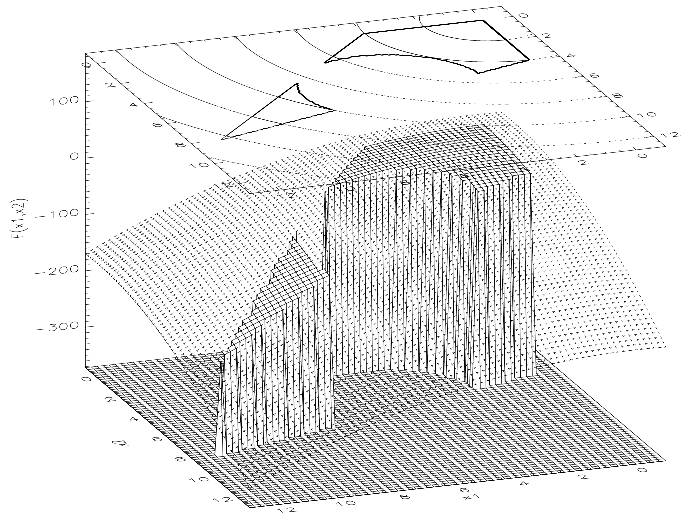

Figure A.31: Graphical representation of Problem 2.48
$F(x) = / -4, -16, -36, -64, -100, -144, -196, -256/$

$$x'' \simeq (6.293, 2.293)\,, \qquad F(x'') \simeq -44.86$$

Start:

$$x^{(0)} = (0, 0)\,, \qquad F(x^{(0)}) = 0$$

There are two feasible regions which are unconnected and closed. The starting point and the global minimum are separated by a non-feasible region. Only the (10 , 100) evolution strategy converged to the global minimum. It sometimes happened with this strategy that one descendant of a generation would jump from one feasible region to the other; however, the group of remaining individuals would converge to one of the other local minima. All other strategies did not converge to the global minimum.

Problem 2.49 after Wolfe (1966)

Objective function:

$$F(x) = \frac{4}{3}\,(x_1^2 + x_2^2 - x_1\,x_2)^{\frac{3}{4}} + x_3$$

Constraints:

$$G_j(x) = x_j \geq 0\,, \qquad \text{for } j = 1(1)3$$

Minimum:

$$x^* = (0, 0, 0)\,, \qquad F(x^*) = 0\,, \qquad \text{all } G_j \text{ active}$$

Start:

$$x^{(0)} = (10, 10, 10)\,, \qquad F(x^{(0)}) \simeq 52.16$$

Problem 2.50

As Problem 2.37, but with some other constraints:

$$G_j(x) = -x_j + 100 \geq 0\,, \qquad \text{for } j = 1(1)n$$

$$G_{n+1}(x) = 1 - \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} (x_j) - x_i \right)^2 \geq 0$$

Minimum:

$$x_i^* = 100\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = -300\,, \qquad \text{for } n = 3\,, \qquad G_1 \text{ to } G_n \text{ active}$$

Start:

$$x_i^{(0)} = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 0$$

Instead of the $2n - 2$ linear constraints of Problem 2.37, a non-linear constraint served here to bound the corridor at its sides. From a geometrical point of view, the cross section of the corridor for $n = 3$ variables is now circular instead of square. For $n = 2$ variables the two problems become equivalent.

A.3 Test Problems for the Third Part of the Strategy Comparison

These are usually n-dimensional extensions of problems from the second set of tests, whose numbers are given in brackets after the new problem number.

Problem 3.1 (analogous to Problem 2.4)

Objective function:

$$F(x) = \sum_{i=1}^{n} \left[(x_1 - x_i^2)^2 + (1 - x_i)^2 \right]$$

Minimum:

$$x_i^* = 1\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

No noteworthy difficulties arose in the solution of this and the following biquadratic problem with any of the comparison strategies. Away from the minimum, the contour patterns of the objective functions resemble those of the n-dimensional sphere problem (Problem 1.1). Nevertheless, the slight differences caused most search methods to converge much more slowly (typically by a factor 1/5). The simplex strategy was particularly affected. The computation time it required were about 10 to 30 times as long as for the sphere problem with the same number of variables. With $n = 100$ and greater, the required accuracy was only achieved in Problem 3.1 after at least one collapse and subsequent reconstruction of the simplex. The evolution strategies on the other hand were all practically unaffected by the difference with respect to Problem 1.1. Also for the complex method the cost was only slightly higher, although with this strategy the computation time increased very rapidly with the number of variables for all problems.

Problem 3.2 (analogous to Problem 2.25)

Objective function:

$$F(x) = \sum_{i=2}^{n} \left[(x_1 - x_i^2)^2 + (1 - x_i)^2 \right]$$

Minimum:

$$x_i^* = 1\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Problem 3.3 (analogous to Problem 2.13)

Objective function:

$$F(x) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (a_{ij} \sin \alpha_j + b_{ij} \cos \alpha_j) - \sum_{j=1}^{n} (a_{ij} \sin x_j + b_{ij} \cos x_j) \right)^2$$

where a_{ij}, b_{ij} for $i, j = 1(1)n$ are integer random numbers from the range $[-100, 100]$, and $\alpha_j, j = 1(1)n$ are random numbers from the range $[-\pi, \pi]$.

Minimum:

$$x_i^* = \alpha_i\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Besides this desired minimum there are numerous others that have the same value (see Problem 2.13). The a_{ij} and b_{ij} require storage space of order $O(n^2)$. For this reason the maximum number of variables for which this problem could be set up had to be limited to $n_{max} = 30$. The computation time per function call also increases as $O(n^2)$. The coordinate strategies ended the search for the minimum before reaching the required accuracy when 10 or more variables were involved. The method of Davies, Swann, and Campey (DSC) with Gram-Schmidt orthogonalization and the complex method failed in the same way for 30 variables. For $n = 30$ the search simplex of the Nelder-Mead strategy also collapsed prematurely, but after a restart the minimum was sufficiently well approximated. Depending on the sequence of random numbers, the two membered evolution strategy converged either to the desired minimum or to one of the others. This was not seen to occur with the multimembered strategies; however, only one attempt could be made in each case because of the long computation times.

Problem 3.4 (analogous to Problem 2.20)

Objective function:

$$F(x) = \sum_{i=1}^{n} |x_i|$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

This problem presented no difficulties to those strategies having a line (one dimensional) search subroutine, since the axes-parallel minimizations are always successful. The simplex method on the other hand required several restarts even for just 30 variables, and for $n = 100$ variables it had to be interrupted, as it exceeded the maximum permitted computation time (8 hours) without achieving the required accuracy. The success or failure of the (1+1) evolution strategy and the complex method depended upon the actual random numbers. Therefore, in this and the following problems, whenever there was any doubt about convergence, several (at least three) attempts were made with different sequences of random numbers. It was seen that the two membered evolution strategy sometimes spent longer near one of the corners formed by the contours of the objective function, where it converged only slowly; however, it finally escaped from this situation. Thus, although the computation times were very varied, the search was never terminated prematurely. The success of the multimembered evolution strategy depended on whether or not recombination was implemented. Without recombination the method sometimes failed for just 30 variables, whereas with recombination it converged safely and with no periods of stagnation. In the latter case the computation times taken were actually no longer than for the sphere problem with the same number of variables.

Problem 3.5 (analogous to Problem 2.21)

Objective function:

$$F(x) = \max_i \{x_i\,,\ i = 1(1)n\}$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

Most of the methods using a one dimensional search failed here, because the value of the objective function is piecewise constant along the coordinate directions. The methods of Rosenbrock and of Davies, Swann, and Campey (whatever the method of orthogonalization) converged safely, since they consider trial steps that do not change the objective function value as successful. If only true improvements are accepted, as in the conjugate gradient, variable metric, and coordinate strategies, the search never even leaves the chosen starting point at one of the corners of the contour surface. The simplex and complex strategies failed for $n > 30$ variables. Even for just 10 variables the search simplex of the Nelder-Mead method had to be constructed anew after collapsing 185 times, before the desired accuracy could be achieved. For the evolution strategy with only one parent and one descendant, the probability of finding from the starting point a point with a better value of the objective function is

$$w_e = 2^{-n}$$

For this reason the (1+1) strategy failed for $n \geq 10$. The multimembered version without recombination, could solve the problem for up to $n = 10$ variables. With recombination, convergence was sometimes still achieved for $n = 30$ variables, but no longer for $n = 100$ in the three attempts made.

Problem 3.6 (analogous to Problem 2.22)

Objective function:

$$F(x) = \sum_{i=1}^{n} |x_i| + \prod_{i=1}^{n} |x_i|$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

In spite of the even sharper corners on the contour surfaces of the objective function all the strategies behaved in much the same way as they did in the minimum search of Problem 3.4. The only notable difference was with the $(10\,,100)$ evolution strategy without recombination. For $n = 30$ variables the minimum search always converged; only for $n = 100$ and above the search was no longer successful.

Problem 3.7 (analogous to Problem 2.23)

Objective function:

$$F(x) = \sum_{i=1}^{n} x_i^{10}$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0$$

The strategy of Powell failed for $n \geq 10$ variables. Since all the step lengths were set to zero the search stagnated and the internal termination criterion did not take effect. The optimization had to be interrupted externally. From $n = 30$, the variable metric method was also ineffective. The quadratic model of the objective function on which it is based led to completely false predictions of suitable search directions. For $n = 10$ the simplex method required 48 restarts, and for $n = 30$ as many as 181 in order to achieve the desired accuracy. None of the evolution strategies had any convergence difficulties in solving the problem. They were not tested further for $n > 300$ simply for reasons of computation time.

Problem 3.8 (similar to Problem 2.37) (corridor model)

Objective function:

$$F(x) = -\sum_{i=1}^{n} x_i$$

Constraints:

$$G_j(x) = \begin{cases} \sqrt{\frac{j+1}{j}} + x_{j+1} - \frac{1}{j} \sum\limits_{i=1}^{j} x_i \geq 0\,, & \text{for } j = 1(1)n-1 \\[2ex] \sqrt{\frac{j-n+2}{j-n+1}} - x_{j-n+2} + \frac{1}{j-n+1} \sum\limits_{i=1}^{j-n+1} x_i \geq 0\,, & \text{for } j = n(1)2\,n-2 \end{cases}$$

The other constraints of Problem 2.37, which bound the corridor in the direction of the minimum being sought, were omitted here. The minimum is thus at infinity.

In comparing the results of this and the following circularly bounded corridor problem with the theoretical rates of progress for this model function, the quantity of interest was the cost, not of reaching a given approximation to an objective, but of covering a given distance along the corridor axis. For the half-width of the corridor, $b = 1$ was taken. The search was started at the origin and terminated as soon as a distance $s \geq 10\,b$ had been covered, or the objective function had reached a value $F \leq -10\sqrt{n}$.

Start:

$$x_i^{(0)} = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^{(0)}) = 0$$

All the tested strategies converged satisfactorily. The number of mutations or generations required by the evolution strategies increased linearly with the number of variables, as expected. Since the number of constraints, as well as the computation time per function call, increased as $O(n)$, the total computation time increased as $O(n^3)$. Because of the maximum of 8 hours per search adopted as a limit on the computation time, the two membered evolution strategy could only be tested to $n = 300$, and the multimembered strategies to $n = 100$. Intermediate results for $n = 300$, however, confirm that the expected trend is maintained.

Problem 3.9 (similar to Problem 2.50)

Objective function:

$$F(x) = -\sum_{i=1}^{n} x_i$$

Constraint:

$$G(x) = 1 - \sum_{i=1}^{n} \left(\frac{1}{n} \sum_{j=1}^{n} (x_j) - x_i \right)^2 \geq 0$$

Minimum, starting point and convergence criterion as in Problem 3.8.

The complex method failed for $n \geq 30$, but the Rosenbrock strategy simply required more objective function evaluations and orthogonalizations compared to the rectangular corridor. The evolution strategies converged safely. They too required more mutations or generations than in the previous problem. However, since only one constraint instead of $2\,n - 2$ was to be tested and respected, the time they took only increased as $O(n^2)$. Recombination in the multimembered version was only a very slight advantage for this and the linearly bounded corridor problem.

Problem 3.10 (analogous to Problem 2.45)

Objective function:

$$F(x) = \sum_{i=1}^{n} x_i^i\, e^{-x_i}$$

Constraints:

$$G_j(x) = \begin{cases} x_j \geq 0\,, & \text{for } j = 1(1)n \\ 2 - x_{j-n} \geq 0\,, & \text{for } j = n+1(1)2\,n \end{cases}$$

Minimum:

$$x_i^* = 0\,, \qquad \text{for } i = 1(1)n\,, \qquad F(x^*) = 0\,, \qquad \text{all } G_1 \text{ to } G_n \text{ active}$$

Besides this global minimum there is a local one within the feasible region:

$$x_i' = \begin{cases} 2\,, & \text{for } i = 1 \\ 0\,, & \text{for } i = 2(1)n \end{cases}\,, \qquad F(x') = 2\,e^{-2}$$

As in the solution of Problem 2.45 with five variables, the search methods only converged if they could adjust the step lengths individually. The strategy of Rosenbrock failed for only $n = 10$. The complex method sometimes converged for the same number of variables after about 1,000 seconds of computation time, but occasionally not even within the allotted 8 hours. For $n = 30$ variables, none of the strategies reached the objective before the time limit expired. The results obtained after 8 hours showed clearly that better progress was being made by the two membered evolution strategy and the multimembered strategy with recombination. The following table gives the best objective function values obtained by each of the strategies compared.

Rosenbrock	$10^{-4\dagger}$
Complex	10^{-7}
$(1+1)$ evolution	10^{-30}
$(10, 100)$ evolution without recombination	10^{-12}
$(10, 100)$ evolution with recombination	10^{-26}

†The Rosenbrock strategy ended the search prematurely after about 5 hours. All the other values are intermediate results after 8 hours of computation time when the strategy's own termination criteria were not yet satisfied. The searches could therefore still have come to a successful conclusion.